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ON SOME FUGLEDE–KADISON DETERMINANT INEQUALITIES OF OPERATOR MEANS

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Abstract. Let $\mathscr{M}$ be a finite von Neumann algebra with finite trace τ. We extend some important matrix determinant inequalities, studied by Lin, Ghabries, Abbas, Mourad and Assi, to the Fuglede-Kadison determinant of τ-measurable operators in the noncommutative algebra $L_{\log_+}(\mathscr{M})$. Some Fuglede-Kadison determinant inequalities are established in $L_{\log_+}(\mathscr{M})$ with different forms to the matrix case.

1. Introduction

Recently, Lin proposed a conjecture concerning determinant inequalities in [19] namely that

$$\det(A^2+|AB|^{\alpha}) \geqslant \det(A^2+|BA|^{\alpha}),\ \alpha>0, \tag{1.1}$$

where A, B are two arbitrary positive semi-definite matrices. In [11], Ghabries, Abbas and Mourad proved that (1.1) holds if A and B are Hermitian matrices. In [12] Ghabries, Abbas, Mourad and Assi showed that, for $0 \leqslant \alpha \leqslant \beta \leqslant \gamma$ and $\frac{\alpha}{\beta} \in [\frac{1}{2}, 1]$,

$$\det(A^{\gamma}+A^{\alpha}B^{\alpha}) \geqslant \det(A^{\gamma}+|B^{\frac{\beta}{2}}A^{\frac{\beta}{2}}|^{\frac{2\alpha}{\beta}}), \tag{1.2}$$

where A, B are two positive semi-definite matrices.

Fuglede and Kadison [9] introduced an operator determinant, called the Fuglede-Kadison determinant, which was used to compute the solution of the invariant subspace problem for operators. Afterwards the Fuglede-Kadison determinant was used to construct Entropy in many varieties of algebras. Now the determinant inequalities play an important role in noncommutative integration theory. The aspect we are interested in is whether inequalities (1.1) and (1.2) hold or not when A, B are two positive τ-measurable operators in the algebra $L_{\log_+}(\mathscr{M})$.

The paper is organized as follows. Section 2 contains some basic facts about logarithmic submajorizations. In section 3, some logarithmic submajorization inequalities

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of geometric mean are discussed and two Fuglede-Kadison determinant inequalities corresponding to (1.2) are proved. Moreover, the inequalities in Theorem 12 are obtained with a slightly more general condition. In the last section, we prove some operator Fuglede-Kadison determinant inequalities which correspond to inequality (1.1). It is necessary to mention that it seems to be impossible for us to present a form completely the same as the above two inequalities for τ-measurable operators case with the noncommutative integral tools on hand.

2. Preliminaries

2.1. τ-measurable operators and singular numbers

Let $\mathscr{H}$ be a separable Hilbert space. Let $\mathscr{M}$ be a finite von Neumann algebra with a faithful normal finite trace τ on $\mathscr{H}$. The identity and the complete lattice of all projections in $\mathscr{M}$ are denoted by 1 and $\mathscr{P}(\mathscr{M})$, respectively. A linear operator

$$x : \mathscr{D}(x) \to \mathscr{H}, \text{ with domain } \mathscr{D}(x) \subseteq \mathscr{H}$$

is said to be affiliated with $\mathscr{M}$, if $ux = xu$ for all unitary u in the commutant $\mathscr{M}'$ of $\mathscr{M}$. For any self-adjoint operator x on $\mathscr{H}$, we denote its spectral measure by e^x. A self-adjoint operator x is affiliated with $\mathscr{M}$ if and only if $e^x(B) \in \mathscr{P}(\mathscr{M})$ for any Borel set $B \subseteq \mathbb{R}$. A densely-defined closed operator x affiliated with $\mathscr{M}$ is called τ-measurable if there exists an $e \in \mathscr{P}(\mathscr{M})$ such that

$$e(\mathscr{H}) \subseteq \mathscr{D}(x) \text{ and } \tau(e^{\perp}) \leqslant \delta \text{ for all } \delta > 0.$$

The set of all τ-measurable operators is denoted by $L_0(\mathscr{M})$ (see [6, 21, 22]). The symbol $x \geqslant 0$ means that x is positive, self adjoint and τ-measurable. For all $x \in L_0(\mathscr{M})$, if

$$x^*x \geqslant xx^*$$

we say an operator x is hyponormal (see [5]). For $x \in L_0(\mathscr{M})$ and $s > 0$, the singular number of x is defined by

$$\mu_s(x) = \inf\{\|xe\| : e \text{ is a projection in } \mathscr{M} \text{ with } \tau(1-e) \leqslant s\}.$$

The function $s \to \mu_s(x)$ is simply denoted by $\mu(x)$. In the following, we give some elementary properties of the generalized singular numbers $\mu_s(x)$ (see [10, 15, 21, 23] for more details).

