Operators
and
Matrices
Volume 16, Number 3 (2022), 909–923
doi:10.7153/oam-2022-16-61

ON SOME ALGEBRAIC PROPERTIES OF BLOCK TOEPLITZ MATRICES WITH COMMUTING ENTRIES

MUHAMMAD AHSAN KHAN* AND AMEUR YAGOUB

(*Communicated by I. M. Spitkovsky*)

Abstract. Toeplitz matrices are ubiquitous and play important roles across many areas of mathematics. In this paper, we present some algebraic results concerning block Toeplitz matrices with block entries belonging to a commutative algebra $\mathscr{A}$. The characterization of normal block Toeplitz matrices with entries from a commutative algebra $\mathscr{A}$ of normal matrices is also obtained.

1. Introduction

Toeplitz matrices are important due to their typical property that the entries in the matrices depend only on the differences of the indices, and as a result, the entries on their main diagonal as well as those lying parallel to main diagonal are constant. These matrices arise naturally in several fields of mathematics, as well as applied areas as signal processing or time series analysis. The monographs dedicated to the subject are [18, 10] and [3].

The corresponding general theory of block Toeplitz matrices is less developed mostly due to intrinsic algebraic difficulties that appear with respect to the scalar case. Block Toeplitz matrices appear very briefly [17] and then in [19, 16, 15, 13].

In [4], the authors have proved a variety of algebraic results concerning scalar Toeplitz matrices. Among other things, they have obtained the necessary and sufficient condition for the product $AB-CD$ to be a Toeplitz matrix, and for $AB-CD=0$, provided that A, B, C, and D are Toeplitz matrices. They have also completely characterized normal Toeplitz matrices. We refer the reader to [6, 1, 7, 8, 9, 2, 11] where characterization of normal Toeplitz matrices have been discussed.

In [14], some generalization of the results of [4] concerning the product $AB-CD$ of block Toeplitz matrices has been made. Apart from this [14] has also classified normal block Toeplitz matrices where the entries are taken from the algebra of scalar diagonal matrices. We pursue here this investigation, obtaining a natural generalization of the results of [4] by taking entries of the block Toeplitz matrices from a fixed commutative algebra of scalar matrices. We will give new proofs, and refinements of some of the results of [14] and [4] in a more natural way than what we obtained in [14]. Most

Mathematics subject classification (2020): 15A27, 15A30, 15B05.
Keywords and phrases: Block Toeplitz matrices, displacement matrix, normal matrices.
* Corresponding author.

Paper OaM-16-61

importantly, we obtain the criteria for characterizing normal block Toeplitz matrices with commuting normal entries.

The remaining paper is organized as follows: Section 1 serves to recall the basic notations and facts we will be using in the forthcoming sections. In section 3, we will provide the generalization and refinements of the results of [4] and [14] concerning the product of block Toeplitz matrices with commuting entries. Section 4 is concerned with the commutation of certain block Toeplitz matrices. The object of section 5 is to present the most important result of this paper; the characterization of normal block Toeplitz matrices with entries from a commutative algebra $\mathscr{A}$ of normal matrices. The last section of the paper, obtains the concrete description of normality of block Toeplitz matrices with entries from the algebra of diagonal matrices.

2. Preliminaries

In this section, we give a summary of the basic notations and facts that we shall encounter throughout this paper. As customary $\mathbb{C}$ stands for the set of complex numbers. We symbolize by $\mathscr{M}_n$ the algebra of $n \times n$ matrices with entries from $\mathbb{C}$ and by $\mathscr{D}_d$ the algebra of $d \times d$ complex diagonal matrices. We will prefer to label the rows and columns of $n \times n$ matrices from 1 to n; so $A \in \mathscr{M}_n$ is written $A = (a_{i,j})_{i,j=1}^n$ with $a_{i,j} \in \mathbb{C}$. Then, we designate by $\mathscr{T}_n \subset \mathscr{M}_n$ the space of scalar Toeplitz matrices.

We will mostly be interested in block matrices, that is, matrices whose elements are matrices of dimension d, instead of complex numbers. Thus a block Toeplitz matrix is actually an $nd \times nd$ matrix, consisting of n^2 blocks of dimension d, and these blocks are constant along the diagonals.

Throughout, we will denote $n \times n$ block matrices by bold capital letters. As we know that the entries of scalar Toeplitz matrices are complex numbers, in order to obtain related results concerning block Toeplitz matrices, we will assume that their entries belong to a fixed commutative algebra of $\mathscr{M}_d$, that we will denote by $\mathscr{A}$.

If $A = \begin{pmatrix} 0 \\ A_1 \\ A_2 \\ \vdots \\ A_{n-1} \end{pmatrix}$ and $\Omega = \begin{pmatrix} 0 \\ \Omega_1 \\ \Omega_2 \\ \vdots \\ \Omega_{n-1} \end{pmatrix}$ are column vectors with entries from $\mathscr{A}$, then let $\mathbf{T}(A,\Omega)$ denote the $n \times n$ block Toeplitz matrix of the form:

$$\mathbf{T}(A,\Omega) = \begin{pmatrix} 0 & \Omega_1^* & \Omega_2^* & \dots & \Omega_{n-1}^* \\ A_1 & 0 & \Omega_1^* & \dots & \Omega_{n-2}^* \\ A_2 & A_1 & 0 & \dots & \Omega_{n-3}^* \\ \vdots & \vdots & \vdots & \ddots & \\ A_{n-1} & A_{n-2} & A_{n-3} & \dots & 0 \end{pmatrix}. \tag{2.1}$$

Then $\mathbf{T}(A,\Omega) + \mathbf{A}_0$ represents the general block Toeplitz matrix with entries from $\mathscr{A}$, where

$$\mathbf{A}_0 = \begin{pmatrix} A_0 & 0 & 0 & \dots & 0 \\ 0 & A_0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \dots & A_0 \end{pmatrix}$$

is a diagonal block Toeplitz matrix with entries from $\mathscr{A}$. If $\mathscr{A}$ is a commutative subalgebra of $\mathscr{M}_d$, then in the sequel we will use the following notations:

- $\mathscr{M}_n \otimes \mathscr{A}$ is the collection of $n \times n$ block matrices whose entries all belong to $\mathscr{A}$;
- $\mathscr{T}_n \otimes \mathscr{A}$ is the collection of $n \times n$ block Toeplitz matrices whose entries all belong to $\mathscr{A}$;
- $\mathscr{D}_n \otimes \mathscr{A}$ is the collection of $n \times n$ diagonal block Toeplitz matrices whose entries all belong to $\mathscr{A}$;
- $\mathscr{C} \otimes \mathscr{A}$ is the collection of all $n \times 1$ block matrices whose entries all belong to $\mathscr{A}$;
- $\mathscr{R} \otimes \mathscr{A}$ is the collection of all $1 \times n$ block matrices whose entries all belong to $\mathscr{A}$.

It is obvious that $\mathscr{D}_n \otimes \mathscr{A} \subset \mathscr{T}_n \otimes \mathscr{A} \subset \mathscr{M}_n \otimes \mathscr{A}$. For block diagonal matrices we will use the notation

$$diag\left(A_1\ A_2\ \cdots\ A_n\right) = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_n \end{pmatrix}.$$

If $A \in \mathscr{C} \otimes \mathscr{A}$ (or $\mathscr{R} \otimes \mathscr{A}$) and $X \in \mathscr{A}$, then we will use the notation $X \diamond A$ to indicate that X is multiplied in a usual way with every entry of A.

