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ON TOEPLITZ–PLUS–HANKEL MATRICES AND TOEPLITZ–PLUS–HANKEL–BEZOUTIANS

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Abstract. The main aim of the present paper is to establish invertibility criteria for Toeplitz-plus-Hankel-Bezoutians based on the analysis of the structure of these matrices. In particular, their inverses have an explicit representation as a sum of a Toeplitz and a Hankel matrix whose symbols are the solution of certain linear systems taking the form of generalized resultant equations. These results generalize previous inversion formulas and criteria for the special cases of centrosymmetric or centroskewsymmetric Toeplitz-plus-Hankel-Bezoutians.

The inversion of Toeplitz-plus-Hankel-Bezoutians considered here is based on the converse problem, the inversion of Toeplitz-plus-Hankel matrices. Consequently, several modifications of known inversion formulas and new results for Toeplitz-plus-Hankel matrices are developed, which allow a deeper insight into the structure of these matrices, too.

1. Introduction

In the present paper we deal with two special types of structured matrices, Toeplitz-plus-Hankel matrices and Toeplitz-plus-Hankel-Bezoutians. *Toeplitz-plus-Hankel matrices* (briefly, *T+H matrices*) are matrices $T_n(\mathbf{a}) + H_n(\mathbf{b})$ which are the sum of a Toeplitz matrix and a Hankel matrix,

$$T_n(\mathbf{a}) = \left[a_{i-j}\right]_{i,j=0}^{n-1}, \qquad H_n(\mathbf{b}) = \left[b_{i+j-n+1}\right]_{i,j=0}^{n-1}. \tag{1.1}$$

Here $\mathbf{a} = (a_j)_{j=-n+1}^{n-1} \in \mathbb{F}^{2n-1}$ and $\mathbf{b} = (b_j)_{j=-n+1}^{n-1} \in \mathbb{F}^{2n-1}$ are the *symbols* of the Toeplitz and the Hankel matrix, respectively. Throughout this paper, the entries of the matrices and vectors are taken from an arbitrary field $\mathbb{F}$, for example from the field of complex numbers.

In order to define T+H-Bezoutians we use polynomial language. A matrix $B = \left[b_{ij}\right]_{i,j=0}^{n-1}$ is called a *Toeplitz-plus-Hankel-Bezoutian* (briefly, *T+H-Bezoutian*) if its generating polynomial can be written in the form

$$B(t,s) := \sum_{i,j=0}^{n-1} b_{ij} t^i s^j = \frac{\sum_{i=1}^{4} \mathbf{u}_i(t)\mathbf{v}_i(s)}{(t-s)(1-ts)} \tag{1.2}$$

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Paper OaM-16-59

for certain polynomials $\mathbf{u}_i(t), \mathbf{v}_i(t)$ $(i = 1,2,3,4)$.

It has been discovered by G. Heinig and one of the authors in 1988 [12] that the inverse of every nonsingular T+H matrix is a T+H-Bezoutian, and, vice versa, the inverse of every nonsingular T+H-Bezoutian is a T+H matrix.

The goal of the present paper is at least two-fold. First of all we want to extend known inversion formulas for T+H matrices. Secondly, and more importantly, we want to establish invertibility criteria for T+H-Bezoutians and compute their inverses. Another, related goal is to gain a deeper insight into the "structure" of T+H-Bezoutians which is not yet completely understood.

Let us mention that there is a vast literature dedicated to the inversion of Toeplitz, Hankel, and also T+H matrices, which started with the papers [17], [7], [15], and [11]. On the other hand, the converse problem – the inversion of Bezoutians – has received little attention in the past (see [10], [9], [8]).

Our considerations are inspired by a series of papers dedicated to the inversion of special classes of T+H-Bezoutians. The starting point was the inversion of Toeplitz-Bezoutians and Hankel-Bezoutians considered in [2]. By a *Hankel-Bezoutian* B_H and a *Toeplitz-Bezoutian* B_T we mean $n \times n$ matrices such that for certain polynomials $\mathbf{u}(t)$ and $\mathbf{v}(t)$, the generating polynomials of the matrices are given by

$$\begin{aligned} B_H(t,s) &= \frac{\mathbf{u}(t)\mathbf{v}(s) - \mathbf{v}(t)\mathbf{u}(s)}{t-s}, \\ B_T(t,s) &= \frac{\mathbf{u}(t)\mathbf{v}(s^{-1})s^n - \mathbf{v}(t)\mathbf{u}(s^{-1})s^n}{1-ts}. \end{aligned} \tag{1.3}$$

Hankel- or Toeplitz-Bezoutians arise as the inverses of Hankel or Toeplitz matrices (and vice versa). With the result of [2], the inversion of these Bezoutians is fairly well understood. It relies on a characterization of the nullspace of generalized resultant matrices in terms of solutions of Bézout equations [1].

In [3], [4], and [5] the inversion of T+H-Bezoutians was discussed for the first time, and inversion formulas and fast inversion algorithms were established, however, only for centrosymmetric and centroskewsymmetric matrices. The reason why these cases could be handled was that such T+H-Bezoutians possess a splitting property. It leads to simpler structured matrices, so-called split-Bezoutians, and their restricted inverses. This kind of generalized inversion was considered in [6]. All these cases are fairly well understood, too.

But up to now the general case of T+H-Bezoutians remains unresolved. This is the challenge and motivation of the present paper. We will obtain some results (in particular, criteria for invertibility and inversion formulas), although they will not be as complete or as simple as in the afore-mentioned cases.

Let us recall some history from [6]. Bezoutians were considered first in connection with elimination theory by Euler in 1748, Bézout in 1764, and Cayley in 1857 (see, e.g., [18]). Much later, in 1974, their importance for the inversion of Hankel and Toeplitz matrices was discovered by Lander [14]. He observed that the inverse of a nonsingular Hankel (Toeplitz) matrix is a Hankel- (Toeplitz-) Bezoutian and vice versa.

As mentioned above, it was discovered in [12] that the same relationship holds between T+H-Bezoutians and T+H matrices. Furthermore, in that paper, an invertibility criterion and an inversion formula for T+H matrices was established. These fundamental results (restated in Theorem 4.1 below) are the starting points of our considerations here. Interestingly, our invertibility theory for T+H-Bezoutians will be based on a modification of this invertibility criterion for T+H matrices. Furthermore, along the way we extend the known invertibility results for T+H matrices as well.

Let us make another observation. Every pair of polynomials $\mathbf{u}(t)$ and $\mathbf{v}(t)$ of appropriate degree gives rise to a Hankel- or Toeplitz-Bezoutian via formula (1.3). Unfortunately, this is no longer the case with T+H-Bezoutians and formula (1.2). The octuple of polynomials $\mathbf{u}_i(t), \mathbf{v}_i(t)$, $i = 1,2,3,4$, vastly over-parameterizes the T+H-Bezoutian, and implicit relations between the polynomials must hold. Thus it is fair to say that the "explicit structure" of T+H-Bezoutians is not yet satisfactorily understood. This state of affairs was another motivation for this paper.

The paper is organized as follows. In Section 2 we introduce notation and make basic observations about T+H matrices. In Section 3 we establish basic results on T+H-Bezoutians. In parts they are taken from [12] and concern the representation of T+H-Bezoutians in terms of a pair (U,V) related to the afore-mentioned polynomials. We introduce the notion of such a pair to be *well-posed* or *normed* and discuss an underlying uniqueness issue. Some simple necessary invertibility condition (Theorem 3.7) is obtained as well.

Section 4 is dedicated to the inversion of T+H matrices. In the first subsection we recall an important result (Theorem 4.1) from [12], on which basically all of what follows is based. In the next two subsections we derive two more invertibility criteria and present a new inversion formula.

In Subsection 4.4 we will elaborate further on this result and eliminate certain redundant parameters. This will lead to three systems of equations (4.17), (4.19) and (4.21) whose solvability is equivalent to the invertibility of the T+H matrix. The (joint) solution (U,V) will give rise to the inverse, a T+H-Bezoutian $B(U,V)$. One intricate point is the non-uniqueness of the solutions, which is necessary to discuss in order to prove the main results of this section (Theorems 4.12 and 4.18). The third system of equations (4.21) is quite cumbersome. We will investigate in Subsection 4.6 whether it can be dropped and this will give rise to the notion of *strictness*.

In Section 5 we then turn to the converse problem, the inversion of T+H-Bezoutians. We start to discuss whether invertibility criteria known for Toeplitz- or Hankel-Bezoutians can be generalized to T+H-Bezoutians. Our main results for the inversion of T+H-Bezoutians will be based on the results of Section 4, which give rise to new systems of equations (5.11), (5.12), and (5.13). The first two of them are systems of generalized resultant matrices. We will also discuss the uniqueness issue and whether the last equation is redundant. This will lead to our main results on the inversion of T+H-Bezoutians in Subsection 5.3.

Finally, let us remark that working on this topic a lot of interesting questions arise which we intend to deal with in a forthcoming paper.

2. Notation

Throughout this paper we consider vectors or matrices whose entries belong to a field $\mathbb{F}$. By $\mathbb{F}^n$ we denote the linear space of all vectors of length n, and $\{\mathbf{e}_i\}_{i=1}^n$ denotes the canonical basis in $\mathbb{F}^n$, i.e., $\mathbf{e}_i = (\delta_{ji})_{j=1}^n$, where δ_{ji} denotes the Kronecker delta. The zero vector in $\mathbb{F}^n$ is denoted by $\mathbf{0}_n$. By $\mathbb{F}^{n\times m}$ we denote the linear space of all $n\times m$ matrices, and I_n stands for the identity matrix in $\mathbb{F}^{n\times n}$.

In what follows we often use polynomial language. We associate with a matrix $A = [\ a_{ij}\ ]_{i,j=0}^{n-1} \in \mathbb{F}^{n\times n}$ the bivariate polynomial

$$A(t,s) := \sum_{i,j=0}^{n-1} a_{ij} t^i s^j. \tag{2.1}$$

and call it the *generating polynomial of* A. Similarly, with a vector $\mathbf{x} = (x_j)_{j=0}^{n-1} \in \mathbb{F}^n$ we associate the polynomial

$$\mathbf{x}(t) := \sum_{j=0}^{n-1} x_j t^j \in \mathbb{F}^n[t], \tag{2.2}$$

where $\mathbb{F}^n[t]$ denotes the linear space of all polynomials in t of degree less than n with coefficients in $\mathbb{F}$.

Hereafter J_n denotes the flip matrix of order n,

$$J_n := \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}. \tag{2.3}$$

For a vector $\mathbf{x} \in \mathbb{F}^n$ we put

$$\mathbf{x}^J := J_n \mathbf{x}.$$

In polynomial language this means $\mathbf{x}^J(t) = \mathbf{x}(t^{-1})t^{n-1}$.

We have already defined Toeplitz and Hankel matrices in the introduction in (1.1). Let us mention that for a Toeplitz matrix we have

$$T_n(\mathbf{a}^J) = T_n(\mathbf{a})^T = J_n T_n(\mathbf{a}) J_n \ \text{ with } \ \mathbf{a}^J = J_{2n-1}\mathbf{a}. \tag{2.4}$$

Furthermore, for a Hankel matrix we have $H_n(\mathbf{b}) = T_n(\mathbf{b})J_n = J_n T_n(\mathbf{b}^J)$. This notation is convenient to represent T+H matrices. Throughout this paper we will write them in the form

$$A = T_n(\mathbf{a}) + T_n(\mathbf{b})J_n, \tag{2.5}$$

where $\mathbf{a} = (a_i)_{i=-n+1}^{n+1}$ and $\mathbf{b} = (b_i)_{i=-n+1}^{n+1}$ are vectors in $\mathbb{F}^{2n-1}$, and we will use the notation

$$A = TH(\mathbf{a},\mathbf{b}).$$

The pair $(\mathbf{a},\mathbf{b}) \in \mathbb{F}^{2n-1}\times\mathbb{F}^{2n-1}$ will be called the *symbol* of the T+H matrix. Let us remark that the transpose is also a T+H matrix given by

$$A^T = TH(\mathbf{a}^J,\mathbf{b}).$$

REMARK 2.1. In contrast to pure Toeplitz or Hankel matrices, the symbol of a T+H matrix is not uniquely determined. The reason is that the intersection of the linear spaces of Toeplitz and Hankel matrices (of order n) consists of all "checkered" matrices and is a two-dimensional space. Therefore, the decomposition (2.5) is unique only up to checkered matrices.

More specifically, if we introduce the vectors of size $2n-1$,

$$\mathbf{e}_{\alpha,\beta} = [\ \alpha,\beta,\alpha,\beta,\ldots,\beta,\alpha,\beta,\alpha\ ], \qquad \alpha,\beta \in \mathbb{F},$$

and the subspace $\mathscr{W}$ of $\mathbb{F}^{2n-1} \times \mathbb{F}^{2n-1}$ of dimension two,

$$\mathscr{W} = \begin{cases} \{ (\mathbf{e}_{\alpha,\beta},\ -\mathbf{e}_{\beta,\alpha}) : \alpha,\beta \in \mathbb{F} \} & \text{if } n \text{ is even} \\ \{ (\mathbf{e}_{\alpha,\beta},\ -\mathbf{e}_{\alpha,\beta}) : \alpha,\beta \in \mathbb{F} \} & \text{if } n \text{ is odd,} \end{cases} \tag{2.6}$$

then we can say that $TH(\mathbf{a},\mathbf{b}) = 0$ if and only if $(\mathbf{a},\mathbf{b}) \in \mathscr{W}$. It follows that the symbol $(\mathbf{a},\mathbf{b})$ of a T+H matrix is determined only up to vectors in $\mathscr{W}$.

Note that, consequently, the set of all $n \times n$ T+H matrices is a linear subspace of $\mathbb{F}^{n\times n}$ of dimension $2(2n-1)-2 = 4n-4$.