PROPOSITION 1. ([10]) *Let $x, y \in L_0(\mathscr{M})$ and $s > 0$.*

1. *$\mu_s(x) = \mu_s(|x|) = \mu_s(x^*)$ and $\mu_s(\alpha x) = |\alpha|\mu_s(x)$ for $\alpha \in \mathbb{C}$.*
2. *If $0 \leqslant x \leqslant y$, then $\mu_s(x) \leqslant \mu_s(y)$.*
3. *For any continuous increasing function on $[0,\infty)$ with $f(0) \geqslant 0$,*

$$\mu_s(f(|x|)) = f(\mu_s(|x|)).$$

2.2. Fuglede-Kadison determinant

Recall that noncommutative $L_p\,(1 \leqslant p \leqslant \infty)$ spaces and $L_1(\mathscr{M})+\mathscr{M}$ (see e.g. [13, 21]) are defined by

$$L_p(\mathscr{M}) = \{x \in L_0(\mathscr{M}) : \mu(x) \in L_p(\mathbb{R})\}$$

and

$$L_1(\mathscr{M})+\mathscr{M} = \{x : x = y+z, y \in L_1(\mathscr{M}), z \in \mathscr{M}\}$$

with norm

$$\|x\|_{L_p(\mathscr{M})} = \|\mu(x)\|_{L_p} \text{ and } \|x\|_{L_1(\mathscr{M})+\mathscr{M}} = \inf_{\substack{x=z+y \\ z\in\mathscr{M},\ y\in L_1(\mathscr{M})}} \left\{\|y\|_{L_1(\mathscr{M})} + \|z\|_{\mathscr{M}}\right\}.$$

Let $L_{\log_+}(\mathscr{M}) = \{x \in L_0(\mathscr{M}) : \log_+|x| \in L_1(\mathscr{M})+\mathscr{M}\}$, where $\log_+\alpha = \max\{\log\alpha, 0\}$ and $\alpha > 0$. From [7], we know that $L_{\log_+}(\mathscr{M})$ is an algebra and

$$L_1(\mathscr{M})+\mathscr{M} \subseteq L_{\log_+}(\mathscr{M}) \subseteq L_0(\mathscr{M}).$$

For $x,\ y \in L_{\log_+}(\mathscr{M})$, if

$$\int_0^t \mu_s(x)ds \leqslant \int_0^t \mu_s(y)ds,\ t>0$$

x is said to be submajorized by y, denoted by $x \prec y$ (or $\mu_s(x) \prec \mu_s(y)$). Similarly, if

$$\int_0^t \log\mu_s(x)ds \leqslant \int_0^t \log\mu_s(y)ds,\ t>0$$

x is said to be logarithmically submajorized by y, which is denoted by $x \prec_{\log} y$ (or $\mu_s(x) \prec_{\log} \mu_s(y)$).

Let $x \in L_{\log_+}(\mathscr{M})$. A Fuglede-Kadison determinant-like function of x is defined by

$$\Lambda_t(x) = \exp\left(\int_0^t \log\mu_s(x)ds\right),\ t>0$$

and the Fuglede-Kadison determinant on $L_{\log_+}(\mathscr{M})$ is $\Delta(x) = \lim_{t\to\infty}\Lambda_t(x)$.

PROPOSITION 2. *Let $x,\ y \in L_{\log_+}(\mathscr{M})$ and $t>0$.*

1. ([10, Theorem 4.2]) $\Lambda_t(xy) \leqslant \Lambda_t(x)\Lambda_t(y)$.
2. ([7, Page 8]) $\Lambda_t(x) = \Lambda_t(x^*) = \Lambda_t(|x|) = \Lambda_t(x^*x)^{\frac{1}{2}}$.
3. ([7, Page 8]) $\Lambda_t(|x|^r) = \Lambda_t(|x|)^r,\ r \in \mathbb{R}^+$.
4. $\Lambda_t(x) \leqslant \Lambda_t(y)$ *if and only if* $x \prec_{\log} y$, *for all* $t>0$.
5. $\Lambda_t(x) \leqslant \Lambda_t(y)$, *if* $0 \leqslant x \leqslant y$, *for all* $t>0$.

It is easy to check that the above properties in Proposition 2 continue to hold for the Fuglede-Kadison determinant of x and y. In particular, $\Delta(xy) = \Delta(x)\Delta(y)$ ([7, Proposition 4.1]).

Next, we enumerate some properties about the Fuglede-Kadison determinant-like function of the product and mean of two operators.