Throughout whenever we will use the notation $\mathbf{T}(A,\Omega)$, A and Ω will be in $\mathscr{C} \otimes \mathscr{A}$ with the first entry 0. Following this notation it is immediate that $\mathbf{T}(A,\Omega)^* = \mathbf{T}(\Omega,A)$. For fixed $1 \leqslant k \leqslant n$, let P_{k-1} be the vectors in $\mathscr{R} \otimes \mathscr{A}$ whose entry at $k-1$ position is I and all other entries are zero; thus

$$\begin{aligned} P_0 &= \left(I,0,0,\cdots,0\right) \\ P_1 &= \left(0,I,0,\cdots,0\right) \\ P_2 &= \left(0,0,I,\cdots,0\right) \end{aligned}$$

etc. Let $I \in \mathscr{A}$ be the identity matrix and $\mathbf{S}$ denote the matrix consisting of I along the subdiagonal and zero elsewhere, i.e.,

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ I & 0 & 0 & \dots & 0 & 0 \\ 0 & I & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & I & 0 \end{pmatrix}.$$

Note that, $\mathbf{S}^n = \mathbf{S}^{*n} = 0$. If $X \in \mathscr{A}$, then we denote the matrix $\mathbf{S} + X \diamond P_0^* P_{n-1}$ by $\mathbf{S}_X$. For $A = \begin{pmatrix} 0 \\ A_1 \\ \vdots \\ A_{n-1} \end{pmatrix} \in \mathscr{C} \otimes \mathscr{A}$, we define $\widetilde{A} := \begin{pmatrix} 0 \\ A_{n-1}^* \\ \vdots \\ A_1^* \end{pmatrix}$. A block Toeplitz matrix $\mathbf{A} = \mathbf{T}(A, \Omega) + \mathbf{A}_0$ is said to be a block circulant matrix if, $A = \widetilde{\Omega}$. The displacement matrix of any square block matrix $\mathbf{A}$ is defined as

$$\triangle(\mathbf{A}) := \mathbf{A} - \mathbf{S}\mathbf{A}\mathbf{S}^*.$$

We refer the reader to [5, 12] for other kinds of displacement matrices. Note, in particular that if $\mathbf{I} \in \mathscr{M}_n \otimes \mathscr{A}$, then $\triangle(\mathbf{I}) = \mathbf{I} - \mathbf{S}\mathbf{S}^* = P_0^* P_0$.

The results below from [4, 14] are also valid for block matrices with entries from $\mathscr{A}$. We are adding their proofs just for completeness.

LEMMA 2.1. *If $\mathbf{A} \in \mathscr{M}_n \otimes \mathscr{A}$, then $\mathbf{A} = \sum_{k=0}^{n-1} \mathbf{S}^k \triangle(\mathbf{A}) \mathbf{S}^{*k}$.*

Proof.

$$\begin{aligned} \sum_{k=0}^{n-1} \mathbf{S}^k (\triangle(\mathbf{A})) \mathbf{S}^{k*} &= \sum_{k=0}^{n-1} \mathbf{S}^k (\mathbf{A} - \mathbf{S}\mathbf{A}\mathbf{S}^*) \mathbf{S}^{k*} \\ &= \sum_{k=0}^{n-1} (\mathbf{S}^k \mathbf{A} \mathbf{S}^{k*} - \mathbf{S}^{k+1} \mathbf{A} \mathbf{S}^{k+1*}) \\ &= \mathbf{A} - \mathbf{S}^n \mathbf{A} \mathbf{S}^{n*} = \mathbf{A}. \quad \square \end{aligned}$$

Thus in order to show that $\mathbf{A} = 0$, it will be enough to show that $\triangle(\mathbf{A}) = 0$. We have the following analog of the Lemma 2.2 of [4] for block Toeplitz matrices with entries from $\mathscr{A}$.

LEMMA 2.2. *$\mathbf{A} \in \mathscr{M}_n \otimes \mathscr{A}$ is in $\mathscr{T}_n \otimes \mathscr{A}$ if and only if there exist vectors $A, \Omega \in \mathscr{C} \otimes \mathscr{A}$ such that $\triangle(\mathbf{A}) = AP_0 + P_0^* \Omega^*$.*

Proof. Suppose that $\mathbf{A} = \mathbf{T}(A,\Omega) + \mathbf{A}_0 \in \mathscr{T}_n \otimes \mathscr{A}$. Since the displacement matrix for $\mathbf{A}$ is defined as $\triangle(\mathbf{A}) = \mathbf{A} - \mathbf{SAS}^*$. Then simple computation yields that

$$\triangle(\mathbf{A}) = \begin{pmatrix} A_0 & \Omega_1^* & \Omega_2^* & \dots & \Omega_{n-1}^* \\ A_1 & 0 & 0 & \dots & 0 \\ A_2 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ A_{n-1} & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{2.2}$$

If we take $A = \begin{pmatrix} A_0 \\ A_{-1} \\ \vdots \\ A_{1-n} \end{pmatrix}$ and $\Omega = \begin{pmatrix} 0 \\ \Omega_1 \\ \vdots \\ \Omega_{n-1} \end{pmatrix}$, then one can easily verify that $\triangle(\mathbf{A}) = AP_0 + P_0^*\Omega^*$.

For the converse, let $\mathbf{A} = (A_{ij})_{i,j=1}^n \in \mathscr{M}_n \otimes \mathscr{A}$. Suppose then that $A = \begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_{n-1} \end{pmatrix}$ and $\Omega = \begin{pmatrix} \Omega_0 \\ \Omega_1 \\ \vdots \\ \Omega_{n-1} \end{pmatrix}$ are vectors in $\mathscr{C} \otimes \mathscr{A}$, such that

$$\triangle(\mathbf{A}) = AP_0 + P_0^*\Omega^*.$$

This means that $\triangle(\mathbf{A})$ is given by (2.2). Computing $\mathbf{A}$ by the formula given in Lemma 2.1 yields that $\mathbf{A}$ is in $\mathscr{T}_n \otimes \mathscr{A}$. $\square$

REMARK 2.3. Note that the condition in the statement of Lemma 2.2 is equivalent to the fact that $\triangle(\mathbf{A})$ may have nonzero entries only on the first row and the first column.

3. Product of block Toeplitz matrices with commuting entries

The goal of this section is to obtain the basic algebraic results concerning the product $\mathbf{AB} - \mathbf{CD}$, where $\mathbf{A}, \mathbf{B}, \mathbf{C}$, and $\mathbf{D}$ are block Toeplitz matrices with entries from $\mathscr{A}$. One can see that the generalization obtained here is more natural than what we obtained in [14]. We start with the following Lemma, which describes the structure of the displacement matrix for the product of two block Toeplitz matrices with entries from $\mathscr{A}$.