3. T+H-Bezoutians: known and basic results

Recall that a T+H-Bezoutian is an $n \times n$ matrix B for which there exist eight vectors $\mathbf{u}_i, \mathbf{v}_i \in \mathbb{F}^{n+2}$ $(i = 1,2,3,4)$ such that, in polynomial language,

$$B(t,s) = \frac{\sum_{i=1}^{4} \mathbf{u}_i(t)\mathbf{v}_i(s)}{(t-s)(1-ts)}. \tag{3.1}$$

As already mentioned earlier, the relationship between T+H-Bezoutians and T+H matrices is shown in the following important theorem, which was established in [12].

THEOREM 3.1. *The inverse of an invertible T+H-Bezoutian is an invertible T+H matrix, and vice versa.*

We introduce a matrix transformation $\nabla : \mathbb{F}^{n\times n} \to \mathbb{F}^{(n+2)\times(n+2)}$ by

$$(\nabla C)(t,s) = (t-s)(1-ts)C(t,s), \qquad C \in \mathbb{F}^{n\times n}, \tag{3.2}$$

which in matrix language can be expressed as

$$\nabla C = \begin{bmatrix} 0 & -\mathbf{r}_1^T & 0 \\ \mathbf{c}_1 & CW_n - W_nC & \mathbf{c}_n \\ 0 & -\mathbf{r}_n^T & 0 \end{bmatrix} \text{ with } W_n = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & 0 & 1 \\ 0 & \cdots & 1 & 0 \end{bmatrix}, \tag{3.3}$$

where $\mathbf{c}_1, \mathbf{c}_n$ $(\mathbf{r}_1, \mathbf{r}_n)$ are the first and last columns (rows) of C.

We know from [12] that an $n \times n$ matrix A is a T+H matrix if and only if the matrix of order $(n-2)$ in the centre of ∇A is the zero matrix. In other words, if we delete the first two and the last two rows and columns of ∇A then we have the zero matrix.

Now we can rephrase the definition of a T+H-Bezoutian by saying that B is a T+H-Bezoutian if and only if $\operatorname{rank} \nabla B \leqslant 4$, or, in other words, if and only if there are $(n+2) \times 4$ matrices

$$U = \left[\begin{array}{cccc} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \end{array}\right], \qquad V = \left[\begin{array}{cccc} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{array}\right] \tag{3.4}$$

such that we have the decomposition

$$\nabla B = UV^T. \tag{3.5}$$

It is obvious from (3.1) that the pair (U,V) uniquely determines the Bezoutian B. Therefore, we will use the notation

$$B = B(U,V). \tag{3.6}$$

Conversely, however, (U,V) is not uniquely determined by B. Indeed, one can always replace the pair (U,V) by $(\widehat{U}, \widehat{V})$ where $\widehat{U} = UC^{-1}$, $\widehat{V} = VC^T$ and C is a nonsingular 4×4 matrix. In case $\operatorname{rank} \nabla B = 4$ this is the only allowed modification.

On the other hand, not every pair (U,V) of the form (3.4) defines a T+H-Bezoutian via (3.1). Clearly, for a T+H-Bezoutian $B = B(U,V)$ the sum

$$(\nabla B)(t,s) = \sum_{i=1}^{4} \mathbf{u}_i(t)\mathbf{v}_i(s)$$

is divisible by both $(t-s)$ and $(1-ts)$, which implies that

$$\sum_{i=1}^{4} \mathbf{u}_i(t)\mathbf{v}_i(t) = 0 \quad \text{and} \quad \sum_{i=1}^{4} \mathbf{u}_i(t)\mathbf{v}_i^J(t) = 0. \tag{3.7}$$

Denoting

$$U(t) = [\mathbf{u}_1(t), \mathbf{u}_2(t), \mathbf{u}_3(t), \mathbf{u}_4(t)] \quad \text{and} \quad V(t) = [\mathbf{v}_1(t), \mathbf{v}_2(t), \mathbf{v}_3(t), \mathbf{v}_4(t)]$$

we can write this condition more compactly as

$$U(t)V^T(t) = U(t)V^T(t^{-1}) = 0. \tag{3.8}$$

DEFINITION 3.2. We will call the pair (U,V) *well-posed* if conditions (3.7) (or, equivalently, (3.8)) are satisfied.

PROPOSITION 3.3. *Suppose the pair* (U,V) *is well-posed. Then* (3.1) *uniquely defines a T+H-Bezoutian* $B = B(U,V)$.

Proof. Consider $c(t,s) = \sum_{i=1}^{4} \mathbf{u}_i(t)\mathbf{v}_i(s)$. Then $c(t,t) = 0$ and $c(t,t^{-1}) = 0$ by (3.7). Hence $c(t,s)$ is divisible by $(t-s)$ and divisible by $(1-ts)$. A simple exercise shows that then $c(t,s)$ is divisible by the product $(t-s)(1-ts)$, which implies that $B(t,s)$ given by (3.1) is a bivariate polynomial of appropriate degree, and thus determines an $n\times n$ matrix. $\square$

We continue to cite two theorems from [12] (Theorems 4.1 and 4.2) and to comment on their proofs.

THEOREM 3.4. *Suppose* B *is a nonsingular matrix with* $\operatorname{rank}\nabla B = 4$. *Then* B *is the inverse of a T+H matrix.*

The proof of this theorem was done in two steps. First, it was established that ∇B admits a certain representation, $\nabla B = UV^T$, in which (U,V) take a special form to be described below. The representation was then used to prove that B^{-1} is a T+H matrix. Notice that this theorem is the main ingredient for the proof of one part of Theorem 3.1.

THEOREM 3.5. *Let* B *be a matrix of order* $n \geqslant 2$. *If* $\operatorname{rank}\nabla B < 4$, *then the first and the last column or the first and the last row of* B *are linearly dependent. Hence, in particular,* B *is singular.*

In fact, the following was proved in [12]: If B is a T+H-Bezoutian (i.e., $\operatorname{rank}\nabla B \leqslant 4$) and the first and last row are linearly independent and the first and last column are linearly independent, too, then $\operatorname{rank}\nabla B = 4$. In addition, the afore-mentioned special form of $\nabla B = UV^T$ was established, which is described now.

DEFINITION 3.6. We say that U is *normed* or V is *normed* (or the pair (U,V) is *normed*) if these matrices are of the form

$$U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{x}_4 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{3.9}$$

and

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ \mathbf{y}_1 & \mathbf{y}_2 & -\mathbf{y}_3 & -\mathbf{y}_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.10}$$

with certain $\mathbf{x}_i, \mathbf{y}_i \in \mathbb{F}^n$ $(i = 1,\dots,4)$.

A T+H-Bezoutian B is called *normable* if it admits a representation $B = B(U,V)$ with a normed (and necessarily well-posed) pair (U,V).

Note that representations $B = B(U,V)$ with normed (U,V) are not unique and that normable T+H-Bezoutians may also have non-normed representations.

Using this terminology we will now summarize the results stated above (see also the first part of proof of Theorem 4.1 in [12]). In fact, one can derive the following from the definition of the Bezoutians without using the relationship to T+H matrices as their inverses.

THEOREM 3.7. *Let B be a T+H-Bezoutian such that the first and last rows are linearly independent and the first and last columns are linearly independent. Then* $\operatorname{rank} \nabla B = 4$ *and B is normable.*

Clearly, in this theorem the assumptions on the rows/columns independence can be replaced by the stronger assumption that B is nonsingular.

There are important nontrivial T+H-Bezoutians B with $\operatorname{rank} \nabla B < 4$. For example, the split-Bezoutians considered in [6] are T+H-Bezoutians with $\operatorname{rank} \nabla B \leqslant 2$ (see the definition (5.5)). While they are singular, they can be inverted with respect to certain subspaces. A particularly simple example of such split-Bezoutians are "checkered" matrices such as $B = [(-1)^{i+j}]_{i,j=0}^{n-1}$. These are T+H-Bezoutians with $\operatorname{rank} \nabla B = 2$ and also T+H matrices, but they are not restrictedly invertible in the sense of [6].

Since the focus in this paper are invertible T+H-Bezoutians, which are always normable, we proceed to establish results on normable Bezoutians.

PROPOSITION 3.8. *Let $B = B(U,V)$, where the pair (U,V) is normed and well-posed. Then, with the notation (3.9) and (3.10), the vectors $\mathbf{x}_3 = (x_{3i})_{i=1}^n$ and $\mathbf{x}_4 = (x_{4i})_{i=1}^n$ are the first and the last column of B and the vectors $\mathbf{y}_1 = (y_{1i})_{i=1}^n$ and $\mathbf{y}_2 = (y_{2i})_{i=1}^n$ are the first and last row of B. In notation,*

$$\mathbf{x}_3 = B\mathbf{e}_1, \quad \mathbf{x}_4 = B\mathbf{e}_n, \quad \mathbf{y}_1 = B^T\mathbf{e}_1, \quad \mathbf{y}_2 = B^T\mathbf{e}_n.$$

In particular,

$$y_{11} = x_{31}, \quad y_{1n} = x_{41}, \quad y_{21} = x_{3n}, \quad y_{2n} = x_{4n}. \tag{3.11}$$

Proof. By assumption we have $\nabla B = UV^T$. Using formula (3.3) we consider the first/last row/column therein to arrive at the first statement. The second statement, (3.11), is a consequence of it. Indeed, the entries in (3.11) are the entries which occur at the four corners of the matrix B. $\square$

In order to discuss the issue of non-uniqueness of (U,V) for a normable Bezoutian $B(U,V)$, let us introduce the following equivalence relation between normed pairs (U,V). We say that two normed pairs (U,V) and $(\widehat{U},\widehat{V})$ are equivalent, written as

$$(U,V) \sim (\widehat{U},\widehat{V}), \tag{3.12}$$

if there is an $X \in \mathbb{F}^{2\times 2}$ so that

$$\widehat{U} = U\begin{bmatrix} I_2 & \mathbf{0} \\ X & I_2 \end{bmatrix}, \quad \widehat{V} = V\begin{bmatrix} I_2 & -X^T \\ \mathbf{0} & I_2 \end{bmatrix}.$$

In view of (3.8) it is easy to see that the property of being well-posed is invariant under this equivalence relation. Furthermore, if $(U,V) \sim (\widehat{U},\widehat{V})$ with both pairs being normed and well-posed, then $UV^T = \widehat{U}\widehat{V}^T$ and therefore $B(U,V) = B(\widehat{U},\widehat{V})$. Under an additional assumption on the rank, the converse is also true.

PROPOSITION 3.9. *Let both (U,V) and $(\widehat{U},\widehat{V})$ be normed and well-posed. If $B = B(U,V) = B(\widehat{U},\widehat{V})$ and* $\operatorname{rank} \nabla B = 4$, *then* $(U,V) \sim (\widehat{U},\widehat{V})$.

Proof. Using the previous proposition and with the appropriate notation we immediately see that $\mathbf{x}_3 = \hat{\mathbf{x}}_3$, $\mathbf{x}_4 = \hat{\mathbf{x}}_4$ and $\mathbf{y}_1 = \hat{\mathbf{y}}_1$, $\mathbf{y}_2 = \hat{\mathbf{y}}_2$ as these vectors are columns and rows of B. From the assumption $B = B(U,V) = B(\widehat{U},\widehat{V})$ it follows that $\nabla B = UV^T = \widehat{U}\widehat{V}^T$. Since $\operatorname{rank} \nabla B = 4$, the columns of U are linear combinations of the columns of $\widehat{U}$ and vice versa. Combining the specific form of the first two columns (notice the -1's therein) with the fact that the last two columns in U and $\widehat{U}$ are the same, it follows that there exists a 2×2 matrix $X_1 \in \mathbb{F}^{2\times 2}$ such that

$$\widehat{U} = U \begin{bmatrix} I_2 & \mathbf{0} \\ X_1 & I_2 \end{bmatrix}.$$

By a similar argumentation, there exists a 2×2 matrix $X_2 \in \mathbb{F}^{2\times 2}$ such that

$$\widehat{V} = V \begin{bmatrix} I_2 & -X_2^T \\ \mathbf{0} & I_2 \end{bmatrix}.$$

Now observe that

$$UV^T = \widehat{U} \begin{bmatrix} I_2 & \mathbf{0} \\ X_1 & I_2 \end{bmatrix} \begin{bmatrix} I_2 & \mathbf{0} \\ -X_2 & I_2 \end{bmatrix} \widehat{V}^T = \widehat{U} \begin{bmatrix} I_2 & \mathbf{0} \\ X_1 - X_2 & I_2 \end{bmatrix} \widehat{V}^T = \widehat{U}\widehat{V}^T$$

and, in particular,

$$\begin{bmatrix} \hat{\mathbf{u}}_3 & \hat{\mathbf{u}}_4 \end{bmatrix} (X_1 - X_2) \begin{bmatrix} \hat{\mathbf{v}}_1^T \\ \hat{\mathbf{v}}_2^T \end{bmatrix} = \mathbf{0}.$$

Since $\begin{bmatrix} \hat{\mathbf{u}}_3 & \hat{\mathbf{u}}_4 \end{bmatrix}$ and $\begin{bmatrix} \hat{\mathbf{v}}_1 & \hat{\mathbf{v}}_2 \end{bmatrix}$ have rank two, this leads to $X_1 - X_2 = \mathbf{0}$. Thus, $(U,V) \sim (\widehat{U},\widehat{V})$. $\square$

4. Inversion of T+H matrices

4.1. Known results and some conclusions

Consider for an $n\times n$ T+H matrix $A = TH(\mathbf{a},\mathbf{b})$ with

$$\mathbf{a} = (a_i)_{i=-n+1}^{n-1}, \qquad \mathbf{b} = (b_i)_{i=-n+1}^{n-1} \tag{4.1}$$

the vectors

$$\begin{aligned} \mathbf{g}_1 &= (a_i + b_{-n-1+i})_{i=1}^{n}, & \mathbf{g}_2 &= (a_{-n-1+i} + b_i)_{i=1}^{n}, \\ \mathbf{f}_1 &= (a_{-i} + b_{-n-1+i})_{i=1}^{n}, & \mathbf{f}_2 &= (a_{n+1-i} + b_i)_{i=1}^{n}, \end{aligned} \tag{4.2}$$

where $a_{\pm n}$ and $b_{\pm n}$ can be chosen arbitrarily. These vectors are defined in a way to ensure that the $n \times (n+2)$ matrices

$$\widehat{\partial} A = \begin{bmatrix} \mathbf{g}_1 & A & \mathbf{g}_2 \end{bmatrix}, \quad \widehat{\partial} A^T = \begin{bmatrix} \mathbf{f}_1 & A^T & \mathbf{f}_2 \end{bmatrix} \tag{4.3}$$

retain the T+H structure, i.e.,

$$\widehat{\partial} A = \begin{bmatrix} a_1 & a_0 & a_{-1} & \cdots & a_{-n+1} & a_{-n} \\ \vdots & a_1 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & a_{-1} & \vdots \\ a_n & a_{n-1} & \cdots & a_1 & a_0 & a_{-1} \end{bmatrix} + \begin{bmatrix} b_{-n} & b_{-n+1} & \cdots & b_{-1} & b_0 & b_1 \\ \vdots & \vdots & \iddots & \iddots & b_1 & \vdots \\ \vdots & b_{-1} & \iddots & \iddots & \vdots & \vdots \\ b_{-1} & b_0 & b_1 & \cdots & b_{n-1} & b_n \end{bmatrix},$$

and likewise for $\widehat{\partial} A^T$, where the transpose $A^T = TH(\mathbf{a}^J, \mathbf{b})$.