Now, we present the definition of weighted geometric means $\#_\alpha$. Suppose that $0 \leqslant x, y \in L_0(\mathscr{M})$ and x is invertible. Then, for $0 \leqslant \alpha \leqslant 1$, $x\#_\alpha y$ is defined by

$$x^{\frac{1}{2}}(x^{-\frac{1}{2}}yx^{-\frac{1}{2}})^\alpha x^{\frac{1}{2}}.$$

The operation $\#_{\frac{1}{2}}$, simply denoted by $\#$, is called the geometric mean. Please refer to ([16, 20]) for more information.

PROPOSITION 3. *Let $x, y \in L_{\log_+}(\mathscr{M})$ and $t > 0$.*

1. ([17, Page 4]) *If xy is self adjoint, then*

$$\Lambda_t(xy) \leqslant \Lambda_t(yx).$$

2. ([18, Page 478]) *If $x, y \geqslant 0$ and $r \geqslant 1$, then*

$$\Lambda_t(|xy|^r) \leqslant \Lambda_t(x^r y^r).$$

3. ([16, Theorem 3.41]) *If $x, y \geqslant 0$ and x is invertible, then*

$$\Lambda_t(x\#_\alpha y) \leqslant \Lambda_t(x^{1-\alpha}y^\alpha),$$

where $0 \leqslant \alpha \leqslant 1$.

For more information on the Fuglede-Kadison determinant, please refer to [1, 2, 3, 7, 9, 10, 16, 18, 17, 21].

3. Geometric mean and logarithmic submajorization

When $\alpha = \frac{1}{2}$, we compare the geometric mean with the arithmetic mean.

REMARK 4. For $0 \leqslant x, y \in L_{\log_+}(\mathscr{M})$, if x is invertible, then

$$x\#y \leqslant \frac{x+y}{2}.$$

Proof. From the definition of $\#$, we have

$$x\#y = x^{\frac{1}{2}}(x^{-\frac{1}{2}}yx^{-\frac{1}{2}})^{\frac{1}{2}}x^{\frac{1}{2}} \text{ and } \frac{x+y}{2} = \frac{x^{\frac{1}{2}}(1+x^{-\frac{1}{2}}yx^{-\frac{1}{2}})x^{\frac{1}{2}}}{2}.$$

So, we just prove $(x^{-\frac{1}{2}}yx^{-\frac{1}{2}})^{\frac{1}{2}} \leqslant \frac{(1+x^{-\frac{1}{2}}yx^{-\frac{1}{2}})}{2}$. Set $c = x^{-\frac{1}{2}}yx^{-\frac{1}{2}} \geqslant 0$. Then

$$1 + c - 2c^{\frac{1}{2}} = (1 - c^{\frac{1}{2}})^2 = |1 - c^{\frac{1}{2}}|^2 \geqslant 0$$

and the remark is proved. $\square$

LEMMA 5. *Let* $0 \leqslant x,\, y \in L_{\log_+}(\mathscr{M})$. *The inequalities*

$$\Lambda_t(1+x) \leqslant \Lambda_t(1+y), \text{ for all } t>0$$

and

$$\Delta(1+x) \leqslant \Delta(1+y)$$

hold if $x \prec_{\log} y$. *Moreover,* $\Lambda_t(1+x) \leqslant \Lambda_t(1+y)$, *for all* $t>0$ *implies* $x \prec y$.

Proof. Let $\varphi(\alpha)=\log(1+\alpha)$. Then $\varphi : [0,\infty) \to [0,\infty)$ is a continuous increasing function such that

$$\varphi(0)=0 \text{ and } \varphi\circ\exp \text{ is convex.}$$

It follows from [7, Proposition 3.2] that $x \prec_{\log} y$ implies

$$\Lambda_t(1+x) \leqslant \Lambda_t(1+y), \text{ for all } t>0$$

and we also have

$$\Delta(1+x) \leqslant \Delta(1+y).$$

Similarly, putting

$$f(s)=\log\mu_s(1+x),\ g(s)=\log\mu_s(1+y) \text{ and } \varphi(\alpha)=e^{\alpha}-1$$

and applying [7, Proposition 3.2] again, we get $x \prec y$. □

By Lemma 5 and Proposition 2, we have the following remark.