LEMMA 3.1. *Let* $C = \begin{pmatrix} 0 \\ C_1 \\ C_2 \\ \vdots \\ C_{n-1} \end{pmatrix}$, $\Gamma = \begin{pmatrix} 0 \\ \Gamma_1 \\ \Gamma_2 \\ \vdots \\ \Gamma_{n-1} \end{pmatrix}$, $D = \begin{pmatrix} 0 \\ D_1 \\ D_2 \\ \vdots \\ D_{n-1} \end{pmatrix}$, *and*

$\Theta = \begin{pmatrix} 0 \\ \Theta_1 \\ \Theta_2 \\ \vdots \\ \Theta_{n-1} \end{pmatrix}$, *be vectors in* $\mathscr{C} \otimes \mathscr{A}$. *If* $\mathbf{C} = \mathbf{T}(C,\Gamma) + \mathbf{C}_0$ *and* $\mathbf{D} = \mathbf{T}(D,\Theta) + \mathbf{D}_0$, *then*

$$\triangle(\mathbf{CD}) = C\Theta^* - \widetilde{\Gamma}\widetilde{D}^* + [\mathbf{C}D + \mathbf{D}_0 C + \mathbf{C}_0\mathbf{D}_0 P_0^*]P_0 + P_0^*[\Gamma^*\mathbf{SDS}^* + \Theta^*\mathbf{C}_0]. \tag{3.1}$$

Proof. Let $\widehat{\mathbf{C}} = \mathbf{T}(C,\Gamma)$ and $\widehat{\mathbf{D}} = \mathbf{T}(D,\Theta)$. Then we have

$$\begin{aligned} \triangle(\mathbf{CD}) &= \triangle[(\widehat{\mathbf{C}} + \mathbf{C}_0)][(\widehat{\mathbf{D}} + \mathbf{D}_0)] \\ &= \triangle[\widehat{\mathbf{C}}\widehat{\mathbf{D}} + \mathbf{C}_0\widehat{\mathbf{D}} + \mathbf{D}_0\widehat{\mathbf{C}} + \mathbf{C}_0\mathbf{D}_0] \\ &= \triangle(\widehat{\mathbf{C}}\widehat{\mathbf{D}}) + \triangle(\mathbf{C}_0\widehat{\mathbf{D}}) + \triangle(\mathbf{D}_0\widehat{\mathbf{C}}) + \triangle(\mathbf{C}_0\mathbf{D}_0). \end{aligned}$$

Since $\mathbf{C}_0, \mathbf{D}_0 \in \mathscr{D}_n \otimes \mathscr{A}$, then $\mathbf{S}$ commute with $\mathbf{C}_0$ and $\mathbf{D}_0$ respectively, therefore last equation above can be written as

$$\triangle(\mathbf{CD}) = \triangle(\widehat{\mathbf{C}}\widehat{\mathbf{D}}) + \mathbf{C}_0\triangle(\widehat{\mathbf{D}}) + \mathbf{D}_0\triangle(\widehat{\mathbf{C}}) + \mathbf{C}_0\mathbf{D}_0\triangle(\mathbf{I}). \tag{3.2}$$

By Lemma 2.2, there exist vectors $D, \Theta \in \mathscr{C} \otimes \mathscr{A}$ such that, $\triangle(\widehat{\mathbf{D}}) = DP_0 + P_0^*\Theta^*$. Similarly $\triangle(\widehat{\mathbf{C}}) = CP_0 + P_0^*\Gamma^*$, with $C, \Gamma \in \mathscr{C} \otimes \mathscr{A}$. Also we have $\triangle(\mathbf{I}) = P_0^*P_0$. Then (3.2) becomes

$$\begin{aligned} \triangle(\mathbf{CD}) &= \triangle(\widehat{\mathbf{C}}\widehat{\mathbf{D}}) + \mathbf{C}_0[DP_0 + P_0^*\Theta^*] + \mathbf{D}_0[CP_0 + P_0^*\Gamma^*] + \mathbf{C}_0\mathbf{D}_0P_0^*P_0 \\ &= \triangle(\widehat{\mathbf{C}}\widehat{\mathbf{D}}) + [\mathbf{C}_0 D + \mathbf{D}_0 C + \mathbf{C}_0\mathbf{D}_0P_0^*]P_0 + P_0^*[\Theta^*\mathbf{C}_0 + \Gamma^*\mathbf{D}_0]. \end{aligned} \tag{$*$}$$

By using the definition of $\triangle$, we have

$$\begin{aligned} \triangle(\widehat{\mathbf{C}}\widehat{\mathbf{D}}) &= \widehat{\mathbf{C}}\widehat{\mathbf{D}} - \mathbf{S}\widehat{\mathbf{C}}\widehat{\mathbf{D}}\mathbf{S}^* \\ &= \widehat{\mathbf{C}}\widehat{\mathbf{D}} - \widehat{\mathbf{C}}\mathbf{S}\widehat{\mathbf{D}}\mathbf{S}^* + \widehat{\mathbf{C}}\mathbf{S}\widehat{\mathbf{D}}\mathbf{S}^* - \mathbf{S}\widehat{\mathbf{C}}[\mathbf{S}^*\mathbf{S} + P_{n-1}^*P_{n-1}]\widehat{\mathbf{D}}\mathbf{S}^* \\ &= \widehat{\mathbf{C}}\triangle\widehat{\mathbf{D}} + \triangle\widehat{\mathbf{C}}(\mathbf{S}\widehat{\mathbf{D}}\mathbf{S}^*) - \mathbf{S}\widehat{\mathbf{C}}P_{n-1}^*P_{n-1}\widehat{\mathbf{D}}\mathbf{S}^* \\ &= \widehat{\mathbf{C}}[DP_0 + P_0^*\Theta^*] + [CP_0 + P_0^*\Gamma^*]\mathbf{S}\widehat{\mathbf{D}}\mathbf{S}^* - \widetilde{\Gamma}\widetilde{D}^*. \end{aligned} \tag{$**$}$$

Since $P_0\mathbf{S} = 0$, so the term $CP_0\mathbf{S}\widehat{\mathbf{D}}\mathbf{S}^* = 0$. Also $\widehat{\mathbf{C}}P_0^*\Theta^* = C\Theta^*$, then ($**$) can be written as

$$\triangle(\widehat{\mathbf{C}}\widehat{\mathbf{D}}) = \widehat{\mathbf{C}}DP_0 + P_0^*\Gamma^*\mathbf{S}\widehat{\mathbf{D}}\mathbf{S}^* + C\Theta^* - \widetilde{\Gamma}\widetilde{D}^*. \tag{3.3}$$

Combining ($*$) and (3.3) we obtained

$$\triangle(\mathbf{CD}) = C\Theta^* - \widetilde{\Gamma}\widetilde{D}^* + [\widehat{\mathbf{C}}D + \mathbf{C}_0 D + \mathbf{D}_0 C + \mathbf{C}_0\mathbf{D}_0 P_0^*]P_0 + P_0^*[\Gamma^*\mathbf{S}\widehat{\mathbf{D}}\mathbf{S}^* + \Theta^*\mathbf{C}_0 + \Gamma^*\mathbf{D}_0]. \tag{3.4}$$

Note that $\widehat{\mathbf{C}}D + \mathbf{C}_0 D = \mathbf{C}D$ and $\Gamma^*\mathbf{S}\widehat{\mathbf{D}}\mathbf{S}^* + \Gamma^*\mathbf{D}_0 = \Gamma^*(\mathbf{S}\widehat{\mathbf{D}}\mathbf{S}^* + \mathbf{D}_0) = \Gamma^*\mathbf{SDS}^*$. Therefore (3.4) becomes

$$\triangle(\mathbf{CD}) = C\Theta^* - \widetilde{\Gamma}\widetilde{D}^* + [\mathbf{C}D + \mathbf{D}_0 C + \mathbf{C}_0\mathbf{D}_0 P_0^*]P_0 + P_0^*[\Gamma^*\mathbf{SDS}^* + \Theta^*\mathbf{C}_0].$$

The proof is finished. □

The following result is the most important result of this section. It gives answers to the questions that when the product $\mathbf{AB} - \mathbf{CD}$ of block Toeplitz matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$, and $\mathbf{D}$ is 0 and is in $\mathscr{T}_n \otimes \mathscr{A}$.