Let us recall Theorem 3.1 of [12], which will be the key result for our considerations in this paper.

THEOREM 4.1. *Suppose $A = TH(\mathbf{a}, \mathbf{b})$ and the equations*

$$A\mathbf{x}_1 = \mathbf{g}_1, \quad A\mathbf{x}_2 = \mathbf{g}_2, \quad A\mathbf{x}_3 = \mathbf{e}_1, \quad A\mathbf{x}_4 = \mathbf{e}_n, \tag{4.4}$$

or the equations

$$A^T\mathbf{y}_1 = \mathbf{e}_1, \quad A^T\mathbf{y}_2 = \mathbf{e}_n, \quad A^T\mathbf{y}_3 = \mathbf{f}_1, \quad A^T\mathbf{y}_4 = \mathbf{f}_2 \tag{4.5}$$

are solvable. Then A is nonsingular, and its inverse is completely given by these solutions,

$$\nabla A^{-1} = U V^T, \tag{4.6}$$

where

$$U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{x}_4 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{4.7}$$

and

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ \mathbf{y}_1 & \mathbf{y}_2 & -\mathbf{y}_3 & -\mathbf{y}_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.8}$$

Clearly, if the assumptions of the theorem are fulfilled, then the pair (U,V) is normed and well-posed, and

$$A^{-1} = B(U,V).$$

Indeed, the latter and the well-posedness are inferred from (4.6).

REMARK 4.2. The theorem implies that the invertibility of A is equivalent to the solvability of (4.4) (or (4.5)). Therefore, the solvability of (4.4) or (4.5) is independent of the choice of $a_{\pm n}, b_{\pm n}$. However, the matrices U and V (i.e., the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_3, \mathbf{y}_4$) depend on them. The precise dependence of U and V on $a_{\pm n}, b_{\pm n}$, which can also be seen by direct inspection, will be analyzed in Lemma 4.16 and Proposition 4.17 below.

REMARK 4.3. Note that the invertibility of A follows e.g. from the solvability of the equations (4.4). In Proposition 3.4 of [12] a recurrence formula for the columns $\mathbf{b}_k = A^{-1}\mathbf{e}_k$, $k = 1, \dots, n$, was given using only the solutions $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$:

$$\begin{aligned} &\mathbf{b}_0 := \mathbf{0}, \quad \mathbf{b}_1 = \mathbf{x}_3, \\ &\mathbf{b}_{k+1} = W_n \mathbf{b}_k - \mathbf{b}_{k-1} + (\mathbf{x}_1 \mathbf{e}_1^T + \mathbf{x}_2 \mathbf{e}_n^T - \mathbf{x}_3 \mathbf{f}_1^T - \mathbf{x}_4 \mathbf{f}_2^T)\mathbf{b}_k, \end{aligned} \tag{4.9}$$

where W_n is defined in (3.3). Analogously, the invertibility of A follows from the solvability of (4.5), and a similar recurrence formula for the rows of A^{-1} can be established.

In what follows we will often use the notation (4.7) and (4.8) instead of referring to the vectors $\mathbf{x}_i$ and $\mathbf{y}_i$ individually. In view of (4.3) it is possible to write the system (4.4) equivalently as

$$(\widehat{\partial} A)\, U = [\,\mathbf{0}\,, \mathbf{0}\,, \mathbf{e}_1,\, \mathbf{e}_n\,] \tag{4.10}$$

with U given by (4.7). Likewise the system (4.5) is equivalent to

$$(\widehat{\partial} A^T)\, V = [\,\mathbf{e}_1,\, \mathbf{e}_n,\, \mathbf{0}\,, \mathbf{0}\,] \tag{4.11}$$

with V given by (4.8). Obviously, Theorem 4.1 amounts to the following.

COROLLARY 4.4. *Suppose that the equations* (4.10) *with normed U or the equations* (4.11) *with normed V are solvable. Then A is nonsingular, and its inverse is given by $A^{-1} = B(U,V)$ where the normed pair (U,V) is the (unique) solution of* (4.10) *and* (4.11).

4.2. The nullspace of ∂A and ∂A^T and invertibility

Notice that when deleting both the first and the last row in (4.10) and (4.11) we arrive at homogeneous equations. To make this more explicit we introduce the $(n-2) \times (n+2)$ matrices ∂A and ∂A^T by deleting the first and last rows of $\widehat{\partial} A$ and $\widehat{\partial} A^T$. More specifically,

$$\begin{aligned} \partial A = &\begin{bmatrix} a_2 & a_1 & a_0 & \cdots & \cdots & a_{-n+2} & a_{-n+1} \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & \cdots & a_0 & a_{-1} & a_{-2} \end{bmatrix} \\ &+ \begin{bmatrix} b_{-n+1} & b_{-n+2} & \cdots & \cdots & b_0 & b_1 & b_2 \\ \vdots & \vdots & & \iddots & & \vdots & \vdots \\ b_{-2} & b_{-1} & b_0 & \cdots & \cdots & b_{n-2} & b_{n-1} \end{bmatrix} \end{aligned} \tag{4.12}$$

and likewise for ∂A^T. Observe that ∂A and ∂A^T do not contain the (arbitrary) coefficients $a_{\pm n}$, $b_{\pm n}$ anymore. Here is what can be immediately said.

COROLLARY 4.5. *Assume $A = TH(\mathbf{a},\mathbf{b})$ is nonsingular. Then the nullspace $\ker \partial A$ is four-dimensional and $\{\mathbf{u}_i\}_{i=1}^4$ given by (4.4) and (4.7) form a basis of $\ker \partial A$. Likewise the nullspace $\ker \partial A^T$ is four-dimensional and $\{\mathbf{v}_i\}_{i=1}^4$ given by (4.5) and (4.8) form a basis of $\ker \partial A^T$.*

We are now going to investigate what an arbitrary basis taken from ∂A or ∂A^T can tell us about the invertibility of $A = TH(\mathbf{a},\mathbf{b})$. The result is based on formulas that have been established previously in [16].

THEOREM 4.6. *The matrix $A = TH(\mathbf{a},\mathbf{b})$ is invertible if and only if $\ker \partial A$ is four dimensional and for an arbitrary basis $\{\hat{\mathbf{u}}_i\}_{i=1}^4$ of $\ker \partial A$ the following 4×4 matrix C is invertible,*

$$C = P^T \begin{bmatrix} \hat{\mathbf{u}}_1 & \hat{\mathbf{u}}_2 & \hat{\mathbf{u}}_3 & \hat{\mathbf{u}}_4 \end{bmatrix},$$

where P is the $(n+2)\times 4$ matrix

$$P = \begin{bmatrix} -\mathbf{e}_1 & -\mathbf{e}_{n+2} & \mathbf{p}_1 & \mathbf{p}_2 \end{bmatrix}$$

with $\mathbf{p}_1 = (a_{1-i} + b_{i-n})_{i=0}^{n+1}$, $\mathbf{p}_2 = (a_{n-i} + b_{i-1})_{i=0}^{n+1}$. Moreover, in this case,

$$\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{u}}_1 & \hat{\mathbf{u}}_2 & \hat{\mathbf{u}}_3 & \hat{\mathbf{u}}_4 \end{bmatrix} C^{-1}$$

is of the form (4.7) and $\{\mathbf{u}_i\}_{i=1}^4$ are the solutions of (4.4).

Proof. In view of Theorem 4.1 and what has been said above, it follows that A is invertible if and only if we can find a basis $\{\mathbf{u}_i\}_{i=1}^4$ of $\ker \partial A$ which is of the form (4.7) and corresponds to a solution of (4.4). On the one hand this means that there exists an invertible matrix C which relates the two bases to each other. On the other hand if we have a basis $\{\mathbf{u}_i\}_{i=1}^4$ of $\ker \partial A$ which is of the from (4.7), then it automatically satisfies all the equations contained in (4.4) excepts possibly the ones corresponding to the first and last rows in the system. Therefore, invertibility of A comes down to finding an invertible matrix C such that the corresponding $\{\mathbf{u}_i\}_{i=1}^4$ is of the form (4.7) and satisfies the equations corresponding to the first and last row in (4.4). These last conditions amount precisely to

$$P^T \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \end{bmatrix} = I_4.$$

Now all of this implies the assertion. $\square$

An analogous results can also established for $\ker \partial A^T$. We just state the corresponding formulas. Given any basis $\{\hat{\mathbf{v}}_i\}_{i=1}^4$ of $\ker \partial A^T$ then a (normed) matrix of the form (4.8) is given by

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{v}}_1 & \hat{\mathbf{v}}_2 & \hat{\mathbf{v}}_3 & \hat{\mathbf{v}}_4 \end{bmatrix} D^{-1},$$

where

$$D = Q^T \begin{bmatrix} \hat{\mathbf{v}}_1 & \hat{\mathbf{v}}_2 & \hat{\mathbf{v}}_3 & \hat{\mathbf{v}}_4 \end{bmatrix}$$

with

$$Q = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{e}_1 & \mathbf{e}_{n+2} \end{bmatrix}$$

and $\mathbf{q}_1 = (a_{i-1} + b_{i-n})_{i=0}^{n+1}$, $\mathbf{q}_2 = (a_{i-n} + b_{i-1})_{i=0}^{n+1}$.

The construction just described together with the fact that the corresponding U or V determine the inverse of A leads to the following result.

COROLLARY 4.7. *Let $A = TH(\mathbf{a},\mathbf{b})$ be a nonsingular T+H matrix. Then $\ker\partial A$ (and $\ker\partial A^T$) is four-dimensional, and its inverse A^{-1} is completely given by any basis of $\ker\partial A$ (or of $\ker\partial A^T$).*

We illustrate this corollary and the formulas contained in Theorem 4.6 with an example.

EXAMPLE 4.8. Let A be the following nonsingular 3×3 T+H matrix,

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}.$$

The corresponding 1×5 matrix ∂A is given by

$$\partial A = \begin{bmatrix} 1, & -2, & 1, & 0, & 1 \end{bmatrix} + \begin{bmatrix} 0, & 1, & -1, & 1, & -1 \end{bmatrix} = \begin{bmatrix} 1, & -1, & 0, & 1, & 0 \end{bmatrix}.$$

Clearly its nullspace is four-dimensional and a possible basis for $\ker\partial A$ is given by the columns of the matrix

$$\widehat{U} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

Choosing $a_{\pm n} = b_{\pm n} = 0$ ($n = 3$) in the vectors $\mathbf{p}_1$ and $\mathbf{p}_2$, the matrices P and $C = P^T\widehat{U}$ become

$$P = \begin{bmatrix} -1 & 0 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ -1 & 0 & 0 & -1 \end{bmatrix}.$$

Hence $U = \widehat{U}C^{-1}$ evaluates to

$$U = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

Thus

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$

which are the solutions of (4.4) with $\mathbf{g}_1 = [-2,1,1]^T$ and $\mathbf{g}_2 = [1,0,0]^T$. Now we obtain the columns $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ of $B = A^{-1}$ by the recursion given in Remark 4.3,

$$B = \left[\ \mathbf{b}_1,\ \mathbf{b}_2,\ \mathbf{b}_3\ \right] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}.$$

4.3. Inversion using an extended matrix

We continue with another modification of Theorem 4.1, a sufficient invertibility criterion and an inversion formula for $A = TH(\mathbf{a},\mathbf{b})$ which uses only solutions of equations the right hand sides of which are unit vectors. The price to pay is that an additional condition has to be satisfied. Furthermore, the symbols $\mathbf{a},\mathbf{b}$ have to be extended by (arbitrarily chosen) entries $a_{\pm n}, a_{\pm(n+1)}, b_{\pm n}, a_{\pm(n+1)}$, so that an extended matrix A_{n+2} of order $n+2$ can be considered:

$$\mathbf{a}_e = \begin{bmatrix} a_{-n-1} \\ a_{-n} \\ \mathbf{a} \\ a_n \\ a_{n+1} \end{bmatrix}, \quad \mathbf{b}_e = \begin{bmatrix} b_{-n-1} \\ b_{-n} \\ \mathbf{b} \\ b_n \\ b_{n+1} \end{bmatrix}, \quad A_{n+2} = TH(\mathbf{a}_e, \mathbf{b}_e). \tag{4.13}$$

Note that removing the first and last row of A_{n+2} results into $\widehat{\partial} A$, while removing the first and the last column of A_{n+2} gives the transpose of $\widehat{\partial} A^T$. In particular, A is in the center of A_{n+2},

$$A_{n+2} = \begin{bmatrix} a_0 + b_{-n-1} & \mathbf{f}_1^T & a_{-n-1} + b_0 \\ \mathbf{g}_1 & A & \mathbf{g}_2 \\ a_{n+1} + b_0 & \mathbf{f}_2^T & a_0 + b_{n+1} \end{bmatrix}.$$

Hereafter for a vector $\mathbf{x} = (x_i)_{i=1}^m$ we denote

$$\mathring{\mathbf{x}} = (x_i)_{i=2}^{m-1} \quad \text{and} \quad \mathbf{x}_0^0 = (x_i)_{i=0}^{m+1} \text{ with } x_0 = x_{m+1} = 0.$$

PROPOSITION 4.9. *Let $A = TH(\mathbf{a},\mathbf{b})$ and $A_{n+2} = TH(\mathbf{a}_e,\mathbf{b}_e)$, where $\mathbf{a}_e, \mathbf{b}_e$ are introduced in (4.13). Suppose that the equations*

$$A\mathbf{x} = \mathbf{e}_1, \quad A\mathbf{z} = \mathbf{e}_n, \quad A_{n+2}\mathbf{x}^{n+2} = \mathbf{e}_1, \quad A_{n+2}\mathbf{z}^{n+2} = \mathbf{e}_{n+2} \tag{4.14}$$

are solvable and that the 2×2 matrix

$$E = \begin{bmatrix} \mathbf{e}_1^T \mathbf{x}^{n+2} & \mathbf{e}_1^T \mathbf{z}^{n+2} \\ \mathbf{e}_{n+2}^T \mathbf{x}^{n+2} & \mathbf{e}_{n+2}^T \mathbf{z}^{n+2} \end{bmatrix} \tag{4.15}$$

is nonsingular. Then A is invertible and A^{-1} is completely given by the solutions $\mathbf{x}$, $\mathbf{z}$, $\mathbf{x}^{n+2}$, $\mathbf{z}^{n+2}$.