REMARK 6. Let $x \in L_{\log_+}(\mathscr{M})$ and $t>0$. Then

$$\Lambda_t(1+|x|)=\Lambda_t(1+|x^*|).$$

THEOREM 7. *Let* $0 \leqslant x,\, y \in L_{\log_+}(\mathscr{M})$ *and* $t>0$. *If* x *is invertible, then*

$$\Lambda_t(x^{\frac{1}{2}}(x\#y)y^{\frac{1}{2}}) \leqslant \Lambda_t(xy).$$

Moreover,

$$\Lambda_t(x^{\frac{1}{2}}(x\#y)y(x\#y)x^{\frac{1}{2}}) \leqslant \Lambda_t(x^2y^2).$$

Proof. By [16, Lemma 3.32 and Proposition 3.33] there exists a contraction $z \in \mathscr{M}$ such that

$$x\#y = x^{\frac{1}{2}}zy^{\frac{1}{2}} = y^{\frac{1}{2}}z^*x^{\frac{1}{2}}.$$

Thus, for all $t>0$, we have

$$\begin{aligned}
\Lambda_t(x^{\frac{1}{2}}(x\#y)y^{\frac{1}{2}}) &= \Lambda_t(x^{\frac{1}{2}}y^{\frac{1}{2}}z^*x^{\frac{1}{2}}y^{\frac{1}{2}}) \\
&\leqslant (\Lambda_t(x^{\frac{1}{2}}y^{\frac{1}{2}}))^2\Lambda_t(z^*) \text{ (by Proposition 2 (1))} \\
&\leqslant \Lambda_t(|x^{\frac{1}{2}}y^{\frac{1}{2}}|^2) \text{ (by Proposition 2 (3) and the contractivity of } z) \\
&\leqslant \Lambda_t(xy) \text{ (by Proposition 3 (2)).}
\end{aligned}$$

Applying Proposition 3 (2) again, we obtain that

$$\begin{aligned}\Lambda_t(x^{\frac{1}{2}}(x\#y)y(x\#y)x^{\frac{1}{2}}) &= (\Lambda_t(x^{\frac{1}{2}}(x\#y)y^{\frac{1}{2}}))^2 \\ &\leqslant \Lambda_t(|xy|)^2 \\ &\leqslant \Lambda_t(x^2y^2). \quad \square\end{aligned}$$

As a result of Lemma 5 and Theorem 7, we have the following corollary.

COROLLARY 8. *Let $0 \leqslant x, y \in L_{\log_+}(\mathscr{M})$. If x is invertible, then*

$$\Lambda_t(1+|x^{\frac{1}{2}}(x\#y)y^{\frac{1}{2}}|) \leqslant \Lambda_t(1+|xy|)$$

and

$$\Lambda_t(1+|y^{\frac{1}{2}}(x\#y)x^{\frac{1}{2}}|^2) \leqslant \Lambda_t(1+|x^2y^2|).$$

In the following theorem, we consider the Fuglede-Kadison determinant-like function of the weighted geometric means.

THEOREM 9. *Suppose that $0 \leqslant x, y \in L_{\log_+}(\mathscr{M})$ and x is invertible. If $\beta \in [\frac{1}{2}, 1]$, $\alpha \geqslant 0$ and $t > 0$, then*

$$\Lambda_t(x^{\frac{\alpha+1}{2}}(x^{-1}\#_\beta y)x^{\frac{\alpha+1}{2}}) \leqslant \Lambda_t(y^\beta x^{\alpha+\beta}).$$

Proof. For $t > 0$, by a simple calculation, we get

$$\begin{aligned}\Lambda_t(x^{\frac{\alpha+1}{2}}(x^{-1}\#_\beta y)x^{\frac{\alpha+1}{2}}) &= \Lambda_t(x^{\frac{\alpha}{2}}(x^{\frac{1}{2}}yx^{\frac{1}{2}})^\beta x^{\frac{\alpha}{2}}) \\ &\leqslant \Lambda_t(x^{\frac{\alpha}{2\beta}}x^{\frac{1}{2}}yx^{\frac{1}{2}}x^{\frac{\alpha}{2\beta}})^\beta \text{ (by [14, Lemma 2.5])} \\ &= \Lambda_t(x^{\frac{\alpha+\beta}{2\beta}}yx^{\frac{\alpha+\beta}{2\beta}})^\beta \\ &= \Lambda_t(|y^{\frac{1}{2}}x^{\frac{\alpha+\beta}{2\beta}}|)^{2\beta} (2\beta \geqslant 1) \\ &\leqslant \Lambda_t(y^\beta x^{\alpha+\beta}) \text{ (by Proposition 3 (2))}.\end{aligned}$$

This completes the proof. $\square$

COROLLARY 10. *Let $0 \leqslant x, y \in L_{\log_+}(\mathscr{M})$ and x be invertible.*

1. *For $\beta \in [\frac{1}{2}, 1]$ and $\alpha \geqslant 0$, we have*

$$\Lambda_t(1+x^{\frac{\alpha+1}{2}}(x^{-1}\#_\beta y)x^{\frac{\alpha+1}{2}}) \leqslant \Lambda_t(1+|y^\beta x^{\alpha+\beta}|).$$