THEOREM 3.2. *If* $\mathbf{A} = \mathbf{T}(A,\Omega) + \mathbf{A}_0$, $\mathbf{B} = \mathbf{T}(B,\Lambda) + \mathbf{B}_0$, $\mathbf{C} = \mathbf{T}(C,\Gamma) + \mathbf{C}_0$, *and* $\mathbf{D} = \mathbf{T}(D,\Theta) + \mathbf{D}_0$, *then:*

(i) $\mathbf{AB} - \mathbf{CD}$ *(or, equivalently* $\mathbf{T}(A,\Omega)\mathbf{T}(B,\Lambda) - \mathbf{T}(C,\Gamma)\mathbf{T}(D,\Theta)$*)* $\in \mathscr{T}_n \otimes \mathscr{A}$ *if and only if*

$$A\Lambda^* - \widetilde{\Omega}\widetilde{B}^* = C\Theta^* - \widetilde{\Gamma}\widetilde{D}^*.$$

(ii) If $\mathbf{AB} - \mathbf{CD} \in \mathscr{T}_n \otimes \mathscr{A}$, *then* $\mathbf{AB} - \mathbf{CD} = 0$ *if and only if*

$$\mathbf{A}B + \mathbf{B}_0 A + \mathbf{A}_0\mathbf{B}_0 P_0^* = \mathbf{C}D + \mathbf{D}_0 C + \mathbf{C}_0\mathbf{D}_0 P_0^*, \tag{3.5}$$

and

$$\mathbf{B}^*\Omega + \mathbf{A}_0^*\Lambda + \mathbf{A}_0^*\mathbf{B}_0^* P_0^* = \mathbf{D}^*\Gamma + \mathbf{C}_0^*\Theta + \mathbf{C}_0^*\mathbf{D}_0^* P_0^*. \tag{3.6}$$

Proof. By Lemma 3.1,

$$\begin{aligned}\triangle(\mathbf{AB}) - \triangle(\mathbf{CD}) = {} & A\Lambda^* - \widetilde{\Omega}\widetilde{B}^* - C\Theta^* + \widetilde{\Gamma}\widetilde{D}^* \\ & + [\mathbf{A}B + \mathbf{B}_0 A + \mathbf{A}_0\mathbf{B}_0 P_0^* - \mathbf{C}D - \mathbf{D}_0 C - \mathbf{C}_0\mathbf{D}_0 P_0^*]P_0 \\ & + P_0^*[\Omega^*\mathbf{SBS}^* + \Lambda^*\mathbf{A}_0 - \Gamma^*\mathbf{SDS}^* - \Theta^*\mathbf{C}_0].\end{aligned}$$

The first four terms are block matrices with 0 on the first row and the first column. On the other hand, the fifth term has nonzero entries only on the first column, and the sixth only on the first row. Therefore, by Lemma 2.2, $\mathbf{AB} - \mathbf{CD} \in \mathscr{T}_n \otimes \mathscr{A}$ if and only if $A\Lambda^* - \widetilde{\Omega}\widetilde{B}^* - C\Theta^* + \widetilde{\Gamma}\widetilde{D}^* = 0$, which is the required relation.

(ii) If $\mathbf{AB} - \mathbf{CD} \in \mathscr{T}_n \otimes \mathscr{A}$, then $\mathbf{AB} = \mathbf{CD}$ if and only if $\triangle(\mathbf{AB} - \mathbf{CD}) = 0$, i.e., $\mathbf{AB} = \mathbf{CD}$ if and only if

$$\mathbf{A}B + \mathbf{B}_0 A + \mathbf{A}_0\mathbf{B}_0 P_0^* = \mathbf{C}D + \mathbf{D}_0 C + \mathbf{C}_0\mathbf{D}_0 P_0^*$$

$$\Omega^*\mathbf{SBS}^* + \Lambda^*\mathbf{A}_0 = \Gamma^*\mathbf{SDS}^* + \Theta^*\mathbf{C}_0$$

or

$$\mathbf{SB}^*\mathbf{S}^*\Omega + \mathbf{A}_0^*\Lambda = \mathbf{SD}^*\mathbf{S}^*\Gamma + \mathbf{C}_0^*\Theta. \tag{3.7}$$

Let us show that (3.5) and (3.6) are equivalent to above equations. We already have (3.5). Then, subtracting (3.7) from (3.6) yields that

$$\triangle(\mathbf{B}^*)\Omega + \mathbf{A}_0^*\mathbf{B}_0^*P_0^* = \triangle(\mathbf{D}^*)\Gamma + \mathbf{C}_0^*\mathbf{D}_0^*P_0^*,$$

Then by Lemma 2.2, the last equation above can be written as

$$(\Lambda P_0 + P_0^*B^*)\Omega + \mathbf{A}_0\mathbf{B}_0P_0^* = (\Theta P_0 + P_0^*D^*)\Gamma + \mathbf{C}_0^*\mathbf{D}_0^*P_0^* \tag{3.8}$$

Since $\Lambda P_0\Omega = \Theta P_0\Gamma = 0$, then (3.8) becomes

$$P_0^*B^*\Omega + \mathbf{A}_0\mathbf{B}_0P_0^* = P_0^*D^*\Gamma + \mathbf{C}_0^*\mathbf{D}_0^*P_0^*$$

which after passing to the adjoint is the zeroth component relation of (3.5). □

The results below gather some consequences of Theorem 3.2.

COROLLARY 3.3. *If* $\mathbf{A} = \mathbf{T}(A,\Omega) + \mathbf{A}_0, \mathbf{B} = \mathbf{T}(B,\Lambda) + \mathbf{B}_0$, *then* $\mathbf{AB} \in \mathscr{T}_n \otimes \mathscr{A}$ *if and only if* $A\Lambda^* = \widetilde{\Omega}\widetilde{B}^*$.

COROLLARY 3.4. *If* $\mathbf{A} = \mathbf{T}(A,\Omega) + \mathbf{A}_0, \mathbf{B} = \mathbf{T}(B,\Lambda) + \mathbf{B}_0$, *then* $\mathbf{AB} \in \mathscr{T}_n \otimes \mathscr{A}$ *if and only if* $\mathbf{BA} \in \mathscr{T}_n \otimes \mathscr{A}$.

Proof. By Theorem 3.2, $\mathbf{AB} \in \mathscr{T}_n \otimes \mathscr{A}$ if and only if $A\Lambda^* = \widetilde{\Omega}\widetilde{B}^*$ if and only if $B\Omega^* = (B_i\Omega_j)_{i,j} = ((A_{n-j-1}\Lambda_{n-i-1})^*)_{i,j} = (\Lambda^*_{n-i-1}A^*_{n-j-1})_{i,j} = \widetilde{\Lambda}\widetilde{A}^*$ if and only if $\mathbf{BA} \in \mathscr{T}_n \otimes \mathscr{A}$. □

Note that Corollary 3.4 can also be obtained from the following stronger Theorem already proved in [16].