Proof. It is easy to see that

$$\begin{bmatrix} \mathbf{g}_1 & A & \mathbf{g}_2 \end{bmatrix} \mathbf{x}^{n+2} = \mathbf{0}, \quad \begin{bmatrix} \mathbf{g}_1 & A & \mathbf{g}_2 \end{bmatrix} \mathbf{z}^{n+2} = \mathbf{0}$$

and, consequently,

$$A\mathring{\mathbf{x}}^{n+2} = -x_1\,\mathbf{g}_1 - x_{n+2}\,\mathbf{g}_2, \quad A\mathring{\mathbf{z}}^{n+2} = -z_1\,\mathbf{g}_1 - z_{n+2}\,\mathbf{g}_2,$$

where $x_1 = \mathbf{e}_1^T\mathbf{x}^{n+2}$, $x_{n+2} = \mathbf{e}_{n+2}^T\mathbf{x}^{n+2}$, $z_1 = \mathbf{e}_1^T\mathbf{z}^{n+2}$, $z_{n+2} = \mathbf{e}_{n+2}^T\mathbf{z}^{n+2}$. Since $\det E \neq 0$, the solutions $\mathbf{x}_1, \mathbf{x}_2$ of the first two equations of (4.4) are linear combinations of $\mathring{\mathbf{x}}^{n+2}$ and $\mathring{\mathbf{z}}^{n+2}$,

$$\begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} -\mathring{\mathbf{x}}^{n+2} & -\mathring{\mathbf{z}}^{n+2} \end{bmatrix} E^{-1}.$$

Taking Remark 4.3 into account completes the proof. □

Analogously, one has a corresponding result assuming that the equations

$$A^T\mathbf{y} = \mathbf{e}_1, \quad A^T\mathbf{w} = \mathbf{e}_n, \quad A_{n+2}^T\mathbf{y}^{n+2} = \mathbf{e}_1, \quad A_{n+2}^T\mathbf{w}^{n+2} = \mathbf{e}_{n+2} \tag{4.16}$$

are solvable and that

$$F = \begin{bmatrix} \mathbf{e}_1^T\mathbf{y}^{n+2} & \mathbf{e}_1^T\mathbf{w}^{n+2} \\ \mathbf{e}_{n+2}^T\mathbf{y}^{n+2} & \mathbf{e}_{n+2}^T\mathbf{w}^{n+2} \end{bmatrix}$$

is nonsingular. It is now easy to establish the following conclusion.

THEOREM 4.10. *Assume that the equations* (4.14) *and* (4.16) *are solvable. Then* $E = F^T$, *and if this matrix is nonsingular, then* $A^{-1} = B(U,V)$, *where*

$$U = \begin{bmatrix} \mathbf{x}^{n+2} & \mathbf{z}^{n+2} & \mathbf{x}_0^0 & \mathbf{z}_0^0 \end{bmatrix} \operatorname{diag}(-E^{-1}, I_2),$$
$$V = \begin{bmatrix} \mathbf{y}_0^0 & \mathbf{w}_0^0 & \mathbf{y}^{n+2} & \mathbf{w}^{n+2} \end{bmatrix} \operatorname{diag}(I_2, F^{-1}),$$

and the pair (U,V) *is normed.*

Note that by a Schur-complement argument the following is clear: Assuming A nonsingular then A_{n+2} is nonsingular if and only if $\det E \neq 0$.

4.4. Equivalent systems for invertibility

We are now going to establish a modification of Theorem 4.1 in which the redundant parameters $a_{\pm n}, b_{\pm n}$ do not occur anymore. This will lead to systems that involve the matrices A and ∂A as well as A^T and ∂A^T. Interestingly, the elimination of $a_{\pm n}, b_{\pm n}$ gives rise to an additional equation which connects the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_3, \mathbf{y}_4$ appearing in (4.4) and (4.5) with one another.

The following proposition provides an equivalent formulation for the systems (4.4) and (4.5), respectively. It is simply obtained by setting apart what correspond to the first and the last row in the first two equations of (4.4), respectively, in the last two equations of (4.5).

PROPOSITION 4.11. *Suppose $A = TH(\mathbf{a},\mathbf{b})$.*

(i) The system (4.4) is equivalent to

$$\partial A \begin{bmatrix} -1 \\ \mathbf{x}_1 \\ 0 \end{bmatrix} = \mathbf{0},\ \partial A \begin{bmatrix} 0 \\ \mathbf{x}_2 \\ -1 \end{bmatrix} = \mathbf{0},\ A\mathbf{x}_3 = \mathbf{e}_1,\ A\mathbf{x}_4 = \mathbf{e}_n, \tag{4.17}$$

and

$$\begin{bmatrix} \mathbf{e}_1^T \\ \mathbf{e}_n^T \end{bmatrix} A \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} - \begin{bmatrix} a_1 & b_1 \\ b_{-1} & a_{-1} \end{bmatrix} = \begin{bmatrix} b_{-n} & a_{-n} \\ a_n & b_n \end{bmatrix}. \tag{4.18}$$

(ii) The system (4.5) is equivalent to

$$A^T\mathbf{y}_1 = \mathbf{e}_1,\ A^T\mathbf{y}_2 = \mathbf{e}_n,\ \partial A^T \begin{bmatrix} 1 \\ -\mathbf{y}_3 \\ 0 \end{bmatrix} = \mathbf{0},\ \partial A^T \begin{bmatrix} 0 \\ -\mathbf{y}_4 \\ 1 \end{bmatrix} = \mathbf{0}, \tag{4.19}$$

and

$$\begin{bmatrix} \mathbf{e}_1^T \\ \mathbf{e}_n^T \end{bmatrix} A^T \begin{bmatrix} \mathbf{y}_3 & \mathbf{y}_4 \end{bmatrix} - \begin{bmatrix} a_{-1} & b_1 \\ b_{-1} & a_1 \end{bmatrix} = \begin{bmatrix} b_{-n} & a_n \\ a_{-n} & b_n \end{bmatrix}. \tag{4.20}$$

Note that the numbers $a_{\pm n}, b_{\pm n}$ do not occur in (4.17) and (4.19) (see also the definition (4.12) of ∂A). Since $a_{\pm n}, b_{\pm n}$ are the same numbers in (4.18) and (4.20) we can combine these two equations into one in order to eliminate these numbers,

$$\begin{bmatrix} \mathbf{e}_1^T \\ \mathbf{e}_n^T \end{bmatrix} A \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} - \begin{bmatrix} a_1 & b_1 \\ b_{-1} & a_{-1} \end{bmatrix} = \begin{bmatrix} \mathbf{y}_3^T \\ \mathbf{y}_4^T \end{bmatrix} A \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_n \end{bmatrix} - \begin{bmatrix} a_{-1} & b_{-1} \\ b_1 & a_1 \end{bmatrix}. \tag{4.21}$$

It is easy to see that regarding the solvability the following can be said:

(i) solvability of (4.4) is equivalent to the solvability of (4.17),

(ii) solvability of (4.5) is equivalent to the solvability of (4.19),

(iii) solvability of (4.4) and (4.5) is equivalent to the solvability of (4.17), (4.19), and (4.21).

Indeed, note that if, for instance, (4.17) is solvable, then system (4.4) is solvable with the numbers $a_{\pm n}, b_{\pm n}$ defined by (4.18). However, as has been observed in Remark 4.2, the solvability of (4.4) is independent of the choice of $a_{\pm n}, b_{\pm n}$. More precisely, changing $a_{\pm n}, b_{\pm n}$ amounts to modifying $\mathbf{x}_1, \mathbf{x}_2$ with linear combinations of $\mathbf{x}_3, \mathbf{x}_4$ (see Lemma 4.16 below).

Combining the previous observation with Theorem 4.1 we obtain the following characterization. Therein, and in what follows, we use the normed matrices U and V defined by (4.7) and (4.8) as notation for and in place of the vectors $\mathbf{x}_i$ and $\mathbf{y}_i$.

THEOREM 4.12. *Let* $A = TH(\mathbf{a}, \mathbf{b})$. *Then the following statements are equivalent:*

(a) *A is invertible.*

(b) *The system* (4.17) *has a normed solution* U.

(c) *The system* (4.19) *has a normed solution* V.

(d) *There exists a well-posed and normed pair* (U,V) *such that* (4.17), (4.19), (4.21) *are true and*

$$A^{-1} = B(U,V).$$

Proof. (b/c)$\Rightarrow$(a): If the system (4.17) has a solution, then also (4.18) is fulfilled for suitably chosen $a_{\pm n}$, $b_{\pm n}$. This being equivalent to (4.4) implies the invertibility of A by Theorem 4.1. The argument is the same in case the system (4.19) has a solution.

(d)$\Rightarrow$(b/c): Obvious.

(a)$\Rightarrow$(d): If A is invertible, then the systems (4.4) and (4.5) have a (unique) solution (U,V) (for any fixed choice of $a_{\pm n}$, $b_{\pm n}$). From (4.6) in Theorem 4.1 it follows that the pair (U,V) is well-posed and $A^{-1} = B(U,V)$. Furthermore, the matrices U and V satisfy (4.17), (4.18), (4.19), (4.20), and thus also (4.21). $\square$

REMARK 4.13. We note that a normed solution U to (4.17) and a normed solution to V to (4.19) as stated in (b/c) does not automatically give the pair (U,V) whose existence is claimed in (d). The reason is that the solution U to (4.17) and solution to V to (4.19) are not unique. Only if U and V 'match' with each other a 'correct' pair (U,V) for (d) is obtained.

This issue of non-uniqueness will be clarified in the next subsection (Proposition 4.17). From there one can infer that is possible to modify either U or V such that the modified pair satisfies the conditions of part (d).

REMARK 4.14. In view of (b/c) of the previous theorem, one can ask the question whether the Bezoutian $B = A^{-1}$ can be obtained solely from the normed solution U of (4.17) (or the normed solution V of (4.19)). This can be done as follows. Given $A = TH(\mathbf{a}, \mathbf{b})$ and, e.g., a solution U of (4.17), one first defines the numbers $a_{\pm n}, b_{\pm n}$ by (4.18), and then considers $\mathbf{f}_1, \mathbf{f}_2$ by (4.2). Now A^{-1} can be obtained by the recursion formula (4.9).

REMARK 4.15. Suppose (U,V) is a normed pair satisfying (only) the equations (4.17) and (4.19). In view of (d) it is natural to ask about the relationship between the following three condition: (i) the pair (U,V) is well-posed, (ii) the pair (U,V) satisfies equation (4.21), and (iii) $B = B(U,V)$ is the inverse of the T+H matrix A. In the following two subsections we will try to clarify the relationship. It is further illustrated in Examples 4.22 and 4.23 below.

4.5. Discussion of the uniqueness of U and V

Throughout this subsection we will assume that $A = TH(\mathbf{a},\mathbf{b})$ is invertible, and we will discuss various combinations of the systems involving (4.17), (4.19), and (4.21).

Before we start notice that the combined system (4.17) and (4.18) (being equivalent to (4.4)) has a unique (normed) solution U for every fixed choice of $a_{\pm n}$, $b_{\pm n}$. Thus the non-uniqueness of the system (4.17) alone only arises from modifications of the parameters $a_{\pm n}$, $b_{\pm n}$. The same holds, of course for the combined system (4.19) and (4.20). The dependence of U and V on these parameters is made explicit in the following lemma.

