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SOME PROPERTIES OF THE p–SPECTRAL RADIUS ON TENSORS FOR GENERAL HYPERGRAPHS AND THEIR APPLICATIONS

JUNHAO ZHANG AND ZHONGXUN ZHU*

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Abstract. Let $H=(V,E)$ be a general hypergraph with rank m and co-rank m_c and A_H be its adjacency tensor, the p-spectral radius $\rho^{(p)}(H)$ of H is defined as $\rho^{(p)}(H)=\max_{x\in S_p^{n-1}} x^T A_H x$, where $S_p^{n-1}=\{x\in R^n | \parallel x \parallel_p=1\}$. For $m=m_c$ and $p\geqslant m$, we know that there is a unique positive eigenvector $x\in S_p^{n-1}$ belonging to $\rho^{(p)}(H)$ and $\rho^{(p)}(H)$ can be computed by α-normal labeling method. In this paper, we generalize these properties to the case for $m\neq m_c$ and some other properties are obtained. At the same time, some applications are also given on the properties attained in this paper.

1. Introduction

Let $H=(V(H),E(H))$ be a general hypergraph with vertex set $V(H)$ and edge set $E(H)\subseteq 2^{V(H)}$, where $2^{V(H)}$ is the power set of $V(H)=[n]=\{1,2,\ldots,n\}$. Denote $r(H)=\max_{e\in E(H)}|e|=m$ (respectively $cr(H)=\min_{e\in E(H)}|e|=m_c$) be the rank (respectively the co-rank) of H. If $r(H)=cr(H)=m$, H is called an m-uniform hypergraph.

For an edge $e=\{l_1,l_2,\ldots,l_s\}\in E(H)$, where $m_c\leqslant s\leqslant m$, an ordered sequence $\eta=(i_1,i_2,\ldots,i_m)$ is called an m-expanded edge from e if its set of the distinct entries is the same as e, denote this by $e\prec\eta$. Let

$$\begin{aligned}
E(H)_{l_1} &= \{e|l_1\in e\in E(H)\};\\
S(e) &= \{\eta|e\prec\eta\};\ \ S(H)=\cup_{e\in E(H)}S(e);\\
S(e)_{l_1} &= \{\eta\in S(e)|\eta=(l_1,i_2,\ldots,i_m)\},\ \ S(H)_{l_1}=\cup_{e\in E(H)_{l_1}}S(e)_{l_1}.
\end{aligned}$$

Then we have

$$|S(e)|=|e||S(e)_{l_1}|=\sum_{k_1+k_2+\cdots+k_s=m}^{k_1,k_2,\ldots,k_s\geqslant 1}\frac{m!}{k_1!k_2!\cdots k_s!},$$

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* Corresponding author.

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and the adjacency tensor $A_H = (a_{i_1 i_2 \dots i_m})$ of H can be defined as in [1]:

$$a_{i_1 i_2 \dots i_m} = \begin{cases} \frac{|e|}{|S(e)|} = a_e, & e \prec (i_1, i_2, \dots, i_m) \text{ and } e \in E(H), \\ 0 & \text{otherwise.} \end{cases}$$

That is, $\forall \eta \in S(e)$, $a_e = a_\eta$. Further,

$$d(i_1) = |E(H)_{i_1}| = \sum_{i_2,\dots,i_m=1}^{n} a_{i_1 i_2 \dots i_m},$$

where $d(i_1)$ is degree of i_1 in H. For $x = (x_1, x_2, \dots, x_n)^T \in R^n$ and an m-expanded edge $\eta = (i_1, i_2, \dots, i_m)$, denote $x^\eta = x_{i_1} x_{i_2} \cdots x_{i_m}$. According to the definition of products of tensors given by Shao [9],

$$\begin{aligned} (A_H x)_{i_1} &= \sum_{i_2,\dots,i_m=1}^{n} a_{i_1 i_2 \dots i_m} x_{i_2} \cdots x_{i_m} = \sum_{e \in E(H)_{i_1}} a_e \sum_{\eta=(i_1,i_2,\dots,i_m) \in S(e)_{i_1}} x_{i_2} \cdots x_{i_m}; \\ P_H(x) &= x^T A_H x = \sum_{i_1,i_2,\dots,i_m=1}^{n} a_{i_1 i_2 \dots i_m} x_{i_1} x_{i_2} \cdots x_{i_m} = \sum_{e \in E(H)} a_e \sum_{\eta \in S(e)} x^\eta \\ &= \sum_{e \in E(H)} a_e \sum_{k_1+k_2+\dots+k_s=m}^{k_1,k_2,\dots,k_s \geqslant 1} \frac{m!}{k_1! k_2! \cdots k_s!} x_{l_1}^{k_1} x_{l_2}^{k_2} \cdots x_{l_s}^{k_s} \\ &= \sum_{i_1=1}^{n} \sum_{e \in E(H)_{i_1}} a_e \sum_{\eta=(i_1,i_2,\dots,i_m) \in S(e)_{i_1}} x_{i_1} x_{i_2} \cdots x_{i_m}. \end{aligned}$$