THEOREM 3.5. *If* $\mathbf{A} = \mathbf{T}(A,\Omega) + \mathbf{A}_0,\ \mathbf{B} = \mathbf{T}(B,\Lambda) + \mathbf{B}_0$, *such that* $\mathbf{AB} \in \mathscr{T}_n \otimes \mathscr{A}$, *then* $\mathbf{AB} = \mathbf{BA}$.

4. Commutants of $\mathbf{S}$, $\mathbf{S}^*$, $\mathbf{S}_X$, and $\mathbf{S}_X^*$

Throughout in this section let $X \in \mathscr{A}$ be a fixed. We start with the following proposition, which gives in terms of $\mathbf{S}_X$ another criterion for characterizing block Toeplitz matrices among all $n \times n$ block matrices.

PROPOSITION 4.1. $\mathbf{A} \in \mathscr{M}_n \otimes \mathscr{A}$ *is in* $\mathscr{T}_n \otimes \mathscr{A}$ *if and only if there exist* $A, B \in \mathscr{C} \otimes \mathscr{A}$, *such that* $\mathbf{A} - \mathbf{S}_X\mathbf{A}\mathbf{S}_X^* = AP_0 + P_0^*B^*$.

Proof. By Lemma 2.2, $\mathbf{A} \in \mathscr{M}_n \otimes \mathscr{A}$ is in $\mathscr{T}_n \otimes \mathscr{A}$ if and only if there exist vectors A', $\Omega' \in \mathscr{C} \otimes \mathscr{A}$ such that

$$\mathbf{A} - \mathbf{SAS}^* = A'P_0 + P_0^*(\Omega')^*.$$

if and only if

$$\mathbf{A}-(\mathbf{S}_X - X\diamond P_0^* P_{n-1})\mathbf{A}(\mathbf{S}_X^* - P_{n-1}^* P_0 \diamond X^*) = A' P_0 + P_0^*(\Omega')^*,$$

if and only if

$$\begin{aligned}\mathbf{A}-\mathbf{S}_X\mathbf{A}\mathbf{S}_X^* = &-\mathbf{S}_X\mathbf{A}P_{n-1}^* P_0\diamond X^* - X\diamond P_0^* P_{n-1}\mathbf{A}\mathbf{S}_X^* + X\diamond P_0^* P_{n-1}\mathbf{A}P_{n-1}^* P_0\diamond X^*\\ &+A' P_0 + P_0^*(\Omega')^*,\end{aligned}$$

if and only if

$$\begin{aligned}\mathbf{A}-\mathbf{S}_X\mathbf{A}\mathbf{S}_X^* &= \left[A' - \mathbf{S}_X\mathbf{A}X^*\diamond P_{n-1}^*\right]P_0 + P_0^*[(\Omega')^* - P_{n-1}\diamond X\mathbf{A}\mathbf{S}_X^* + X\diamond P_{n-1}\mathbf{A}P_{n-1}^* P_\diamond X^*],\\ &= AP_0 + P_0^* B^*,\end{aligned}$$

where $A = A' - \mathbf{S}_X\mathbf{A}X^*\diamond P_{n-1}^*$ and $B = \Omega' - \mathbf{S}_X\mathbf{A}^* X^*\diamond P_{n-1}^* + X\diamond P_0^* P_{n-1}\mathbf{A}^* P_{n-1}^* \diamond X^*$. □

REMARK 4.2. Since $P_{n-1}\mathbf{S}^* = 0$, then

$$\begin{aligned}I - \mathbf{S}_X\mathbf{S}_X^* &= \mathbf{I} - (\mathbf{S} + X\diamond P_0^* P_{n-1})(\mathbf{S}^* + P_{n-1}^* P_0 \diamond X^*)\\ &= \mathbf{I} - \mathbf{S}\mathbf{S}^* - \mathbf{S}P_{n-1}^* P_0\diamond X^* - X\diamond P_0^* P_{n-1}\mathbf{S}^* - X\diamond P_0^* P_{n-1} P_{n-1}^* P_0\diamond X^*\\ &= P_0^* P_0 - \mathbf{S}X^*\diamond P_{n-1}^* P_0 - P_0^* P_{n-1}\diamond X\mathbf{S}^* - X\diamond P_0^* P_{n-1} P_{n-1}^* P_0 \diamond X^*\\ &= P_0^* P_0 - P_0^* P_{n-1}\diamond X\mathbf{S}^* - \mathbf{S}_X P_{n-1}^* P_0\diamond X^*\\ &= P_0^* P_0 - \mathbf{S}_X P_{n-1}^* P_0 \diamond X^*.\end{aligned}$$

REMARK 4.3. $\mathbf{S}, \mathbf{S}_X \in \mathscr{T}_n \otimes \mathscr{A}$, with $\mathbf{S} = \mathbf{T}(P_1, 0)$ and $\mathbf{S}_X = \mathbf{T}(P_1, X^*\diamond \widetilde{P_1})$.

The following result characterized lower (upper) triangular block Toeplitz matrices among all $n\times n$ block matrices with entries from $\mathscr{A}$.

THEOREM 4.4. *If $\mathbf{A}\in\mathscr{M}_n\otimes\mathscr{A}$, then the following hold:*

(i) $\mathbf{A}\mathbf{S} = \mathbf{S}\mathbf{A}$ *if and only if* $\mathbf{A} = \mathbf{T}(A, 0) + \mathbf{A}_0$.

(ii) $\mathbf{A}\mathbf{S}^* = \mathbf{S}^*\mathbf{A}$ *if and only if* $\mathbf{A} = \mathbf{T}(0, \Omega) + \mathbf{A}_0$.

Proof. We will give the proof only for (i) and leave (ii) as an easy exercise for the reader.

(i) Suppose that $\mathbf{A}\mathbf{S} = \mathbf{S}\mathbf{A}$, then $\triangle(\mathbf{A}) = \mathbf{A}\triangle(\mathbf{I}) = \mathbf{A}P_0^* P_0$. By Lemma 2.2, $\mathbf{A}\in\mathscr{T}_n\otimes\mathscr{A}$, with $\mathbf{A} = \mathbf{T}(A,0)+\mathbf{A}_0$. For the converse, let $\mathbf{A} = \mathbf{T}(A,0)+\mathbf{A}_0$, also $\mathbf{S}\in\mathscr{T}_n\otimes\mathscr{A}$, then by Corollary 3.3 and Theorem 3.5, it is immediate that $\mathbf{A}\mathbf{S} = \mathbf{S}\mathbf{A}$. □

THEOREM 4.5. *If $\mathbf{A}\in\mathscr{M}_n\otimes\mathscr{A}$, then*

(i) $\mathbf{A}\mathbf{S}_X = \mathbf{S}_X\mathbf{A}$ *if and only if* $\mathbf{A} = \mathbf{T}(A, X^*\diamond\widetilde{A}) + \mathbf{A}_0$.

(ii) $\mathbf{A}\mathbf{S}_X^* = \mathbf{S}_X^*\mathbf{A}$ *if and only if* $\mathbf{A} = \mathbf{T}(X^*\diamond\widetilde{A}, A) + \mathbf{A}_0^*$.