2. *For $\frac{\alpha}{\beta} \in [\frac{1}{2}, 1]$ and $\gamma \leqslant 0 \leqslant \alpha \leqslant \beta$, we have*

$$\Lambda_t(1+x^{\frac{\beta-\gamma}{2}}(x^{-\beta}\#_{\frac{\alpha}{\beta}}y^\beta)x^{\frac{\beta-\gamma}{2}}) \leqslant \Lambda_t(1+|y^\alpha x^{\alpha-\gamma}|).$$

Proof. Based on Lemma 5 and Theorem 9, we obtain the first desired result (1). Replacing α, β, x and y by $-\frac{\gamma}{\beta}$, $\frac{\alpha}{\beta}$, x^{β} and y^{β} respectively in Corollary 10 (1), we get (2). $\square$

An analogous result to Corollary 10 (2) is as below in Lemma 11.

LEMMA 11. *Let $0 \leqslant x,\ y \in L_{\log_+}(\mathscr{M})$ and let x be invertible. If $0 \leqslant \alpha \leqslant \beta \leqslant \gamma$, $\frac{\alpha}{\beta} \in [\frac{1}{2}, 1]$ and $t > 0$ then*

$$\Lambda_t(1 + x^{\frac{\beta-\gamma}{2}}(x^{-\beta} \#_{\frac{\alpha}{\beta}} y^{\beta})x^{\frac{\beta-\gamma}{2}}) \leqslant \Lambda_t(1 + |y^{\alpha}x^{\alpha-\gamma}|).$$

Proof. According to Lemma 5, we only need to prove

$$\Lambda_t(x^{\frac{\beta-\gamma}{2}}(x^{-\beta} \#_{\frac{\alpha}{\beta}} y^{\beta})x^{\frac{\beta-\gamma}{2}}) = \Lambda_t(x^{-\frac{\gamma}{2}}(x^{\frac{\beta}{2}} y^{\beta} x^{\frac{\beta}{2}})^{\frac{\alpha}{\beta}} x^{-\frac{\gamma}{2}}) \leqslant \Lambda_t(y^{\alpha}x^{\alpha-\gamma}),\ t > 0.$$

For all $t > 0$, we have

$$\begin{aligned}
\Lambda_t(x^{-\frac{\gamma}{2}}(x^{\frac{\beta}{2}} y^{\beta} x^{\frac{\beta}{2}})^{\frac{\alpha}{\beta}} x^{-\frac{\gamma}{2}}) &\leqslant \Lambda_t(x^{-\frac{\gamma\beta}{2\alpha}}(x^{\frac{\beta}{2}} y^{\beta} x^{\frac{\beta}{2}})x^{-\frac{\gamma\beta}{2\alpha}})^{\frac{\alpha}{\beta}} \text{ (by [14, Lemma 2.5])} \\
&= \Lambda_t(x^{\frac{\alpha\beta-\gamma\beta}{2\alpha}} y^{\beta} x^{\frac{\alpha\beta-\gamma\beta}{2\alpha}})^{\frac{\alpha}{\beta}} \\
&= \Lambda_t(|y^{\frac{\beta}{2}} x^{\frac{\alpha\beta-\gamma\beta}{2\alpha}}|^2)^{\frac{\alpha}{\beta}} \\
&= \Lambda_t(|y^{\frac{\beta}{2}} x^{\frac{\alpha\beta-\gamma\beta}{2\alpha}}|)^{\frac{2\alpha}{\beta}} \text{ (by Proposition 2 (3))} \\
&\leqslant \Lambda_t(y^{\alpha}x^{\alpha-\gamma}) \text{ (by Proposition 3 (2) and } \frac{2\alpha}{\beta} \geqslant 1).
\end{aligned}$$

This completes the proof. $\square$

We show some analogues of inequality (1.2) in $L_{\log_+}(\mathscr{M})$ and the first one is valid under a completely new condition.

THEOREM 12. *Let $0 \leqslant x,\ y \in L_{\log_+}(\mathscr{M})$. If x is invertible, then the following inequalities hold.*

1. *For $\gamma \leqslant 0 \leqslant \alpha \leqslant \beta$ and $\frac{\alpha}{\beta} \in [\frac{1}{2}, 1]$, we have*

$$\Delta(x^{\gamma} + |y^{\frac{\beta}{2}} x^{\frac{\beta}{2}}|^{\frac{2\alpha}{\beta}}) \leqslant \Delta(x^{\gamma} + x^{\gamma}|y^{\alpha}x^{\alpha-\gamma}|).$$

2. *For $0 \leqslant \alpha \leqslant \beta \leqslant \gamma$, $\frac{\alpha}{\beta} \in [\frac{1}{2}, 1]$, we have*

$$\Delta(x^{\gamma} + |y^{\frac{\beta}{2}} x^{\frac{\beta}{2}}|^{\frac{2\alpha}{\beta}}) \leqslant \Delta(x^{\gamma} + x^{\gamma}|y^{\alpha}x^{\alpha-\gamma}|).$$