For any real number $p \geqslant 1$ and $x = (x_1, x_2, \dots, x_n)^T \in R^n$, let $\| x \|_p = (|x_1|^p + |x_2|^p + \dots + |x_n|^p)^{\frac{1}{p}}$, $S_p^{n-1} = \{x \in R^n | \| x \|_p = 1\}$, $S_{p,+}^{n-1} = \{x \in R_+^n | \| x \|_p = 1\}$ and $S_{p,++}^{n-1} = \{x \in R_{++}^n | \| x \|_p = 1\}$, where $R_+^n = \{x \in R^n | x_i \geqslant 0, i \in [n]\}$ and $R_{++}^n = \{x \in R^n | x_i > 0, i \in [n]\}$, respectively. The p-spectral radius $\rho^{(p)}(H)$ of H is defined as [4]

$$\rho^{(p)}(H) = \max_{x \in S_p^{n-1}} P_H(x) \tag{1.1}$$

if $x \in S_p^{n-1}$ is a vector satisfing $P_H(x) = \rho^{(p)}(H)$, x is called an eigenvector belonging to $\rho^{(p)}(H)$. Obviously, if H is an m-uniform hypergraph, we have

$$P_H(x) = x^T A_H x = m \sum_{e=\{u_1,u_2,\dots,u_m\} \in E(H)} x_{u_1} x_{u_2} \cdots x_{u_m},$$

and $\rho^{(p)}(H)$ is the same as $\lambda^{(p)}(H)$ introduced in [8] by V. Nikiforov.

By Lagrange's method, let

$$L(x, \lambda) = P_H(x) - \lambda(|x_1|^p + |x_2|^p + \dots + |x_n|^p - 1).$$

Let x be the optimal solution of (1.1), then

$$\begin{cases} \frac{\partial(L(x,\lambda))}{\partial x_i} = \frac{\partial(P_H(x))}{\partial x_i} - \lambda p x_i |x_i|^{p-2} = 0, & i \in [n]; \\ \frac{\partial(L(x,\lambda))}{\partial \lambda} = |x_1|^p + |x_2|^p + \cdots + |x_n|^p - 1 = 0. \end{cases}$$

Further, we have

$$\lambda p = \lambda p \sum_{i=1}^{n} |x_i|^p = \sum_{i=1}^{n} x_i \lambda p x_i |x_i|^{p-2} = \sum_{i=1}^{n} x_i \frac{\partial (P_H(x))}{\partial x_i} = m P_H(x) = m\rho^{(p)}(H),$$

$$\frac{\partial (P_H(x))}{\partial x_i} = m\rho^{(p)}(H) x_i |x_i|^{p-2}.$$

If $x \in R_+^n$ and $i \in [n]$, we have

$$\frac{1}{m} \frac{\partial (P_H(x))}{\partial x_i} = \rho^{(p)}(H) x_i^{p-1} = (A_H x)_i. \tag{1.2}$$

The following is a list of inequalities which will be used to prove our main results.

LEMMA 1.1. (Generalized Cauchy-Schwarz Inequality [3, 8]) *Let* $x^{(j)} = (x_1^{(j)}, x_2^{(j)}, \ldots, x_d^{(j)})^T \in R_+^d$ *for* $j \in [k]$*, then* $\sum_{i=1}^{d} \prod_{j=1}^{k} x_i^{(j)} \leqslant \| x^{(1)} \|_k \cdots \| x^{(k)} \|_k$*, equality holds if and only if all vectors are collinear to one of them.*

LEMMA 1.2. (Power Mean Inequality [3, 8]) *Let* x_j *be a nonnegative real number for* $j \in [k]$*, if* $0 < p < q$*, then* $(\frac{x_1^p + x_2^p + \cdots + x_k^p}{k})^{\frac{1}{p}} \leqslant (\frac{x_1^q + x_2^q + \cdots + x_k^q}{k})^{\frac{1}{q}}$*, equality holds if and only if* $x_1 = x_2 = \cdots = x_k$.