LEMMA 4.16. *Let $A = TH(\mathbf{a},\mathbf{b})$ be invertible. If we change*

$$\begin{bmatrix} \widehat{b}_{-n} & \widehat{a}_{-n} \\ \widehat{a}_n & \widehat{b}_n \end{bmatrix} = \begin{bmatrix} b_{-n} & a_{-n} \\ a_n & b_n \end{bmatrix} + X, \qquad X \in \mathbb{F}^{2\times 2}, \tag{4.22}$$

then

(i) the solution to the combined system (4.17) and (4.18) changes to

$$\widehat{U} = U \begin{bmatrix} I_2 & \mathbf{0} \\ X & I_2 \end{bmatrix}, \tag{4.23}$$

(ii) the solution to the combined system (4.19) and (4.20) changes to

$$\widehat{V} = V \begin{bmatrix} I_2 & -X^T \\ \mathbf{0} & I_2 \end{bmatrix}. \tag{4.24}$$

Proof. Let us prove (i). The proof of (ii) is analogous, but notice the slight difference in signs and the transposed being involved. Write the two solutions corresponding to the different parameters as

$$\begin{aligned} U &= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{x}_4 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \\ \widehat{U} &= \begin{bmatrix} \hat{\mathbf{u}}_1 & \hat{\mathbf{u}}_2 & \hat{\mathbf{u}}_3 & \hat{\mathbf{u}}_4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ \hat{\mathbf{x}}_1 & \hat{\mathbf{x}}_2 & \hat{\mathbf{x}}_3 & \hat{\mathbf{x}}_4 \\ 0 & -1 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{4.25}$$

It follows that $\mathbf{x}_3 = \hat{\mathbf{x}}_3, \mathbf{x}_4 = \hat{\mathbf{x}}_4$ since the last two equations in (4.17) do not involve $a_{\pm n}$, $b_{\pm n}$. From the first two equations and (4.18) we get

$$A \begin{bmatrix} \hat{\mathbf{x}}_1 - \mathbf{x}_1 & \hat{\mathbf{x}}_2 - \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \widehat{b}_{-n} - b_{-n} & \widehat{a}_{-n} - a_{-n} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \widehat{a}_n - a_n & \widehat{b}_n - b_n \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_n \end{bmatrix} X$$

which equals $A\left[\begin{array}{cc}\mathbf{x}_3 & \mathbf{x}_4\end{array}\right]X$, by again the last two equations in (4.17). Thus

$$\left[\begin{array}{cc}\hat{\mathbf{x}}_1 & \hat{\mathbf{x}}_2\end{array}\right]=\left[\begin{array}{cc}\mathbf{x}_1 & \mathbf{x}_2\end{array}\right]+\left[\begin{array}{cc}\mathbf{x}_3 & \mathbf{x}_4\end{array}\right]X$$

which implies (4.23). $\square$

We can apply the previous lemma to immediately obtain the following result. However, we will also give an independent proof avoiding the use of $a_{\pm n}$, $b_{\pm n}$.

PROPOSITION 4.17. *Let $A = TH(\mathbf{a},\mathbf{b})$ be invertible. Then*

(i) *solutions to (4.17) with normed U are unique up to changes*

$$U\mapsto U\left[\begin{array}{cc}I_2 & \mathbf{0}\\ X_1 & I_2\end{array}\right]\quad \text{with } X_1\in\mathbb{F}^{2\times 2}, \tag{4.26}$$

(ii) *solutions to (4.19) with normed V are unique up to changes*

$$V\mapsto V\left[\begin{array}{cc}I_2 & -X_2^T\\ \mathbf{0} & I_2\end{array}\right]\quad \text{with } X_2\in\mathbb{F}^{2\times 2}, \tag{4.27}$$

(iii) *solutions to the combined system (4.17), (4.19), and (4.21) are unique up to changes*

$$\widehat{U}=U\left[\begin{array}{cc}I_2 & \mathbf{0}\\ X & I_2\end{array}\right],\quad \widehat{V}=V\left[\begin{array}{cc}I_2 & -X^T\\ \mathbf{0} & I_2\end{array}\right],\ \text{where } X\in\mathbb{F}^{2\times 2}. \tag{4.28}$$

Recalling (3.12) this means $(U,V)\sim(\widehat{U},\widehat{V})$.

Proof. First of all notice that if we apply transformations (4.26), (4.27), or (4.28) to solutions U, V of corresponding systems, then the resulting matrices $\widehat{U}$, $\widehat{V}$ are also solutions. Thus it remains to show the converse, i.e., whether different solutions are related to each other by the given transformations.

(i): As in the proof of the previous lemma, using the notation (4.25) to denote two solutions of the system (4.17), we conclude that $\mathbf{x}_3=\hat{\mathbf{x}}_3$, $\mathbf{x}_4=\hat{\mathbf{x}}_4$. Furthermore from the first equations in (4.17) we obtain

$$A\left[\begin{array}{c}\hat{\mathbf{x}}_1-\mathbf{x}_1\end{array}\right], A\left[\begin{array}{c}\hat{\mathbf{x}}_2-\mathbf{x}_2\end{array}\right]\in \operatorname{lin}\{\mathbf{e}_1,\mathbf{e}_n\},$$

noting that ∂A does not contain the first and the last row of A. The latter is equal to $\operatorname{lin}\{A\mathbf{x}_3,A\mathbf{x}_4\}$, and thus $\hat{\mathbf{x}}_1-\mathbf{x}_1$ and $\hat{\mathbf{x}}_2-\mathbf{x}_2$ belong to $\operatorname{lin}\{\mathbf{x}_3,\mathbf{x}_4\}$ so that

$$\left[\begin{array}{cc}\hat{\mathbf{x}}_1 & \hat{\mathbf{x}}_2\end{array}\right]=\left[\begin{array}{cc}\mathbf{x}_1 & \mathbf{x}_2\end{array}\right]+\left[\begin{array}{cc}\mathbf{x}_3 & \mathbf{x}_4\end{array}\right]X_1$$

for some matrix X_1. This implies the relationship (4.26).

(ii): We proceed similarly, and using the appropriate notation we arrive at $\hat{\mathbf{y}}_1=\mathbf{y}_1,\hat{\mathbf{y}}_2=\mathbf{y}_2$ and

$$\left[\begin{array}{cc}\hat{\mathbf{y}}_3 & \hat{\mathbf{y}}_4\end{array}\right]=\left[\begin{array}{cc}\mathbf{y}_3 & \mathbf{y}_4\end{array}\right]+\left[\begin{array}{cc}\mathbf{y}_1 & \mathbf{y}_2\end{array}\right]X_2^T$$

for some matrix $X_2 \in \mathbb{F}^{2\times 2}$. This implies the (4.27).

(iii): We continue with the arguments made in (i) and (ii). The goal is to show that $X_1 = X_2$ by invoking equation (4.21). Indeed, assuming that both normed pairs (U,V) and $(\widehat{U},\widehat{V})$ satisfy (4.21) it follows that

$$\begin{bmatrix} \mathbf{e}_1^T \\ \mathbf{e}_n^T \end{bmatrix} A \begin{bmatrix} \hat{\mathbf{x}}_1 - \mathbf{x}_1 & \hat{\mathbf{x}}_2 - \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{y}}_3^T - \mathbf{y}_3^T \\ \hat{\mathbf{y}}_4^T - \mathbf{y}_4^T \end{bmatrix} A \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_n \end{bmatrix}.$$

Using what we established above we conclude that

$$\begin{bmatrix} \mathbf{e}_1^T \\ \mathbf{e}_n^T \end{bmatrix} A \begin{bmatrix} \mathbf{x}_3 & \mathbf{x}_4 \end{bmatrix} X_1 = X_2 \begin{bmatrix} \mathbf{y}_1^T \\ \mathbf{y}_2^T \end{bmatrix} A \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_n \end{bmatrix}$$

which is, using (4.17) and (4.19), equivalent to

$$\begin{bmatrix} \mathbf{e}_1^T \\ \mathbf{e}_n^T \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_n \end{bmatrix} X_1 = X_2 \begin{bmatrix} \mathbf{e}_1^T \\ \mathbf{e}_n^T \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_n \end{bmatrix}.$$

Since, clearly, $\begin{bmatrix} \mathbf{e}_1^T \\ \mathbf{e}_n^T \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_n \end{bmatrix} = I_2$ we have $X_1 = X_2$. $\square$

Using the previous considerations we are now able to establish the following result, which complements and should be appreciated in connection with Theorem 4.12.

THEOREM 4.18. *Let $A = TH(\mathbf{a},\mathbf{b})$ be invertible. Then any (normed) solution (U,V) to* (4.17)*,* (4.19)*, and* (4.21) *is well-posed and*

$$A^{-1} = B(U,V).$$

Proof. From Theorem 4.12(d) we know that one solution to these three systems exists which is also well-posed and identifies the inverse of A as the corresponding Bezoutian. Any other solution is related to this one by the transformation described in (4.28). However, such a transformation leaves the property of being well-posed invariant and leads to the same Bezoutian (as has been noted in the paragraph before Proposition 3.9). $\square$

4.6. Strict pairs and strict Bezoutians

The question we are now going to address is whether the third condition (4.21) can be dropped. One might be tempted to just rely on (4.17) and (4.19) to determine the solutions U and V. However, the problem is that the corresponding pair (U,V) may not satisfy (4.21). In fact, this pair may not even be well-posed and thus does not define a Bezoutian. In practice, i.e., for the purpose of computing the Bezoutian from (U,V), one can easily remedy the situation. One simply needs to modify either U by a suitable transformation (4.26) or V by a suitable transformation (4.27). The question which matrix X_1 or X_2 has to be applied can be answered by using equations (4.18), (4.20), and Lemma 4.16. We leave the details to the reader.

Elaborating on these lines, one is led to the following question. Suppose the (normed) pair (U,V) is a solution to (4.17) and (4.19) and is well-posed. Does it follow that (U,V) also satisfies (4.21), and thus is the correct pair giving raise to the inverse $A^{-1}=B(U,V)$? It turns out that this may or may not be the case. This dichotomy is reflected in the following definition.

DEFINITION 4.19. We say that a normed pair (U,V) is *strict* if the two polynomial equations

$$\begin{bmatrix} \mathbf{x}_3(t) & \mathbf{x}_4(t) \end{bmatrix} X \begin{bmatrix} \mathbf{y}_1(t) \\ \mathbf{y}_2(t) \end{bmatrix} = 0, \qquad \begin{bmatrix} \mathbf{x}_3(t) & \mathbf{x}_4(t) \end{bmatrix} X \begin{bmatrix} \mathbf{y}_1^J(t) \\ \mathbf{y}_2^J(t) \end{bmatrix} = 0, \tag{4.29}$$

involving a matrix $X \in \mathbb{F}^{2\times 2}$ can only be satisfied simultaneously when X is the zero matrix. Therein we use the notation (4.7) and (4.8) for normed pairs.

We note that the vectors $\mathbf{x}_3,\mathbf{x}_4,\mathbf{y}_1,\mathbf{y}_2$ are invariant under transformation (4.26) and (4.27). Thus the notion of strictness is invariant under such transformations, too.

Observing that well-posed and normed pairs (U,V) define a T+H-Bezoutian in which the vectors $\mathbf{x}_3,\mathbf{x}_4,\mathbf{y}_1,\mathbf{y}_2$ occur as the first/last columns and rows (see Proposition 3.8), we can extend this notion to any T+H-Bezoutian.

DEFINITION 4.20. A T+H-Bezoutian B is called *strict* if the two polynomial equations (4.29) can only be satisfied by the zero matrix X, where $\mathbf{x}_3,\mathbf{x}_4$ are the first and last column and $\mathbf{y}_1,\mathbf{y}_2$ are the first and the last row of B.

PROPOSITION 4.21. *Suppose $A = TH(\mathbf{a},\mathbf{b})$ be invertible. Let U and V be any (normed) solutions to* (4.17) *and* (4.19)*. If the pair (U,V) is strict and well-posed, then it satisfies* (4.21) *and $A^{-1}=B(U,V)$.*

Proof. Since A is invertible we can conclude from Theorem 4.12 that the combined system (4.17), (4.19), and (4.21) has a well-posed and normed pair $(\widehat{U},\widehat{V})$ as its solution with $A^{-1}=B(\widehat{U},\widehat{V})$. However, we cannot immediately say whether this pair coincides with the given pair (U,V). What we can say is that since both U and $\widehat{U}$ satisfy (4.17), and since both V and $\widehat{V}$ satisfy (4.19), we can infer from parts (i) and (ii) of Proposition 4.17 that

$$\widehat{U} = U \begin{bmatrix} I_2 & \mathbf{0} \\ X_1 & I_2 \end{bmatrix}, \qquad \widehat{V} = V \begin{bmatrix} I_2 & -X_2^T \\ \mathbf{0} & I_2 \end{bmatrix} \tag{4.30}$$

with certain $X_1,X_2 \in \mathbb{F}^{2\times 2}$. Both pairs are also well-posed. Therefore

$$\widehat{U}(t)\widehat{V}^T(t) = U(t) \begin{bmatrix} I_2 & \mathbf{0} \\ X_1 - X_2 & I_2 \end{bmatrix} V^T(t) = 0 = U(t)V^T(t),$$

and putting $X = X_1 - X_2$, we obtain that the first condition in (4.29) is satisfied. Likewise

$$\widehat{U}(t)\widehat{V}^T(t^{-1}) = U(t)\begin{bmatrix} I_2 & \mathbf{0} \\ X_1 - X_2 & I_2 \end{bmatrix} V^T(t^{-1}) = 0 = U(t)V^T(t^{-1}),$$

and this implies that the second condition in (4.29) holds. Since (U,V) is assumed to be strict it follows that $X = 0$, i.e., $X_1 = X_2$. We can conclude that the pair (U,V) is equivalent to the pair $(\widehat{U},\widehat{V})$, i.e, $(U,V) \sim (\widehat{U},\widehat{V})$. Therefore, by part (iii) of Proposition 4.17, we see that the pair (U,V) also satisfies (4.21). Furthermore, $A = B(\widehat{U},\widehat{V}) = B(U,V)$ by direct computation (see also the paragraph before Proposition 3.9). $\square$

The conclusions in the previous proposition fail if the pair (U,V) is not strict. Indeed, let $X \in \mathbb{F}^{2\times 2}$ be a nonzero matrix such that (4.29) is satisfied. Then one can consider a new pair $(\widehat{U},\widehat{V})$ defined by (4.30) with $X_1 - X_2 = X$. This new pair, along with (U,V), satisfies the conditions (4.17) and (4.19) (see Proposition 4.17), and it is also well-posed. However, the corresponding Bezoutians differ. Indeed, with $C = B(\widehat{U},\widehat{V}) - B(U,V)$ we obtain that

$$C(t,s) = \frac{ts}{(t-s)(1-ts)}\left(\begin{bmatrix} \mathbf{x}_3(t) & \mathbf{x}_4(t) \end{bmatrix} X \begin{bmatrix} \mathbf{y}_1(s) \\ \mathbf{y}_2(s) \end{bmatrix}\right), \tag{4.31}$$

which is a well-defined Bezoutian due to (4.29), but nonzero. Clearly, $B(\widehat{U},\widehat{V})$ and $B(U,V)$ cannot both be the inverse of A, and thus the conclusions in Proposition 4.21 fail without the assumption of strictness.