Proof. (i) Suppose that $\mathbf{A}\mathbf{S}_X = \mathbf{S}_X\mathbf{A}$, then we have

$$\begin{aligned}\triangle(\mathbf{A}) &= \mathbf{A} - \mathbf{S}\mathbf{A}\mathbf{S}^* \\ &= \mathbf{A} - [\mathbf{S}_X - X \diamond P_0^* P_{n-1}]\mathbf{A}[\mathbf{S}_X^* - P_{n-1}^* P_0 \diamond X^*] \\ &= \mathbf{A} - \mathbf{S}_X\mathbf{A}\mathbf{S}_X^* + \left[\mathbf{S}_X\mathbf{A}X^* \diamond P_{n-1}^*\right] P_0 + P_0^* [P_{n-1} \diamond X\mathbf{A}\mathbf{S}_X^* - X \diamond P_{n-1}\mathbf{A}P_{n-1}^* P_0 \diamond X^*] \\ &= \mathbf{A}(\mathbf{I} - \mathbf{S}_X\mathbf{S}_X^*) + \mathbf{A}\left[\mathbf{S}_X X^* \diamond P_{n-1}^*\right] P_0 + P_0^* \left[X \diamond P_{n-1}\mathbf{A}\mathbf{S}^*\right] \\ &= \mathbf{A}P_0^* P_0 + P_0^* \left[X \diamond P_{n-1}\mathbf{A}\mathbf{S}^*\right].\end{aligned}$$

By Lemma 2.2, $\mathbf{A} \in \mathscr{T}_n \otimes \mathscr{A}$, since $X \diamond P_{n-1}\mathbf{A}\mathbf{S}^* = X \diamond \widetilde{A}^*$, then $\mathbf{A} = \mathbf{T}(A, X^* \diamond \widetilde{A}) + \mathbf{A}_0$. For the converse, since $\mathbf{A}, \mathbf{S}_X \in \mathscr{T}_n \otimes \mathscr{A}$, then Corollary 3.3 and Theorem 3.5 imply that, $\mathbf{A}\mathbf{S}_X = \mathbf{S}_X\mathbf{A}$. The proof of (ii) is similar to the proof of (i). $\square$

COROLLARY 4.6. *If $\mathbf{A} = \mathbf{T}(A,\Omega) + \mathbf{A}_0$, $\mathbf{B} = \mathbf{T}(B,\Lambda) + \mathbf{B}_0$, such that $\mathbf{A}$ and $\mathbf{B}$ commutes with $\mathbf{S}_X$ for some $X \in \mathscr{A}$, then $\mathbf{A}\mathbf{B} \in \mathscr{T}_n \otimes \mathscr{A}$.*

Proof. Suppose that $\mathbf{A} = \mathbf{T}(A,\Omega) + \mathbf{A}_0$, $\mathbf{B} = \mathbf{T}(B,\Lambda) + \mathbf{B}_0$, such that for some $X \in \mathscr{A}$, $\mathbf{A}$ and $\mathbf{B}$ commutes with $\mathbf{S}_X$ then it follows from Theorem 4.5, that $\Omega = X^* \diamond \widetilde{A}$ and $\Lambda = X^* \diamond \widetilde{B}$, we have then

$$A\Lambda^* = A(X^* \diamond \widetilde{B})^* = X \diamond A\widetilde{B}^* = \widetilde{\Omega}\widetilde{B}^*.$$

Therefore by Corollary 3.3, $\mathbf{A}\mathbf{B} \in \mathscr{T}_n \otimes \mathscr{A}$. $\square$

5. Characterization of normal block Toeplitz matrices

In this section, we take $\mathscr{A}$ to be a commutative algebra of normal matrices. Then we will characterized normal block Toeplitz matrices with entries from such $\mathscr{A}$. We start with the following Lemma.

LEMMA 5.1. *If $\mathbf{A} = (A_{i,j})_{i,j=1}^n \in \mathscr{M}_n \otimes \mathscr{A}$, then $\mathbf{A}$ is normal if and only if*

$$\sum_{k=1,k\neq p}^{n} [A_{k,p}^* A_{k,p} - A_{p,k}A_{p,k}^*] = 0 \quad \textit{for every } p = 1,2,\cdots,n,$$

and

$$\sum_{k=1}^{n} [A_{k,i}^* A_{k,j} - A_{i,k}A_{j,k}^*] = 0 \quad \textit{for every } 1 \leqslant i < j \leqslant n.$$

The following result is the main result of this paper. It gives us the criteria for classifying normal block Toeplitz matrices with block entries belonging to $\mathscr{A}$.

THEOREM 5.2. *If $\mathbf{A} = \mathbf{T}(A,\Omega) + \mathbf{A}_0 \in \mathscr{T}_n \otimes \mathscr{A}$, then $\mathbf{A}$ is normal if and only if for every s and k, with $1 \leqslant s,k \leqslant n-1$,*

$$A_s A_k^* + A_{n-s}^* A_{n-k} = \Omega_s \Omega_k^* + \Omega_{n-s}^* \Omega_{n-k}. \tag{5.1}$$

Proof. Suppose that $\mathbf{A}$ is normal and let $\mathbf{N} = (N_{i,j})_{i,j=1}^n = \mathbf{A}^*\mathbf{A} - \mathbf{A}\mathbf{A}^*$. Since $\mathbf{A} \in \mathscr{T}_n \otimes \mathscr{A}$, it follows from Lemma 5.1 that $\mathbf{A}$ is normal if and only if

$$N_{p,p} = \sum_{k=1}^{p-1} \left(\Omega_{p-k}\Omega_{p-k}^* - A_{p-k}^* A_{p-k}\right) + \sum_{k=p+1}^{n} \left(A_{k-p}^* A_{k-p} - \Omega_{k-p}\Omega_{k-p}^*\right) = 0$$

and

$$\begin{aligned} N_{i,j} = &\sum_{k=1}^{i-1} \left(\Omega_{i-k}\Omega_{j-k}^* - A_{i-k}A_{j-k}^*\right) + \sum_{k=i+1}^{j-1} \left(A_{k-i}^*\Omega_{j-k}^* - \Omega_{k-i}^* A_{j-k}^*\right) \\ &+ \sum_{k=j+1}^{n} \left(A_{k-i}^* A_{k-j} - \Omega_{k-i}^*\Omega_{k-j}\right) = 0, \end{aligned}$$

if and only if

$$N_{p,p} = \sum_{k=1}^{p-1} (\Omega_k\Omega_k^* - A_k^* A_k) - \sum_{k=1}^{n-p} (\Omega_k\Omega_k^* - A_k^* A_k) = 0 \tag{5.2}$$

and

$$\begin{aligned} N_{i,j} = &\sum_{k=1}^{i-1} \left(\Omega_k\Omega_{j-i+k}^* - A_k A_{j-i+k}^*\right) + \sum_{k=1}^{j-i-1} \left(A_k^*\Omega_{j-i-k}^* - \Omega_k^* A_{j-i-k}^*\right) \\ &+ \sum_{k=1}^{n-j} \left(A_{j-i+k}^* A_k - \Omega_{j-i+k}^*\Omega_k\right) = 0, \end{aligned}$$

for every $p = 1, 2, \cdots, n$ and $1 \leqslant i < j \leqslant n$, respectively. We first calculate the equation (5.2), when $n = 2m$ for some fixed positive integer m. If we calculate the diagonal entries of $\mathbf{N}$, then

$$N_{p,p} = \sum_{k=1}^{p-1} B_k - \sum_{k=1}^{2m-p} B_k = -\left[\sum_{k=1}^{(2m-p+1)-1} B_k - \sum_{k=1}^{2m-(2m-p+1)} B_k\right] = -N_{2m-p+1,2m-p+1}. \tag{5.3}$$

for every p, where $B_k = \Omega_k\Omega_k^* - A_k^* A_k$. Thus, we know that $N_{p,p} = -N_{2m-p+1,2m-p+1}$ for every p. So it suffices to consider the diagonal entries (p,p) of $\mathbf{N}$ for $p = 1, 2, \cdots, m$. A simple computation shows from (5.3) that

$$N_{m,m} = \sum_{k=1}^{m-1} B_k - \sum_{k=1}^{m} B_k = -B_m = \Omega_m\Omega_m^* - A_m^* A_m = 0.$$