LEMMA 1.3. (Hölder Inequality [3, 8]) *Let* $\frac{1}{p} + \frac{1}{q} = 1$ *for* $p, q > 1$*, and* $x = (x_1, x_2, \ldots, x_n)^T, y = (y_1, y_2, \ldots, y_n)^T \in R_+^n$*, then* $\sum_{k=1}^{n} x_k y_k \leqslant (\sum_{k=1}^{n} x_k^p)^{\frac{1}{p}} (\sum_{k=1}^{n} y_k^q)^{\frac{1}{q}}$*, equality holds if and only if one of* x *and* y *is* 0*, or there exist* c_1 *and* c_2 *such that* $c_1 x_k^p = c_2 y_k^q$ *for* $k \in [n]$.

In the following sections, we will focus to study on properties of p-spectral radius and its eigenvector for general hypergraphs. We find that the uniform property of hypergraphs is not essential for the applications of tensors related to hypergraphs. For general hypergraphs, some classical results in hypergraph theory can also hold.

2. Some properties from the p-spectral radius and its eigenvector of general hypergraphs

It is easy to see that if $x = (x_1, x_2, \ldots, x_n)^T \in S_p^{n-1}$, then $|x| \in S_{p,+}^{n-1}$, where $|x| = (|x_1|, |x_2|, \ldots, |x_n|)^T$. Further if $x \in S_p^{n-1}$ satisfing $P_H(x) = \rho^{(p)}(H)$, we also have $P_H(|x|) = \rho^{(p)}(H)$, that is, there is always a nonnegative vector x satisfing $\| x \|_p = 1$ and $\rho^{(p)}(H) = P_H(x)$.

LEMMA 2.1. *Let $p \geqslant 1$, if H is a general hypergraph with rank m and $x \in R^n$, then*

$$P_H(x) \leqslant \rho^{(p)}(H) \parallel x \parallel_p^m .$$

Proof. By (1.1), we have

$$\rho^{(p)}(H) = \max_{x \in R^n} P_H\left(\frac{x}{\parallel x \parallel_p}\right) \geqslant P_H\left(\frac{x}{\parallel x \parallel_p}\right) = \frac{1}{\parallel x \parallel_p^m} P_H(x). \quad \square$$

In the study of the largest eigenvalue of m-uniform connected hypergraphs, a Perron-Frobenius-type theory of nonnegative tensors is very useful tool. For the adjacency tensor of an m-uniform connected hypergraph H, the authors in [2] obtained that there is unique positive eigenvector $x \in S_{m,+}^{n-1}$ belonging to $\rho^{(m)}(H)$ and Nikiforov [8] showed that there is unique positive eigenvector $x \in S_{p,+}^{n-1}$ belonging to $\rho^{(p)}(H)$ for $p \geqslant m$. It is natural that we can ask the following problem:

PROBLEM 2.2. *Can we generalize the Perron-Frobenius-type theory on uniform hypergraphs to general hypergraphs?*

Using the idea of Lemma 3.3 in [2] and Theorem 5.2 in [8] with more detailed techniques, we can also obtain the following result.

THEOREM 2.3. *Let H be a general connected hypergraph with rank m and $p \geqslant m$. If $x = (x_1, x_2, \ldots, x_n)^T \in S_{p,+}^{n-1}$ satisfies the equations*

$$\frac{1}{m} \frac{\partial (P_H(x))}{\partial x_i} = \rho^{(p)}(H) x_i^{p-1}, \ \ i = 1, \ldots, n. \tag{2.1}$$

Then x is the unique positive vector satisfying equations (2.1).

Proof. By (2.1), we have

$$\rho^{(p)}(H) = \rho^{(p)}(H) \sum_{i=1}^n x_i^p = \frac{1}{m} \sum_{i=1}^n x_i \frac{\partial (P_H(x))}{\partial x_i} = P_H(x),$$

then x is an eigenvector belonging to $\rho^{(p)}(H)$.

Claim 1. x is a positive vector.