While we plan to investigate the notions of strictness and non-strictness in more detail in a future paper, let us give two examples of a non-strict and strict, resp., pair (U,V) related to nonsingular T+H-Bezoutians $B = B(U,V)$.

EXAMPLE 4.22. We consider the same matrices as in Example 4.8, a T+H-Bezoutian B and its inverse, a T+H matrix $A = B^{-1}$,

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}. \tag{4.32}$$

Using one of the methods discussed earlier in this section, it is possible to determine a pair (U,V) such that $B = B(U,V)$,

$$U = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \qquad V = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.33}$$

This pair is normed, well-posed and satisfies (4.17), (4.19), and (4.21). From this pair or from the first/last rows/columns of B, the underlying vectors $\mathbf{x}_3, \mathbf{x}_4, \mathbf{y}_1, \mathbf{y}_2$ can be determined:

$$\mathbf{x}_3(t) = 1 + t^2, \;\; \mathbf{x}_4(t) = 1 - t + t^2, \;\; \mathbf{y}_1(t) = 1 + t^2, \;\; \mathbf{y}_2(t) = 1 + t + t^2.$$

For X equal to

$$X_\lambda = \lambda \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \lambda \in \mathbb{F},$$

it is easily seen that the two corresponding polynomial equations (4.29) are satisfied:

$$-2(1+t^2)^2 + (1+t^2)(1+t+t^2) + (1-t+t^2)(1+t^2) = 0.$$

Note that due to symmetry ($\mathbf{y}_1 = \mathbf{y}_1^J,\ \mathbf{y}_2 = \mathbf{y}_2^J$) both equation amount to the same. Therefore, the T+H-Bezoutian B and the pair (U,V) are non-strict.

Normed and well-posed solutions to (4.17) and (4.19) are not unique. Indeed, one can consider the modified pairs $(U_{\lambda_1}, V_{\lambda_2})$ with

$$U_{\lambda_1} = U \begin{bmatrix} I_2 & \mathbf{0} \\ X_{\lambda_1} & I_2 \end{bmatrix}, \qquad V_{\lambda_2} = V \begin{bmatrix} I_2 & -X_{\lambda_2}^T \\ \mathbf{0} & I_2 \end{bmatrix}, \qquad \lambda_1, \lambda_2 \in \mathbb{F}.$$

In contrast, $(U_{\lambda_1}, V_{\lambda_2})$ satisfies (4.21) if and only if $\lambda_1 = \lambda_2$. The T+H-Bezoutians corresponding to these pairs are given by

$$B(U_{\lambda_1}, V_{\lambda_2}) = B(U,V) + C_{\lambda_1 - \lambda_2}$$

where the matrix C_λ corresponds to the matrix considered in (4.31) and evaluates to

$$C_\lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus we obtain a one-parameter family of T+H-Bezoutian

$$B_\lambda = B + C_\lambda = \begin{bmatrix} 1 & 0 & 1 \\ 0 & \lambda & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

which arise from normed and well-posed pairs $(U_{\lambda_1}, V_{\lambda_2})$, all satisfying the same equations (4.17) and (4.19) (but not equation (4.21) unless $\lambda = \lambda_1 - \lambda_2$ is zero). It is interesting to note that these Bezoutians are invertible and their inverses represent a one-parameter family of T+H matrices,

$$(B_\lambda)^{-1} = \begin{bmatrix} -\lambda+1 & 1 & \lambda \\ -1 & 0 & 1 \\ \lambda & -1 & -\lambda \end{bmatrix} = A + \lambda \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

This concrete example confirms what has been stated above, namely that the conclusions of Proposition 4.21 fail if the assumption of strictness is not satisfied.

EXAMPLE 4.23. Consider the T+H matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 0 \end{bmatrix} \tag{4.34}$$

with inverse

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix}, \tag{4.35}$$

where $B = B(U,V)$ with, for example,

$$U = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \qquad V = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 2 \\ 1 & 2 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.36}$$

One can show by direct verification that the two polynomial equations (4.29) involving

$$\mathbf{x}_3(t) = 1+t^2, \quad \mathbf{x}_4(t) = 1-t+2t^2, \quad \mathbf{y}_1(t) = 1+t^2, \quad \mathbf{y}_2(t) = 1+t+2t^2$$

have only the trivial solution $X = 0$. Therefore, the T+H-Bezoutian B and the pair (U,V) are strict.

This normed pair satisfies equations (4.17) and (4.19). Any other normed pair $(\widehat{U},\widehat{V})$ satisfying the same equations is given by (4.30) with arbitrary $X_1, X_2 \in \mathbb{F}^{2\times 2}$.

Now, due to the strictness, if we require in addition the pair $(\widehat{U},\widehat{V})$ to be well-posed or to satisfy equation (4.21), then necessarily $X_1 = X_2$. This then entails that $(\widehat{U},\widehat{V})$ is both well-posed and satisfies equation (4.21). Moreover, we obtain the same Bezoutian $B = B(\widehat{U},\widehat{V})$. Notice that all this is in accordance with Theorem 4.18 and Proposition 4.21.

5. Inversion of T+H-Bezoutians

After having discussed the inversion of T+H matrices, we now turn to the problem of inverting T+H-Bezoutians. In the first subsection we obtain simple necessary (but not sufficient) criteria for the invertibility of Bezoutians. In the later subsections we are then going to establish necessary and sufficient criteria, which are of a different kind, namely, in terms of the solvability of a certain system of equations of generalized resultant form. From the solution to these systems, the symbol $(\mathbf{a},\mathbf{b})$ of the T+H matrix $TH(\mathbf{a},\mathbf{b})$ being the inverse of $B = B(U,V)$ can be determined.

5.1. Necessary criteria for invertibility

In what follows we need the following $m\times(m+k)$ matrix which is associated with a vector $\mathbf{w} = (w_i)_{i=0}^k \in \mathbb{F}^{k+1}$,

$$D_{m,m+k}(\mathbf{w}) = \begin{bmatrix} w_0 & w_1 & \dots & w_k & & & 0 \\ & w_0 & w_1 & \dots & w_k & & \\ & & \ddots & \ddots & & \ddots & \\ 0 & & & w_0 & w_1 & \dots & w_k \end{bmatrix}. \tag{5.1}$$

The equation $D_{m,m+k}(\mathbf{w})\mathbf{x} = \mathbf{y}$ can be interpreted in the language of rational functions as follows (see [6]):

$$\mathbf{w}(t^{-1})\mathbf{x}(t) \equiv \mathbf{y}(t) \mod \operatorname{lin}\{\dots,t^{-2},t^{-1},t^m,t^{m+1},\dots\}. \tag{5.2}$$

Indeed, write

$$\mathbf{w}(t^{-1}) = w_0 + w_1 t^{-1} + \dots + w_k t^{-k}$$

and

$$\mathbf{x}(t) = x_0 + x_1 t + \dots + x_{m+k-1} t^{m+k-1}.$$

The coefficients of the powers $1,t,\dots,t^{m-1}$ in the product $\mathbf{w}(t^{-1})\mathbf{x}(t)$ have to coincide with those of the polynomial $\mathbf{y}(t) = y_0 + y_1 t + \dots + y_{m-1} t^{m-1}$.

Further for $\mathbf{x} \in \mathbb{F}^m$ we have

$$\left(D_{m,m+k}(\mathbf{w})^T \mathbf{x}\right)(t) = \mathbf{w}(t)\mathbf{x}(t), \tag{5.3}$$

which means that the transpose $D_{m,m+k}(\mathbf{w})^T$ can be identified with the operator of multiplication by $\mathbf{w}(t)$ acting from $\mathbb{F}^m[t]$ to $\mathbb{F}^{m+k}[t]$.

Apart from the trivial case ($\mathbf{w} \equiv \mathbf{0}$) the matrix $D_{m,m+k}(\mathbf{w})$ has full rank. Thus the dimension of its kernel (nullspace) is k,

$$\dim\ker D_{m,m+k}(\mathbf{w}) = k. \tag{5.4}$$

To put the following considerations into context, recall a known criterion for the invertibility of Hankel- or Toeplitz-Bezoutians (see [10]).

PROPOSITION 5.1. *The $n \times n$ Hankel-Bezoutian B_H (or Toeplitz-Bezoutian B_T) introduced by its generating polynomial in (1.3) is nonsingular if and only if $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are generalized coprime, which means that the polynomials $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are coprime in the usual sense and $\deg \mathbf{u}(t) = n$ or $\deg \mathbf{v}(t) = n$.*

The question is whether this result can be generalized to T+H-Bezoutians. First results in this direction for special, namely centrosymmetric or centroskewsymmetric, T+H-Bezoutians of order n were obtained in [13] (see also [4, 5]), where $\mathbb{F}$ is a field with $\operatorname{char}(\mathbb{F}) \neq 2$.

Let us recall here only the centrosymmetric case for odd n. A matrix $B \in \mathbb{F}^{n\times n}$ is centrosymmetric if $B = J_n B J_n$. Moreover we need the definition of a split-Bezoutian of order n,

$$B_{\mathrm{sp},n}(\mathbf{g},\mathbf{f})(t,s) = \frac{\mathbf{g}(t)\mathbf{f}(s) - \mathbf{f}(t)\mathbf{g}(s)}{(t-s)(1-ts)}, \tag{5.5}$$

where $\mathbf{g}(t),\mathbf{f}(t) \in \mathbb{F}^{n+2}[t]$ are symmetric polynomials, i.e., $\mathbf{g} = J_{n+2}\mathbf{g}$ and $\mathbf{f} = J_{n+2}\mathbf{f}$. The following result was first established in [13].

PROPOSITION 5.2. *Let n be odd and $B \in \mathbb{F}^{n\times n}$ be a centrosymmetric T+H-Bezoutian. Then B can be represented in the form*

$$B = B_{\mathrm{sp},n}(\mathbf{g}_1,\mathbf{f}_1) + D_{n-2,n}^T(\mathbf{w}) B_{\mathrm{sp},n-2}(\mathbf{g}_2,\mathbf{f}_2) D_{n-2,n}(\mathbf{w}), \tag{5.6}$$

where $\mathbf{w}(t)=t^2-1$ *and* $\{\mathbf{g}_i(t),\mathbf{f}_i(t)\}$ *are two pairs of symmetric polynomials of degree* $n+1$ *for* $i=1$ *and of degree* $n-1$ *for* $i=2$.

Furthermore, the T+H-Bezoutian B is nonsingular if and only if both $\{\mathbf{g}_1(t),\mathbf{f}_1(t)\}$ *and* $\{\mathbf{g}_2(t),\mathbf{f}_2(t)\}$ *are pairs of coprime polynomials.*

For general T+H-Bezoutians we now have the following result.

PROPOSITION 5.3. *Let* $B=B(U,V)$ *be an* $n\times n$ *T+H-Bezoutian given by a well-posed pair* (U,V) *of the form* (3.4). *If the polynomials* $\{\mathbf{u}_i(t)\}_{i=1}^4$ *or* $\{\mathbf{v}_i(t)\}_{i=1}^4$ *have a nonconstant common divisor* $\mathbf{w}(t)\in\mathbb{F}^{k+1}[t]$, *then* B *is singular and allows the following representation*

$$B=D_{n-k,n}^T(\mathbf{w})B(P,V) \quad or \quad B=B(U,Q)D_{n-k,n}(\mathbf{w}), \tag{5.7}$$

where $P=\begin{bmatrix}\mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & \mathbf{p}_4\end{bmatrix}$ *with* $\mathbf{p}_i$ *being defined by* $\mathbf{u}_i(t)=\mathbf{w}(t)\mathbf{p}_i(t)$ *or where* $Q=\begin{bmatrix}\mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 & \mathbf{q}_4\end{bmatrix}$ *with* $\mathbf{q}_i$ *being defined by* $\mathbf{v}_i(t)=\mathbf{w}(t)\mathbf{q}_i(t)$.

Proof. The proof of the identity is straightforward by taking the definition (3.1) of a (possibly nonsquare) T+H-Bezoutian and properties of $D_{n,n-k}(\mathbf{w})$ into account. Due to the sizes of the matrices involved in (5.7) and $k\geqslant 1$, the nonsingularity of B follows. $\square$

Let us remark that we also have a representation (5.7) if either all of $\{\mathbf{u}_i(t)\}_{i=1}^4$ or all of $\{\mathbf{v}_i(t)\}_{i=1}^4$ are polynomials which do not attain their maximal degree $n+1$. In this case, clearly, the last row or the last column of B is zero. Correspondingly, we call the a quadruple $\{\mathbf{w}_i(t)\}_{i=1}^4$ of polynomials in $\mathbb{F}^{n+2}[t]$ *generalized coprime* if they do not have a nonconstant common factor and at least one of them attains their maximal degree, i.e.,

$$\max\{\deg\mathbf{w}_i(t)\,:\,i=1,\dots,4\}=n+1.$$

Combining this result with Theorem 3.7 we obtain the following necessary criteria for the invertibility of a T+H-Bezoutian.

COROLLARY 5.4. *Let* $B=B(U,V)$ *be a nonsingular* $n\times n$ *T+H-Bezoutian. Then* $\operatorname{rank}\nabla B=4$ *and each of the two quadruples* $\{\mathbf{u}_i(t)\}_{i=1}^4$ *and* $\{\mathbf{v}_i(t)\}_{i=1}^4$ *is generalized coprime. Moreover,* B *is normable.*

It turns out that in contrast to the pure Toeplitz or Hankel case (as stated in Proposition 5.1) the converse of this corollary fails. In particular, assuming that $\{\mathbf{u}_i(t)\}_{i=1}^4$ and $\{\mathbf{v}_i(t)\}_{i=1}^4$ are two quadruples of (generalized) coprime polynomials does not imply that the corresponding Bezoutian is invertible. Furthermore, the rank condition is independent of the coprimeness condition.