Proof. Notice that

$$x^{\gamma}+(|y^{\frac{\beta}{2}}x^{\frac{\beta}{2}}|)^{\frac{2\alpha}{\beta}}=x^{\frac{\gamma}{2}}(1+x^{\frac{\beta-\gamma}{2}}(x^{-\beta}\#_{\frac{\alpha}{\beta}}y^{\beta})x^{\frac{\beta-\gamma}{2}})x^{\frac{\gamma}{2}}$$

and

$$x^{\gamma}+x^{\gamma}|y^{\alpha}x^{\alpha-\gamma}|=x^{\gamma}(1+|y^{\alpha}x^{\alpha-\gamma}|).$$

Then the theorem follows from Corollary 10 (2) and Lemma 11. □

Considering the results in above theorem, we give a conjecture as follows.

CONJECTURE 13. Let A, B be two positive semi-definite matrices and $\gamma\leqslant 0\leqslant\alpha\leqslant\beta$, $\frac{\alpha}{\beta}\in[\frac{1}{2},1]$. If A is invertible, then

$$\det(A^{\gamma}+A^{\alpha}B^{\alpha})\geqslant\det(A^{\gamma}+|B^{\frac{\beta}{2}}A^{\frac{\beta}{2}}|^{\frac{2\alpha}{\beta}}).$$

4. Some other important determinant inequalities

First we recall some basic properties of $\mathbb{M}_2(\mathscr{M})$ in [8]. Let $\mathbb{M}_2(\mathscr{M})$ be the von Neumann algebra of all 2×2 operator matrices equipped with the trace $\tau_2=\mathrm{tr}\otimes\tau$. We denote the Fuglede-Kadison determinant on $\mathbb{M}_2(\mathscr{M})$ corresponding to τ_2 by Δ_2, i.e.,

$$\Delta_2(A)=\exp(\tau_2(\log|A|))$$

where $A\in\mathbb{M}_2(\mathscr{M})$. According to the proof of in [9, Theorem 1], we conclude that

$$\Delta_2(AB)=\Delta_2(A)\Delta_2(B).$$

This equation plays a crucial role in proving $\Delta(1+ab)=\Delta(1+ba)$ in the following remark.

Some results in the following remark are well-known. We also give brief calculations for convenience.

REMARK 14. 1. Let $A,B\in\mathbb{M}_2(\mathscr{M})$. Then

$$\tau_2(AB)=\tau_2(BA).$$

2. Let $a,b\in\mathscr{M}$ and $A=\mathrm{diag}(a,b)\in\mathbb{M}_2(\mathscr{M})$. Then

$$\Delta_2(A)=\Delta(a)\Delta(b).$$

3. Let $a\in\mathscr{M}$ and

$$A=\begin{bmatrix}1 & a\\ 0 & 1\end{bmatrix}\in\mathbb{M}_2(\mathscr{M}).$$

Then $\Delta_2(A)=1$. Similarly, when A is a lower triangular matrix, the conclusion is also valid.

4. Let $a, b \in \mathscr{M}$. Then
$$\Delta(1+ab) = \Delta(1+ba).$$

Proof. (2) We conclude from the definition of $\Delta_2(A)$ that

$$\begin{aligned}\Delta_2(A) &= \exp(\tau_2 \log(\mathrm{diag}(|a|,|b|)) \\ &= \exp((\mathrm{tr}\otimes\tau)(\mathrm{diag}(1,0)\otimes\log|a|) + (\mathrm{tr}\otimes\tau)(\mathrm{diag}(0,1)\otimes\log|b|)) \\ &= \Delta(a)\Delta(b).\end{aligned}$$