Otherwise, assume that H_0 is the hypergraph induced by the verices with zero entries in x, it is easy to see that $H_0 \neq \emptyset$. There must exist an edge e satisfying $S = e \cap V(H_0) \neq \emptyset$ and $T = e \backslash V(H_0) \neq \emptyset$ since H is connected. Let $t \in T$ and ε be a sufficiently small positive real number such that

$$\delta(\varepsilon) = x_t - \sqrt[p]{x_t^p - |S| \varepsilon^p}.$$

Now we can define a new vector $y \in S^{n-1}_{p,+}$ as follows.

$$y_i = \begin{cases} \varepsilon, & \text{if } i \in S, \\ x_i - \delta(\varepsilon), & \text{if } i = t, \\ x_i, & \text{if } i \notin S \cup \{t\}. \end{cases}$$

Let $|S|^{\frac{1}{p}} f(\varepsilon) = [\frac{|S|}{1-(1-\varepsilon^p)^{\frac{p}{k}}}]^{\frac{1}{p}}\varepsilon$, where k is a positive integer. Note that

$$\frac{df^p(\varepsilon)}{d\varepsilon} = \frac{p\varepsilon^{p-1}[1-(1-\varepsilon^p)^{\frac{p}{k}-1}(1-\varepsilon^p+\frac{p}{k}\varepsilon^{p-1})]}{[1-(1-\varepsilon^p)^{\frac{p}{k}}]^2} > 0,$$

that is, $f(\varepsilon)$ is an increasing function on ε. Further from the sufficiently small property of ε, we can set

$$\min_{v\in T}\{x_v\} > |S|^{\frac{1}{p}} f(\varepsilon), \;\; \delta(\varepsilon) < \frac{1}{2}\min_{v\in T}\{x_v\}. \tag{2.2}$$

From $x_t \geqslant \min_{v\in T}\{x_v\}$, we can obtain

$$1-\left(1-\frac{\delta(\varepsilon)}{x_t}\right)^k < \varepsilon^p. \tag{2.3}$$

Let

$$D = \sum_{\{t,u_2,\dots,u_h\}=f\in E(H), f\neq e, f\cap S=\emptyset} a_f \sum_{k+k_2+\cdots+k_h=m}^{k,k_2,\dots,k_h\geqslant 1} \frac{m!}{k!k_2!\cdots k_h!} x_t^k x_{u_2}^{k_2}\cdots x_{u_h}^{k_h}$$

By (2.2), we have $x_t - \delta(\varepsilon) > \frac{x_t}{2}$ and $x_v > \varepsilon$ for $v \in T$. Further by a direct calculation, we have

$$\begin{aligned}
& P_H(y) - P_H(x) \\
= & \sum_{f\in E(H), f\subset V(H)\setminus(S\cup\{t\})} a_f \sum_{\eta\in S(f)} (y^\eta - x^\eta) + \sum_{f\in E(H), f\neq e, f\cap S\neq\emptyset} a_f \sum_{\eta\in S(f)} (y^\eta - x^\eta) + \\
& \sum_{f\in E(H), t\in f, f\neq e, f\cap S=\emptyset} a_f \sum_{\eta\in S(f)} (y^\eta - x^\eta) + a_e \sum_{\eta\in S(e)} (y^\eta - x^\eta) \\
= & \sum_{f\in E(H), f\neq e, f\cap S\neq\emptyset} a_f \sum_{\eta\in S(f)} y^\eta + \sum_{f\in E(H), t\in f, f\neq e, f\cap S=\emptyset} a_f \sum_{\eta\in S(f)} (y^\eta - x^\eta) + a_e \sum_{\eta\in S(e)} y^\eta \\
\geqslant & \sum_{\{t,u_2,\dots,u_h\}=f\in E(H), f\neq e, f\cap S=\emptyset} a_f \sum_{\eta\in S(f)} (y^\eta - x^\eta) + a_e \sum_{\eta\in S(e)} y^\eta \\
= & \sum_{\{t,u_2,\dots,u_h\}=f\in E(H), f\neq e, f\cap S=\emptyset} a_f \sum_{k+k_2+\cdots+k_h=m}^{k,k_2,\dots,k_h\geqslant 1} \frac{m!}{k!k_2!\cdots k_h!} [(x_t-\delta(\varepsilon))^k - x_t^k] x_{u_2}^{k_2}\cdots x_{u_h}^{k_h} \\
& + a_e \sum_{\eta\in S(e)} y^\eta
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\{t,u_2,\ldots,u_h\}=f\in E(H),f\neq e,f\cap S=\emptyset} a_f \sum_{k+k_2+\cdots+k_h=m}^{k,k_2,\ldots,k_h\geqslant 1} \frac{m!}{k!k_2!\cdots k_h!}\left[\left(1-\frac{\delta(\varepsilon)}{x_t}\right)^k-1\right] \\
&\quad \times x_t^k x_{u_2}^{k_2}\cdots x_{u_h}^{k_h} + a_e \sum_{\eta\in S(e)} y^\eta \\
&\geqslant a_e \sum_{\eta\in S(e)} y^\eta - D\varepsilon^p \ \ \text{(by (2.3))} \\
&\geqslant \frac{|S|x_t}{2}\varepsilon^{m-1} - D\varepsilon^p \geqslant \left(\frac{|S|x_t}{2} - D\varepsilon^{p-m+1}\right)\varepsilon^{m-1}.
\end{aligned}
$$

From $p-m+1>0$ and by taking ε sufficiently small, we have $P_H(y)-P_H(x)>0$, a contradiction.