We will illustrate these failures with some examples.

EXAMPLE 5.5. We consider centrosymmetric T+H-Bezoutian B of odd order n represented in the form (5.6). Then both quadruples $\{\mathbf{u}_i(t)\}_{i=1}^4$ and $\{\mathbf{v}_i(t)\}_{i=1}^4$ are (up to certain minus signs) equal to

$$\{\mathbf{g}_1(t),\mathbf{f}_1(t),(t^2-1)\mathbf{g}_2(t),(t^2-1)\mathbf{f}_2(t)\}. \tag{5.8}$$

It is possible to find a choice such that the four polynomials in (5.8) are coprime (i.e., they do not share a nonconstant common factor), while on the other hand, the pair $\{\mathbf{g}_1(t),\mathbf{f}_1(t)\}$ is not coprime. Then Proposition 5.2 shows that B is singular.

For instance, with $n=3$ and $\mathbb{F}=\mathbb{C}$ we can take

$$\mathbf{g}_1(t)=(1+t^2)^2,\ \mathbf{f}_1(t)=(1+t^2)t,\ \mathbf{g}_2(t)=1+t^2,\ \mathbf{f}_2(t)=t.$$

Using the fact that $\{\mathbf{g}_1,\mathbf{f}_1\}$ and $\{\mathbf{g}_2,\mathbf{f}_2\}$ are linearly independent one can show that the four polynomials in (5.8) are linearly independent. This implies $\operatorname{rank} U=\operatorname{rank} V=\operatorname{rank}\nabla B=4$. In fact, a straightforward computation gives that

$$B=-\begin{bmatrix} 2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 2\end{bmatrix},$$

which has linearly independent first and last rows/columns, and thus B is normable (see Theorem 3.7).

One can modify this example, by choosing $\mathbf{f}_1=\mathbf{g}_1$ (and leaving the other polynomials unchanged). In this case $\operatorname{rank}\nabla B=2$ (in fact, $B_{\mathrm{sp},n}(\mathbf{g}_1,\mathbf{f}_1)=0$ in (5.6)), while still the four polynomials in (5.8) do not share a nonconstant common factor.

Let us now give an example for a representation similar to (5.7).

EXAMPLE 5.6. Consider the singular T+H-Bezoutian $B(U,V)$ of order 5,

$$B(U,V)=\frac{1}{3}\begin{bmatrix} 1 & 2 & 1 & 0 & 0\\ 0 & 0 & 1 & 1 & 0\\ -1 & -2 & 0 & 2 & 1\\ -1 & 1 & -1 & 0 & 3\\ -1 & 1 & -1 & -1 & 2\end{bmatrix},$$

where we choose the corresponding matrices U,V as normed in the form (3.9), (3.10),

$$\mathbf{u}_1=\frac{1}{3}\begin{bmatrix} -3\\0\\3\\1\\0\\-1\\0\end{bmatrix},\quad \mathbf{u}_2=\frac{1}{3}\begin{bmatrix} 0\\1\\3\\1\\-4\\-6\\-3\end{bmatrix},\quad \mathbf{u}_3=\frac{1}{3}\begin{bmatrix} 0\\1\\0\\-1\\-1\\-1\\0\end{bmatrix},\quad \mathbf{u}_4=\frac{1}{3}\begin{bmatrix} 0\\0\\0\\1\\3\\2\\0\end{bmatrix},$$

$$\mathbf{v}_1=\frac{1}{3}\begin{bmatrix} 0\\1\\2\\1\\0\\0\\0\end{bmatrix},\quad \mathbf{v}_2=\frac{1}{3}\begin{bmatrix} 0\\-1\\1\\-1\\-1\\2\\0\end{bmatrix},\quad \mathbf{v}_3=\frac{1}{3}\begin{bmatrix} 3\\-1\\-5\\2\\5\\2\\0\end{bmatrix},\quad \mathbf{v}_4=\frac{1}{3}\begin{bmatrix} 0\\-4\\-2\\2\\4\\7\\3\end{bmatrix}.$$

It is easy to see that the polnomials $\mathbf{u}_i(t)$ and $\mathbf{v}_i(t), i = 1,2,3,4$ are zero for $t = -1$. We define matrices P and Q such that the i-th column of it is the coefficient vector of the polynomials $\mathbf{p}_i(t) = \frac{\mathbf{u}_i(t)}{t+1}$ and $\mathbf{q}_i(t) = \frac{\mathbf{v}_i(t)}{t+1}$,

$$P = \frac{1}{3}\begin{bmatrix} -3 & 0 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ -1 & -3 & -1 & 2 \\ 0 & -3 & 0 & 0 \end{bmatrix}, \quad Q = \frac{1}{3}\begin{bmatrix} 0 & 0 & 3 & 0 \\ 1 & -1 & -4 & -4 \\ 1 & 2 & -1 & 2 \\ 0 & -3 & 3 & 0 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix},$$

and compute

$$B(P,Q) = \frac{1}{3}\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 2 & -3 & 2 \end{bmatrix}.$$

Using (5.7) we obtain the representation

$$B(U,V) = D_{4,5}^T(\mathbf{w})B(P,Q)D_{4,5}(\mathbf{w}),$$

where

$$D_{4,5}(\mathbf{w}) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

In summary, this example exhibits a singular T+H-Bezoutian, in which the quadruples of polynomials have a common factor. On the other hand, the Bezoutian is normable and $\operatorname{rank} \nabla B = 4$.

5.2. Systems equivalent to the invertibility of T+H-Bezoutians

Despite the counter-examples of the previous subsection we want to establish necessary and sufficient criteria for the inversion of T+H-Bezoutians. These criteria will be in terms of equivalent systems. Recall that for the inversion of T+H matrices we had come up with systems (4.17), (4.19), and (4.21), in which the vector $(\mathbf{a},\mathbf{b})$ is given and (U,V) is the solutions to be sought. We now need to take the opposite point of view: the pair (U,V) is given and the vector $(\mathbf{a},\mathbf{b})$ is sought. Accordingly, we will first rewrite these systems in an equivalent way.

Hereafter we use the notation of the previous sections. In particular, $\mathbf{a} = (a_i)_{i=-n+1}^{n-1}$, $\mathbf{b} = (b_i)_{i=-n+1}^{n-1}$, $A = TH(\mathbf{a},\mathbf{b})$, and ∂A is defined in (4.12).

LEMMA 5.7. *For $\mathbf{a},\mathbf{b} \in \mathbb{F}^{2n-1}$, $A = TH(\mathbf{a},\mathbf{b})$, and $\mathbf{w},\mathbf{z} \in \mathbb{F}^n$ the system*

$$A\mathbf{w} = \mathbf{z}$$

is equivalent to

$$D_{n,2n-1}(\mathbf{w}^J)\mathbf{a} + D_{n,2n-1}(\mathbf{w})\mathbf{b} = \mathbf{z}$$

Proof. First observe that

$$T_n(\mathbf{a})\mathbf{w} = D_{n,2n-1}(\mathbf{w}^J)\mathbf{a}.$$

Indeed, writing out the matrix vector multiplications it is straightforward to see that

$$\begin{bmatrix} a_0 & a_{-1} & \dots & a_{-n+1} \\ a_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{-1} \\ a_{n-1} & \dots & a_1 & a_0 \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ \vdots \\ w_n \end{bmatrix}$$

coincides with

$$\begin{bmatrix} w_n & \dots & w_1 & 0 & & & 0 \\ 0 & w_n & \dots & w_1 & 0 & & \\ & \ddots & \ddots & & \ddots & \ddots & \\ & & 0 & w_n & \dots & w_1 & 0 \\ 0 & & & 0 & w_n & \dots & w_1 \end{bmatrix} \begin{bmatrix} a_{-n+1} \\ \vdots \\ a_0 \\ \vdots \\ a_{n-1} \end{bmatrix}.$$

More formally, this is of course just the identity

$$\sum_{j=1}^{n} a_{k-j} w_j = \sum_{l=k}^{k+n-1} w_{n+k-l} a_{-n+l}, \qquad k = 1, \dots, n.$$

In the above identity we now replace $\mathbf{w}$ by $\mathbf{w}^J = J_n \mathbf{w}$ and $\mathbf{a}$ by $\mathbf{b}$. We get

$$T_n(\mathbf{b}) J_n \mathbf{w} = D_{n,2n-1}(\mathbf{w})\mathbf{b}.$$

Adding the two identities and noting that $A = T_n(\mathbf{a}) + T_n(\mathbf{b})J_n$, proves the claim. □

LEMMA 5.8. *For* $\mathbf{a}, \mathbf{b} \in \mathbb{F}^{2n+1}$, $A = TH(\mathbf{a}, \mathbf{b})$, *and* $\mathbf{w} \in \mathbb{F}^{n+2}$ *the system*

$$(\partial A)\,\mathbf{w} = \mathbf{0}_{n-2}$$

is equivalent to

$$D_{n-2,2n-1}(\mathbf{w}^J)\mathbf{a} + D_{n-2,2n-1}(\mathbf{w})\mathbf{b} = \mathbf{0}_{n-2}.$$

Proof. Note that the matrix ∂A is of size $(n-2) \times (n+2)$, see also (4.12). Here we observe that

$$(\partial T_n(\mathbf{a}))\,\mathbf{w} = D_{n-2,2n-1}(\mathbf{w}^J)\mathbf{a}.$$

Indeed, along the same lines as in the previous proof,

$$\begin{bmatrix} a_2 & a_1 & a_0 & \cdots & \cdots & a_{-n+2} & a_{-n+1} \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & \cdots & a_0 & a_{-1} & a_{-2} \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_n \\ w_{n+1} \end{bmatrix}$$

coincides with

$$\begin{bmatrix} w_{n+1} & w_n & \dots & w_1 & w_0 & 0 & & & 0 \\ 0 & w_{n+1} & w_n & \dots & w_1 & w_0 & 0 & & \\ & \ddots & \ddots & \ddots & & \ddots & \ddots & \ddots & \\ & & 0 & w_{n+1} & w_n & \dots & w_1 & w_0 & 0 \\ 0 & & & 0 & w_{n+1} & w_n & \dots & w_1 & w_0 \end{bmatrix} \begin{bmatrix} a_{-n+1} \\ \vdots \\ a_0 \\ \vdots \\ a_{n-1} \end{bmatrix},$$

or, formally,

$$\sum_{j=0}^{n+1} a_{k-j} w_j = \sum_{l=k-1}^{k+n} w_{n+k-l} a_{-n+l}, \qquad k = 2, \dots, n-1.$$

Replacing $\mathbf{w}$ by $\mathbf{w}^J = J_{n+2}\mathbf{w}$ and $\mathbf{a}$ by $\mathbf{b}$, we get

$$(\partial T_n(\mathbf{b}))\,J_{n+2}\mathbf{w} = D_{n-2,2n-1}(\mathbf{w})\mathbf{b}.$$

Noting that $\partial(T_n(\mathbf{b})J_n) = (\partial T_n(\mathbf{b}))J_{n+2}$ and adding the two equations again proves the claim. $\square$

In what follows we are going to consider a normed pair (U,V), i.e., where $U = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4]$ and $V = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4]$ are of the form (4.7) and (4.8). Introducing the entries $\mathbf{u}_i = (u_{ij})_{j=0}^{n+1}$ and $\mathbf{v}_i = (v_{ij})_{j=0}^{n+1}$, recall that the first and the last entry of $\mathbf{u}_3$, $\mathbf{u}_4$, $\mathbf{v}_1$, $\mathbf{v}_2$ are zero. In these cases we denote the middle parts by

$$\mathring{\mathbf{u}}_i = (u_{ij})_{j=1}^{n}, \qquad \mathring{\mathbf{v}}_i = (v_{ij})_{j=1}^{n}.$$

Note that these coincide with $\mathbf{x}_i$ and $\mathbf{y}_i$, respectively.

Let us put these parts of U (for $i = 1,2,3,4$) into a $(4n-4)\times(4n-2)$ matrix, where we omit the second subscript $2n-1$ in each block,

$$D(U) = \begin{bmatrix} D_{n-2}(\mathbf{u}_1^J) & D_{n-2}(\mathbf{u}_1) \\ D_{n-2}(\mathbf{u}_2^J) & D_{n-2}(\mathbf{u}_2) \\ D_n(\mathring{\mathbf{u}}_3^J) & D_n(\mathring{\mathbf{u}}_3) \\ D_n(\mathring{\mathbf{u}}_4^J) & D_n(\mathring{\mathbf{u}}_4) \end{bmatrix}. \tag{5.9}$$

Likewise, we introduce a similar (though slightly different) matrix for V,

$$D(V) = \begin{bmatrix} J_n D_n(\mathring{\mathbf{v}}_1) & D_n(\mathring{\mathbf{v}}_1) \\ J_n D_n(\mathring{\mathbf{v}}_2) & D_n(\mathring{\mathbf{v}}_2) \\ J_{n-2} D_{n-2}(\mathbf{v}_3) & D_{n-2}(\mathbf{v}_3) \\ J_{n-2} D_{n-2}(\mathbf{v}_4) & D_{n-2}(\mathbf{v}_4) \end{bmatrix}. \tag{5.10}$$

We note that $D(U)$ and $D(V)$ can be considered as generalized resultant matrices.

As the upshot of all this we have the following result. Therein $\mathbf{0}_{n-2}$ denotes the zero vector in $\mathbb{F}^{n-2}$ and $\mathbf{e}_1, \mathbf{e}_n \in \mathbb{F}^n$.