(3) Because
$$\Delta_2\left(\begin{bmatrix} u & 0 \\ 0 & 1\end{bmatrix} A \begin{bmatrix} u^* & 0 \\ 0 & 1\end{bmatrix}\right) = \Delta_2(A),$$
where u is a unitary operator, we only need to consider the case $A = \begin{bmatrix} 1 & |a| \\ 0 & 1\end{bmatrix}$. Let $\lambda^2 - \lambda(2+|a|^2)+1 = 0$. Then by functional calculus, we obtain two operators
$$\lambda_1 = \frac{2+|a|^2+|a|\sqrt{|a|^2+4}}{2}$$
and
$$\lambda_2 = \frac{2+|a|^2-|a|\sqrt{|a|^2+4}}{2}$$
where $\lambda_1+\lambda_2 = 2+|a|^2$ and $\lambda_1\lambda_2 = 1$. We may assume that $|a|$ is invertible. By a direct computation, we have
$$U = \begin{bmatrix} (\lambda_1-1)^{-1}(|\lambda_1-1|^{-2}+|a|^{-2})^{-\frac{1}{2}} & (\lambda_2-1)^{-1}(|\lambda_2-1|^{-2}+|a|^{-2})^{-\frac{1}{2}} \\ |a|^{-1}(|\lambda_1-1|^{-2}+|a|^{-2})^{-\frac{1}{2}} & |a|^{-1}(|\lambda_2-1|^{-2}+|a|^{-2})^{-\frac{1}{2}}\end{bmatrix}$$
which entails that $U^*U = UU^* = \mathrm{diag}(1,1)$ and $U^*|A|U = \mathrm{diag}(\lambda_1^{\frac{1}{2}}, \lambda_2^{\frac{1}{2}})$. Therefore, we have

$$\begin{aligned}\Delta_2(A) &= \exp(\tau_2(U\mathrm{diag}(\log\lambda_1^{\frac{1}{2}}, \log\lambda_2^{\frac{1}{2}})U^*)) \\ &= \exp(\tau_2(\mathrm{diag}(\log\lambda_1^{\frac{1}{2}}, \log\lambda_2^{\frac{1}{2}}))(\text{by trace property in previous (1)}) \\ &= \exp((\mathrm{tr}\otimes\tau)(\mathrm{diag}(1,0)\otimes\log\lambda_1^{\frac{1}{2}}) + (\mathrm{tr}\otimes\tau)(\mathrm{diag}(0,1)\otimes\log\lambda_2^{\frac{1}{2}})) \\ &= \exp(\tau(\log\lambda_1^{\frac{1}{2}}\lambda_2^{\frac{1}{2}})) \\ &= \exp(\tau(\log(\lambda_1\lambda_2)^{\frac{1}{2}})) \\ &= 1.\end{aligned}$$

(4) We have the following decomposition
$$\begin{bmatrix} 1 & 0 \\ -b & 1\end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1+ba\end{bmatrix}\begin{bmatrix} 1 & a \\ 0 & 1\end{bmatrix} = \begin{bmatrix} 1 & a \\ -b & 1\end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1\end{bmatrix}\begin{bmatrix} 1+ab & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix} 1 & 0 \\ -b & 1\end{bmatrix}.$$

Using the multiplicativity of Δ_2 and the conclusion in the previous (2) and (3), we conclude that

$$\Delta(1+ab) = \Delta(1+ba). \quad \square$$

Immediately, we obtain the following remark.

REMARK 15. Let $x, y \in \mathscr{M}$. Then for any positive integer n, we have

$$\Delta(x^2 + (yx)^n) = \Delta(x^2 + (xy)^n).$$

LEMMA 16. *Suppose that* $0 \leqslant x, y \in L_{\log_+}(\mathscr{M})$. *If* x, y *are invertible and* $\alpha \in \mathbb{R}$, *then*

$$(x^*y^*yx)^\alpha = x^*y^*(yxx^*y^*)^{\alpha-1}yx.$$

Proof. Let $x^*y^* = u|x^*y^*|$ be the polar decomposition of the operator x^*y^*, where u is a unitary. Then

$$u = x^*y^*|x^*y^*|^{-1},\ u^* = |x^*y^*|^{-1}yx \text{ and } yx = |x^*y^*|u^*.$$

Hence,

$$\begin{aligned}(x^*y^*yx)^\alpha &= (u|x^*y^*|^2u^*)^\alpha \\ &= u|x^*y^*|^{2\alpha}u^* \\ &= x^*y^*|x^*y^*|^{-1}|x^*y^*|^{2\alpha}|x^*y^*|^{-1}yx \\ &= x^*y^*(yxx^*y^*)^{\alpha-1}yx. \quad \square\end{aligned}$$

LEMMA 17. *Let* $0 \leqslant x, y \in L_{\log_+}(\mathscr{M})$ *and* $p, q \geqslant 0$.

1. *If* xy^q *is hyponormal, then*

$$xy^pxy^q \prec_{\log} x^2y^{p+q}.$$

2. *If* $y^{-q}x$ *is hyponormal and* y *is invertible, then*

$$\Lambda_t(xy^{p-q}x) \leqslant \Lambda_t(xy^pxy^{-q}).$$

Proof. (1) Let $a = xy^px \geqslant 0$ and $b = y^q \geqslant 0$. Since $\mu_s(ab) = \mu_s(ba)$ ([4, Corollary 3.6]), we have

$$\mu_s(xy^pxy^q) = \mu_s(y^qxy^px).$$

Let $c = xy^q$ and $d = y^px$. Then $c^* = y^qx$. Since c is hyponormal, applying [5, Proposition 4.3] to $\mu_s(y^qxy^px)$, we see that $\mu_s(c^*d) \leqslant \mu_s(cd)$, i.e.,

$$\mu_s(y^qxy^px) \leqslant \mu_s(xy^qy^px) = \mu_s(xy^{p+q}x).$$

Hence,

$$\int_0^t \log \mu_s(xy^p xy^q)ds \leqslant \int_0^t \log \mu_s(xy^{p+q}x)ds \leqslant \int_0^t \log \mu_s(x^2y^{p+q})ds.$$