By recurrence for every $p = 1, 2, \cdots, m$, we have

$$\begin{aligned} N_{p,p} &= \sum_{k=1}^{p-1} B_k - \sum_{k=1}^{2m-p} B_k = \sum_{k=1}^{(p+1)-1} B_k - \sum_{k=1}^{2m-(p+1)} B_k - (B_p + B_{2m-p}) \\ &= N_{p+1,p+1} - (B_p + B_{2m-p}). \end{aligned}$$

Which implies that $B_p+B_{2m-p}=0$, for every $p=1,2,\cdots,m$. Therefore

$$A_p^*A_p+A_{2m-p}^*A_{2m-p}=\Omega_p\Omega_p^*+\Omega_{2m-p}\Omega_{2m-p}^*. \tag{5.4}$$

Next we consider the case $i<j$. we write (5.3) as

$$N_{i,j}=\sum_{k=1}^{i-1}C_{r,k}+\sum_{k=1}^{r-1}D_{r,k}+\sum_{k=1}^{2m-j}E_{r,k},$$

where for every $1\leqslant r=j-i\leqslant n-1$,

$$\begin{aligned}C_{r,k}&=\Omega_k\Omega_{r+k}^*-A_kA_{r+k}^*.\\ D_{r,k}&=A_k^*\Omega_{r-k}^*-\Omega_k^*A_{r-k}^*.\\ E_{r,k}&=A_{r+k}^*A_k-\Omega_{r+k}^*\Omega_k.\end{aligned}$$

Since entries are commuting and normal, then

$$\begin{aligned}N_{i,i+1}&=\sum_{k=1}^{i-1}C_{1,k}+0+\sum_{k=1}^{2m-i-1}E_{1,k}\\ &=\sum_{k=1}^{i}C_{1,k}+\sum_{k=1}^{2m-i-2}E_{1,k}-C_{1,i}+E_{1,2m-i-1}\\ &=N_{i+1,i+2}-C_{1,i}+E_{1,2m-i-1}.\end{aligned}$$

Then $C_{1,i}-E_{1,2m-i-1}=0$, therefore

$$\Omega_i\Omega_{i+1}^*+\Omega_{2m-i}^*\Omega_{2m-(i+1)}=A_iA_{1+i}^*+A_{2m-i}^*A_{2m-(i+1)}, \tag{5.5}$$

and

$$\begin{aligned}N_{i,i+2}&=\sum_{k=1}^{i-1}C_{2,k}+D_{2,1}+\sum_{k=1}^{2m-i-2}E_{2,k}\\ &=\sum_{k=1}^{i}C_{2,k}+D_{2,1}+\sum_{k=1}^{2m-i-3}E_{2,k}-C_{2,i}+E_{2,2m-i-2}\\ &=N_{i+1,i+3}-C_{2,i}+E_{2,2m-i-2}.\end{aligned}$$

Then $C_{2,i}=E_{2,2m-i-2}=0$, therefore

$$\Omega_i\Omega_{i+2}^*+\Omega_{2m-i}^*\Omega_{2m-(i+2)}=A_iA_{2+i}^*+A_{2m-i}^*A_{2m-(i+2)}. \tag{5.6}$$

Similar computations for $N_{i,i+r}$, $r=3,4,\cdots,2m-i$, yields that

$$\Omega_i\Omega_{i+r}^*+\Omega_{2m-i}^*\Omega_{2m-(i+r)}=A_iA_{i+r}^*+A_{2m-i}^*A_{2m-(i+r)}. \tag{5.7}$$

Hence by equations (5.3)–(5.7), we conclude that if $\mathbf{A}$ is normal then for every $1\leqslant s,k\leqslant n-1$,

$$\Omega_s\Omega_k^*+\Omega_{n-s}^*\Omega_{n-k}=A_sA_k^*+A_{n-s}^*A_{n-k}. \tag{5.8}$$

Conversely suppose that for every $1 \leqslant s,k \leqslant n-1$, (5.1) is true. We show that $N_{p,p}=0$ and $N_{i,j}=0$ for every $p=1,2,\ldots,n$ and $1 \leqslant i < j \leqslant n$ respectively. We have, by the proof of first implication, that $N_{p,p}=N_{p+1,p+1}$, for every $p=1,2,\ldots,m$, and $N_{p,p}=-N_{2m-p+1,2m-p+1}$ for all p, since $N_{m,m}=-B_m=\Omega_m\Omega_m^*-A_m^*A_m=0$, then $N_{p,p}=0$, for every $p=1,2,\ldots,2m$.

On the other hand for the case $i<j$, we have $N_{i,i+1}=N_{i+1,i+2}$, $N_{i,i+2}=N_{i+1,i+3},\ldots,$ $N_{i,i+r}=N_{i+1,i+r+1}$, therefore $\mathbf{N}\in\mathscr{T}_n\otimes\mathscr{A}$. Since entries belong to $\mathscr{A}$, then

$$N_{m,m+1}=\sum_{k=1}^{m-1}C_{1,k}+\sum_{k=1}^{2m-m-1}E_{1,k}=0.$$

Consequently $N_{i,i+1}=0$ for every i, also since

$$N_{m,m+2}=\sum_{k=1}^{m-1}C_{2,k}+D_{2,1}+\sum_{k=1}^{2m-m-2}E_{2,k}=0.$$

Consequently $N_{i,i+2}=0$, for every i. In general $N_{i,i+r}=0$, for all $r=1,2,\ldots,2m-i$. A similar method works for the case $n=2m+1$. Hence, we complete the proof. $\square$

In some cases of $\mathscr{A}$, one may obtain more concrete description of normality. The case when $\mathscr{A}$ is taken to be the algebra of diagonal matrices is discussed in detail in the next section.

COROLLARY 5.3. *If* $\mathbf{A}=\mathbf{T}(A,\Omega)+\mathbf{A}_0\in\mathscr{T}_n\otimes\mathscr{A}$, *such that* $\mathbf{A}$ *commute with* $\mathbf{S}_X$ *for some unitary* $X\in\mathscr{A}$, *then* $\mathbf{A}$ *is normal.*

Proof. If $\mathbf{A}\mathbf{S}_X=\mathbf{S}_X\mathbf{A}$, then it follows from Theorem 4.5, that $\Omega=X^*\diamond\widetilde{A}$. Since $X\in\mathscr{A}$ is unitary, then for every $1\leqslant s,k\leqslant n-1$,

$$\begin{aligned}\Omega_s\Omega_k^*+\Omega_{n-s}^*\Omega_{n-k}&=X^*A_{n-s}^*A_{n-k}X+A_sXX^*A_k^*\\&=A_sA_k^*+A_{n-s}^*A_{n-k}.\end{aligned}$$

Therefore, Theorem 5.2 implies that $\mathbf{A}$ is normal. $\square$

REMARK 5.4. For $X=I$ Corollary 5.3, implies the normality of block circulant matrices defined in section 2.