PROPOSITION 5.9. *Let* $\mathbf{a},\mathbf{b}\in\mathbb{F}^{2n-1}$. *Then*

(i) the system (4.17) *is equivalent to*

$$D(U)\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}=\begin{bmatrix}\mathbf{0}_{n-2}\\ \mathbf{0}_{n-2}\\ \mathbf{e}_1\\ \mathbf{e}_n\end{bmatrix}, \tag{5.11}$$

where $D(U)$ *is a* $(4n-4)\times(4n-2)$ *matrix.*

(ii) The system (4.19) *is equivalent to*

$$D(V)\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}=\begin{bmatrix}\mathbf{e}_1\\ \mathbf{e}_n\\ \mathbf{0}_{n-2}\\ \mathbf{0}_{n-2}\end{bmatrix}, \tag{5.12}$$

where $D(V)$ *is a* $(4n-4)\times(4n-2)$ *matrix.*

(iii) The system (4.21) *is equivalent to*

$$C(U,V)\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}=\mathbf{0}_4, \tag{5.13}$$

where $C(U,V)$ *is a certain* $4\times(4n-2)$ *matrix.*

It is a simple exercise to write down $C(U,V)$ explicitly, but we refrain from doing so as its precise form is not very illuminating. However we mention that this matrix only depends on $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$, but not on $\mathring{\mathbf{u}}_3$, $\mathring{\mathbf{u}}_4$, $\mathring{\mathbf{v}}_1$, or $\mathring{\mathbf{v}}_2$.

Proof. (i): Using the previously introduced notations, system (4.17) reads

$$\partial A\mathbf{u}_1=\mathbf{0},\quad \partial A\mathbf{u}_2=\mathbf{0},\quad A\mathring{\mathbf{u}}_3=\mathbf{e}_1,\quad A\mathring{\mathbf{u}}_4=\mathbf{e}_4$$

with $A=TH(\mathbf{a},\mathbf{b})$. Application of Lemmas 5.7 and 5.8 leads to four systems, which can be conveniently written in block form using (5.11).

(ii): The system (4.19) is a similar system, but with $A^T=TH(\mathbf{a}^J,\mathbf{b})$ instead of A. This amounts to changing $\mathbf{a}$ to $\mathbf{a}^J=J_{2n-1}\mathbf{a}$. Furthermore, the role of the first two and last two of the indices $i=1,\dots,4$ is reversed. Therefore, instead of, e.g.,

$$D_n(\mathring{\mathbf{u}}_3^J)\mathbf{a}+D_n(\mathring{\mathbf{u}}_3)\mathbf{b}=\mathbf{e}_1$$

we have

$$D_n(\mathring{\mathbf{v}}_1^J)J_{2n-1}\mathbf{a}+D_n(\mathring{\mathbf{v}}_1)\mathbf{b}=\mathbf{e}_1.$$

Taking into account that

$$D_n(\mathring{\mathbf{v}}^J)J_{2n-1}=J_nD_n(\mathring{\mathbf{v}}),\qquad D_{n-2}(\mathbf{v}^J)J_{2n-1}=J_{n-2}D_n(\mathbf{v})$$

it is seen straightforwardly that we arrive at equation (5.12) which involves the block matrix $D(V)$.

(iii): Equation (4.21), which is an identity of 2×2 matrices, can be written as four equations. These equations depend linearly of the vector $(\mathbf{a},\mathbf{b})$, and the system is homogeneous. $\square$

REMARK 5.10. Recall from Remark 2.1 that $TH(\mathbf{a},\mathbf{b})=0$ if and only if $(\mathbf{a},\mathbf{b})\in\mathscr{W}$, where $\mathscr{W}$ is the two-dimensional subspace of $\mathbb{F}^{2n-1}\times\mathbb{F}^{2n-1}$ defined in (2.6). Either by the equivalence stated in the previous proposition or by direct inspection, it can be seen that the subspace $\mathscr{W}$ is in the kernel of each of the matrices $D(U)$, $D(V)$, and $C(U,V)$, i.e.,

$$\mathscr{W}\subset\ker D(U),\qquad \mathscr{W}\subset\ker D(V),\qquad \mathscr{W}\subset\ker C(U,V).$$

In view of the dimensions of the systems (5.11) and (5.12), one might conjecture that – generically – each of these two systems is solvable and that the solution is unique modulo $\mathscr{W}$. However, we have reasons to believe that this may not be the case.

5.3. Computation of the symbol of the inverses of T+H-Bezoutians

Finally, we are now going to analyse how the invertibility of a T+H-Bezoutian $B(U,V)$ given by a well-posed and normed pair (U,V) is related to the solvability of the three equations (5.11), (5.12), and (5.13). We also address the question how the symbol $(\mathbf{a},\mathbf{b})$ can be computed and whether the third equation (5.13) can be dropped.

PROPOSITION 5.11. *Let $B=B(U,V)$ be a nonsingular T+H-Bezoutian given by a well-posed and normed pair (U,V). Then there exists an $(\mathbf{a},\mathbf{b})$ which satisfies the equations* (5.11)*,* (5.12)*,* (5.13) *and for which*

$$B^{-1}=TH(\mathbf{a},\mathbf{b}).$$

Proof. Since B has an inverse, its inverse is a T+H matrix by Theorem 3.1. So $A=B^{-1}=TH(\mathbf{a},\mathbf{b})$ for some $(\mathbf{a},\mathbf{b})$. Since A itself is invertible, we can conclude from Theorem 4.12 that equations (4.17), (4.19), and (4.21) are satisfied for some normed and well-posed pair $(\widehat{U},\widehat{V})$, possibly different from (U,V), such that $A^{-1}=B=B(\widehat{U},\widehat{V})$. Since B is invertible we have $\operatorname{rank}\nabla B=4$ by Theorem 3.7 and therefore, we can apply Proposition 3.9 to conclude that $(U,V)\sim(\widehat{U},\widehat{V})$. This implies that the equations (4.17), (4.19), and (4.21) are also satisfied for the pair (U,V) (either by direct verification or by Proposition 4.17(iii)). But the equations (4.17), (4.19), and (4.21) satisfied for (U,V) and $(\mathbf{a},\mathbf{b})$ are equivalent to the equations (5.11), (5.12), and (5.13). $\square$

This proposition shows that the invertibility of the T+H-Bezoutian implies the solvability of the systems of equations (5.11), (5.12), and (5.13). Regarding the converse, the following can be said.

PROPOSITION 5.12. *Let (U,V) be a normed pair and assume that $(\mathbf{a},\mathbf{b})$ is a solution to the equations* (5.11), (5.12)*, and* (5.13)*. Then the pair (U,V) is well-posed, the T+H-Bezoutian $B = B(U,V)$ is nonsingular, and*

$$B^{-1} = TH(\mathbf{a},\mathbf{b}).$$

Proof. Assume that (U,V) along with $(\mathbf{a},\mathbf{b})$ satisfies the equations (5.11), (5.12), and (5.13). This means that they satisfy the equations (4.17), (4.19), and (4.21). In particular, $A = TH(\mathbf{a},\mathbf{b})$ is invertible by the equivalence of (a) and (b/c) in Theorem 4.12. Theorem 4.18 now implies that (U,V) is well-posed and $A^{-1} = B(U,V)$, from which we conclude that the Bezoutian is nonsingular. □

THEOREM 5.13. *Let (U,V) be a normed pair. Then the following statements are equivalent:*

(a) The system of equations (5.11), (5.12)*, and* (5.13) *has a solution $(\mathbf{a},\mathbf{b})$.*

(b) The pair (U,V) is well-posed and the Bezoutian $B = B(U,V)$ is nonsingular.

In this case,

$$B^{-1} = TH(\mathbf{a},\mathbf{b}).$$

Proof. The direction (a)⇒(b) follows from Proposition 5.12, which also implies that B^{-1} is given by any solution $(\mathbf{a},\mathbf{b})$ to the system (5.11), (5.12), and (5.13). The direction (b)⇒(a) is a consequence of Proposition 5.11. □

COROLLARY 5.14. *Let (U,V) be a normed pair. Then solutions $(\mathbf{a},\mathbf{b})$ to the equations* (5.11), (5.12)*, and* (5.13) *are unique modulo $\mathscr{W}$, where $\mathscr{W}$ is defined in* (2.6).

Proof. Let $(\mathbf{a},\mathbf{b})$ and $(\hat{\mathbf{a}},\hat{\mathbf{b}})$ be two solutions to the system. Then either the previous theorem or Proposition 5.12 imply that $B = B(U,V)$ is well-defined and nonsingular, and

$$B^{-1} = TH(\mathbf{a},\mathbf{b}) = TH(\hat{\mathbf{a}},\hat{\mathbf{b}}).$$

Hence $(\mathbf{a},\mathbf{b}) - (\hat{\mathbf{a}},\hat{\mathbf{b}}) \in \mathscr{W}$ by Remark 2.1. □

Furthermore, the three systems of equations (5.11), (5.12), and (5.13) are highly over-determined. The question arises if one can drop at least the third system. As we will see, this can be done in the case of strictness, the notion of which has been introduced in Subsection 4.6.

PROPOSITION 5.15. *Let (U,V) be a normed pair which is strict and well-posed. If $(\mathbf{a},\mathbf{b})$ is a solution to* (5.11) *and* (5.12)*, then $B = B(U,V)$ is invertible and $B^{-1} = TH(\mathbf{a},\mathbf{b})$. In particular, the solutions to* (5.11) *and* (5.12) *are unique modulo $\mathscr{W}$.*

Proof. Assuming that $(\mathbf{a},\mathbf{b})$ is a solution to (5.11) and (5.12), we can consider $A = TH(\mathbf{a},\mathbf{b})$, which is invertible by Theorem 4.12, parts (a)-(c). Now apply Proposition 4.21 to conclude that $A^{-1} = B(U,V)$, which implies the invertibility of $B = B(U,V)$. Furthermore, if we have another solution $(\hat{\mathbf{a}},\hat{\mathbf{b}})$, then by repeating the arguments we have just given we obtain $TH(\mathbf{a},\mathbf{b}) = TH(\hat{\mathbf{a}},\hat{\mathbf{b}})$ which means uniqueness modulo $\mathscr{W}$. $\square$

In view of Example 4.22 one can show that the previous result fails if the assumption (U,V) being strict is dropped. We leave the details to the reader.

Let us illustrate the previous results with two examples, featuring a strict and a non-strict, respectively, T+H Bezoutian.

EXAMPLE 5.16. Let us first consider the strict Bezoutian B given by (4.35) in Example 4.23. Then $B = B(U,V)$ with (U,V) given by (4.36). The corresponding matrices $D(U)$ and $D(V)$ become

$$D(U) = \left[\begin{array}{ccccc|ccccc} 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 \\ \hline -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ \hline 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline 2 & -1 & 1 & 0 & 0 & 1 & -1 & 2 & 0 & 0 \\ 0 & 2 & -1 & 1 & 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 2 & -1 & 1 & 0 & 0 & 1 & -1 & 2 \end{array}\right],$$

$$D(V) = \left[\begin{array}{ccccc|ccccc} 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 1 & 2 & 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & 0 & 1 & 1 & 2 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ \hline 1 & -1 & -1 & -2 & 0 & 1 & -1 & -1 & -2 & 0 \\ \hline 0 & 1 & 2 & 2 & 1 & 0 & 1 & 2 & 2 & 1 \end{array}\right].$$

The two systems (5.11) and (5.12) have a solution $(\mathbf{a},\mathbf{b})$ given by

$$\mathbf{a} = \big[\,0,\,0,\,1,\,-2,\,0\,\big]^T, \quad \mathbf{b} = \big[\,0,\,1,\,0,\,1,\,-1\,\big]^T,$$

which can be seen directly but is also obvious since the inverse $B^{-1} = A = TH(\mathbf{a},\mathbf{b})$ is given by (4.34). However, both the nullspace of $D(U)$ and $D(V)$ are three-dimensional,

$$\ker D(U) = \mathscr{W} \dotplus \operatorname{lin}\big\{\big[0,-1,1,0,0\,\big|\,0,0,-1,1,0\big]^T\big\},$$
$$\ker D(V) = \mathscr{W} \dotplus \operatorname{lin}\big\{\big[0,-1,1,0,0\,\big|\,0,1,-1,0,0\big]^T\big\}.$$

This shows, somewhat surprisingly, that only one equation (5.11) or (5.12) is not enough to uniquely (modulo $\mathscr{W}$) determine the correct symbol of the inverse $A = TH(\mathbf{a},\mathbf{b})$. However, taking both equations into account, the solution is unique (modulo $\mathscr{W}$) since $\ker D(U) \cap \ker D(V) = \mathscr{W}$. This is consistent with Proposition 5.15.

EXAMPLE 5.17. We now consider the non-strict Bezoutian B given by (4.32) in Example 4.22. Then $B = B(U,V)$ with (U,V) given by (4.33). The corresponding matrices $D(U)$ and $D(V)$ become

$$D(U) = \left[\begin{array}{ccccc|ccccc} 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 \\ \hline -1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & -1 \\ \hline 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline 1 & -1 & 1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 1 & -1 & 1 \end{array}\right],$$

$$D(V) = \left[\begin{array}{ccccc|ccccc} 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 1 & -1 & -1 & 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ \hline 0 & 1 & 2 & 2 & 1 & 0 & 1 & 2 & 2 & 1 \end{array}\right].$$

The two systems (5.11) and (5.12) have a solution $(\mathbf{a}, \mathbf{b})$ given by

$$\mathbf{a} = \left[\, 1,\ 0,\ 1,\ -2,\ 1 \,\right]^T, \quad \mathbf{b} = \left[\, 0,\ 1,\ -1,\ 1,\ -1 \,\right]^T.$$

The nullspaces of $D(U)$ and $D(V)$ are three-dimensional and coincide,

$$\ker D(U) = \ker D(V) = \mathscr{W} \dot{+} \operatorname{lin}\left\{ \left[0, 0, 1, 0, 0 \,\middle|\, 0, 0, -1, 0, 0\right]^T \right\}.$$

Therefore, as expected, the (simultaneous) solution $(\mathbf{a}, \mathbf{b})$ to both equations (5.11) and (5.12) is not unique modulo $\mathscr{W}$. The set of solutions gives rise to a one-parameter family of T+H matrices $TH(\mathbf{a}, \mathbf{b})$ given by

$$A_\lambda = A + \lambda \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix},$$

which are the inverses of the one-parameter family of T+H-Bezoutians B_λ described in Example 4.22. Notice that uniqueness can be achieved by taking into account the third equation (4.21), in accordance with Corollary 5.14.

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