(2) By Proposition 1 (1), we get

$$\mu_s(xy^{p-q}x) = \mu_s(xy^p y^{-q}x) = \mu_s(xy^{-q}y^p x).$$

Since $y^{-q}x$ is hyponormal, applying [5, Proposition 4.3] to $\mu_s(xy^{-q}y^p x)$ we have

$$\mu_s(xy^{-q}y^p x) \leqslant \mu_s(y^{-q}xy^p x) = \mu_s(xy^p xy^{-q}).$$

Therefore,

$$\int_0^t \log \mu_s(xy^{p-q}x)ds \leqslant \int_0^t \log \mu_s(xy^p xy^{-q})ds.$$

The proof is completed. □

Combining Lemma 5 with Lemma 17, we have the following result.

REMARK 18. Let $0 \leqslant x, y \in L_{\log_+}(\mathscr{M})$ and $p, q \geqslant 0$.

1. If xy^q is hyponormal, then

$$\Delta(1 + |xy^p xy^q|) \leqslant \Delta(1 + |x^2 y^{p+q}|).$$

2. If $y^{-q}x$ is hyponormal and y is invertible, then

$$\Delta(1 + |xy^{p-q}x|) \leqslant \Delta(1 + |xy^p xy^{-q}|).$$

We show some operator Fuglede-Kadison determinant inequalities which are corresponding to inequality (1.1) in the next two theorems.

THEOREM 19. *Let* $0 \leqslant x,\ y \in L_{\log_+}(\mathscr{M})$. *If* x^2y *is hyponormal and* $x,\ y$ *are invertible, then we have*

$$\Delta(x^2 + |yx|^p) \leqslant \Delta(x^2 + x^2|y(yx^2y)^{\frac{p}{2}}x^{-2}y^{-1}|)$$

for all $p \in [0, \infty)$.

Proof. In fact, since x^2y is hyponormal, we conclude that $(yx^2y)^{-1}y$ is hyponormal. So, for every $p \in [0, \infty)$ and $t > 0$, we have

$$\begin{aligned}
\Lambda_t(x^{-1}(|yx|^2)^{\frac{p}{2}}x^{-1}) &= \Lambda_t(x^{-1}(xy^2x)^{\frac{p}{2}}x^{-1}) \\
&= \Lambda_t(y(yx^2y)^{\frac{p}{2}-1}y) \text{ (by Lemma 16)} \\
&\leqslant \Lambda_t(y(yx^2y)^{\frac{p}{2}}y(yx^2y)^{-1}) \text{ (by Lemma 17 (2))} \\
&= \Lambda_t(y(yx^2y)^{\frac{p}{2}}x^{-2}y^{-1}).
\end{aligned}$$

Next, by Lemma 5, we can assert that

$$\Lambda_t(1+x^{-1}(xy^2x)^{\frac{p}{2}}x^{-1}) \leqslant \Lambda_t(1+|y(yx^2y)^{\frac{p}{2}}x^{-2}y^{-1}|),$$

and

$$\Delta(1+x^{-1}(xy^2x)^{\frac{p}{2}}x^{-1}) \leqslant \Delta(1+|y(yx^2y)^{\frac{p}{2}}x^{-2}y^{-1}|).$$

Finally, multiplying both sides of the above determinant inequality by $\Delta(x^2) > 0$, we obtain

$$\Delta(x^2+|yx|^p) \leqslant \Delta(x^2+x^2|y(yx^2y)^{\frac{p}{2}}x^{-2}y^{-1}|). \quad \square$$

Actually, using Theorem 12 in Section 3, we get another analogue of inequality (1.1). In order to compare with the above theorem, we exhibit it in the following theorem.

THEOREM 20. *Suppose that $0 \leqslant x, y \in L_{\log_+}(\mathscr{M})$ and x is invertible. Then*

$$\Delta(x^2+|yx|^p) \leqslant \Delta(x^2+x^2|y^p x^{p-2}|),\ 1 \leqslant p \leqslant 2.$$

Proof. For $1 \leqslant p \leqslant 2$, replacing α by p, β by 2 and γ by 2 in Theorem 12 (2), respectively, we get the result. $\square$

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