6. Normality: the case $\mathscr{A}=\mathscr{D}_d$

Theorem 5.2 is valid for a general algebra $\mathscr{A}$. We may expect that for particular cases of $\mathscr{A}$ one can obtain more precise description of normal Block-Toeplitz matrices. As an example, we study in detail the case when $\mathscr{A}=\mathscr{D}_d$. We start with the following result, which is Lemma 5.1 of [13].

LEMMA 6.1. *Suppose $\mathbf{A} = (A_{i-j})_{i,j=1}^{n} \in \mathscr{T}_n \otimes \mathscr{D}_d$. Then there is a change of basis that brings $\mathbf{A}$ into the following form*

$$\mathbf{A}' = diag\left(A_1'\ A_2'\ \cdots A_d'\right),$$

where for every $k = 1,2,\cdots d$, $A_k' \in \mathscr{T}_n$ is given by the formula $A_k' = (a_{r-s,k})_{r,s=1}^{n}$.

THEOREM 6.2. *If $\mathbf{A} \in \mathscr{T}_n \otimes \mathscr{D}_d$, then $\mathbf{A}$ is normal if and only if there exist scalars $\lambda_1, \lambda_2, \cdots \lambda_d$ with $|\lambda_k| = 1$, such that, if we denote, for $k = 1,2,\cdots d$, $\alpha_k = \begin{pmatrix} 0 \\ a_{-1,k} \\ \vdots \\ a_{-n,k} \end{pmatrix}$, and $\beta_k = \begin{pmatrix} 0 \\ a_{1,k} \\ \vdots \\ a_{n,k} \end{pmatrix}$, then, for each $k = 1,2,\cdots d$, either $\alpha_k = \lambda_k \widetilde{\beta_k}$ or $\alpha_k = \lambda_k \overline{\widetilde{\beta}}_k$.*

Proof. If $\mathbf{A} \in \mathscr{T}_n \otimes \mathscr{D}_d$, then by Lemma 6.1, $\mathbf{A}$ has the form

$$\mathbf{A}' = diag\left(A_1'\ A_2'\ \cdots A_d'\right),$$

where for every $k = 1,2,\cdots d$, $A_k' = (a_{r-s,k})_{r,s=1}^{n}$ is a scalar Toeplitz matrix.

Since $\mathbf{A}'$ is a block diagonal matrix, then $\mathbf{A}'\mathbf{A}^{*\prime} = \mathbf{A}^{*\prime}\mathbf{A}'$ if and only if $A_k' A_k'^{*} = A_k'^{*} A_k'$. Applying Theorem 3.4 of [4], which gives criteria for normality of scalar Toeplitz matrices, each A_k' is normal if and only if there exist scalars $\lambda_1, \lambda_2, \cdots, \lambda_d$, with $|\lambda_k| = 1$, such that either $\alpha_k = \lambda_k \widetilde{\beta_k}$ or $\alpha_k = \lambda_k \overline{\widetilde{\beta}}_k$, where $\alpha_k = \begin{pmatrix} 0 \\ a_{-1,k} \\ \vdots \\ a_{-n,k} \end{pmatrix}$, and $\beta_k = \begin{pmatrix} 0 \\ a_{1,k} \\ \vdots \\ a_{n,k} \end{pmatrix}$. This ends the proof of the theorem. $\square$

Acknowledgements. The authors would like to thank the referee who kindly reviewed the earlier version of this manuscript and provided valuable suggestions.

REFERENCES

[1] D. FARENICK, M. KRUPNIK, N. KRUPNIK, W. LEE, *Normal Toeplitz matrices*, SIAM J. Matrix Anal. Appl., **17**, (1996), 1037–1043.

[2] V. GEL'FGAT, *A normality criterion for Toeplitz matrices*, Comput. Math. Math. Phys., **35**, (1995), 1147–1150.

[3] U. GRENANDER, G. SZEGO, *Toeplitz Forms and Their Applications*, University of Calif. Press, Berkeley and Los Angeles, 1958.

[4] C. GU, L. PATTON, *Commutation relation for Toeplitz and Hankel matrices*, SIAM J. Matrix Anal. Appl. **24** (2003), 728–746.

[5] G. HEINIG, K. ROST, *Algebraic methods for Toeplitz-like matrices and operators*, Birkhauser Verlag, Basel, 1984.

[6] D. IKRAMOV, *On a description of normal Toeplitz matrices*, Comput. Math. Math. Phys., **34**, (1994), 399–404.

[7] D. IKRAMOV, *Classification of normal Toeplitz matrices with real elements*, Math. Notes, **57**, (1995), 463–469.

[8] D. IKRAMOV, N. CHUGUNOV, *A criterion for the normality of a complex Toeplitz matrix*, Comput. Math. Math. Phys., **36**, (1996), 131–137.

[9] D. IKRAMOV, N. CHUGUNOV, *On the skew-symmetric part of Toeplitz matrices*, Math. Notes, **63**, (1998), 124–127.

[10] I. S. IOHVIDOV, *Hankel and Toeplitz matrices and forms*, Birkhuser Boston, Cambridge, MA, 1982.

[11] K. ITO, *Every normal Toeplitz matrix is either of type I or of type II*, SIAM J. Matrix Anal. Appl., **17**, (1996), 998–1006.

[12] T. KAILATH, H. SAYED, *Displacement structure: Theory and applications*, SIAM Rev., **37**, (1995), 297–386.

[13] M. A. KHAN, *A family of maximal algebras of block Toeplitz matrices*, An. St. Univ. Ovidius, Constanta Seria Matematica, **26**, 3 (2018), 127–142.

[14] M. A. KHAN, *Some results concerning block Toeplitz matrices*, arXiv:2103.14827.

[15] M. A. KHAN, *Product of matrix valued truncated Toeplitz operators*, Hacettepe Journal of Mathematics and Statistics, **51**, 3, (2022), 700–711.

[16] M. A. KHAN, D. TIMOTIN, *Algebras of block Toeplitz Matrices with commuting entries*, Linear and Multilinear Algebra **69**, 14 (2021), 2702–2716.

[17] T. SHALOM, *On Algebras of Toeplitz matrices*, Linear Algebra and its applications, **96**, (1987), 211–226.

[18] H. WIDOM, *Toeplitz Matrices*, in Studies in Real and Complex Analysis, (J. I. I. Hirschmann, ed.), MAA Studies in Mathematics, Prentice-Hall, Englewood Cliffs, NJ, 1965.

[19] D. L. ZIMMERMAN, *Block Toeplitz products of block Toeplitz matrices*, Linear and Multilinear Algebra, **25**, (1989), 185–190.

(Received March 26, 2022)

Muhammad Ahsan Khan
Department of Mathematics
University of Kotli Azad Jammu & Kashmir
Kotli 11100, Azad Jammu & Kashmir, Pakistan
e-mail: ahsankhan388@hotmail.com

Ameur Yagoub
Laboratoire de mathématiques pures et appliquées
Université de Amar Telidji
Laghouat, 03000, Algeria
e-mail: a.yagoub@lagh-univ.dz