Claim 2. x is a unique positive eigenvector.

Let $w=(w_1,w_2,\ldots,w_n)^T\in S_{p,++}^{n-1}$ be another positive eigenvector belonging to $\rho^{(p)}(H)$. Then

$$
\begin{aligned}
\rho^{(p)}(H)x_{i_1}^p &= \sum_{e\in E(H)_{i_1}} a_e \sum_{\eta=(i_1,i_2,\ldots,i_m)\in S(e)_{i_1}} x_{i_1}x_{i_2}\cdots x_{i_m}, \\
\rho^{(p)}(H)w_{i_1}^p &= \sum_{e\in E(H)_{i_1}} a_e \sum_{\eta=(i_1,i_2,\ldots,i_m)\in S(e)_{i_1}} w_{i_1}w_{i_2}\cdots w_{i_m},
\end{aligned}
$$

for $i_1\in[n]$. Denote $q=(q_1,q_2,\ldots,q_n)^T\in S_{p,++}^{n-1}$, where $q_i=(\frac{x_i^p+w_i^p}{2})^{\frac{1}{p}}$ for $i\in[n]$. Now we have

$$
\begin{aligned}
\rho^{(p)}(H)q_{i_1}^p &= \rho^{(p)}(H)\frac{x_{i_1}^p+w_{i_1}^p}{2} \\
&= \sum_{e\in E(H)_{i_1}} a_e \sum_{\eta=(i_1,i_2,\ldots,i_m)\in S(e)_{i_1}} \frac{x_{i_1}x_{i_2}\cdots x_{i_m}+w_{i_1}w_{i_2}\cdots w_{i_m}}{2}.
\end{aligned}
$$

For vectors $(x_{i_1},w_{i_1})^T,\cdots,(x_{i_m},w_{i_m})^T$, by Lemmas 1.1 and 1.2, we have

$$
\begin{aligned}
\frac{x_{i_1}x_{i_2}\cdots x_{i_m}+w_{i_1}w_{i_2}\cdots w_{i_m}}{2} &\leqslant \prod_{j=1}^m\left(\frac{x_{i_j}^m+w_{i_j}^m}{2}\right)^{\frac{1}{m}} \\
&\leqslant \prod_{j=1}^m\left(\frac{x_{i_j}^p+w_{i_j}^p}{2}\right)^{\frac{1}{p}} = \prod_{j=1}^m q_{i_j}. \qquad (2.4)
\end{aligned}
$$

Then

$$
\rho^{(p)}(H)q_{i_1}^p \leqslant \sum_{e\in E(H)_{i_1}} a_e \sum_{\eta=(i_1,i_2,\ldots,i_m)\in S(e)_{i_1}} q_{i_1}q_{i_2}\cdots q_{i_m}.
$$

Further we have

$$\begin{aligned}\rho^{(p)}(H) &= \sum_{l_1=1}^{n} \rho^{(p)}(H) q_{l_1}^{p} \\ &\leqslant \sum_{i_1=1}^{n} \sum_{e\in E(H)_{i_1}} a_e \sum_{\eta=(i_1,i_2,\ldots,i_m)\in S(e)_{i_1}} q_{i_1}q_{i_2}\cdots q_{i_m} \\ &= P_H(q) \leqslant \rho^{(p)}(H).\end{aligned}$$

Hence the equalities hold in (2.4). By the condition for equality in Lemma 1.1 and the connectivity of H, we have $\frac{x_i}{w_i} = c$ for $i \in [n]$. Further from $x, w \in S^{n-1}_{p,++}$, we have $c = 1$, that is, $x = w$. $\square$

3. The α-normal labeling method for general hypergraphs

To compute the spectral radius $\rho^{(m)}(H)$ of m-uniform hypergraph $H = (V,E)$, Lu and Man [7] introduced a very useful weighted incidence matrix $M_{|V|\times|E|} = (M(v,e))$, where $M(v,e) > 0$ if $v \in e$, $M(v,e) = 0$ otherwise, and proposed a highly skilled method, named α-normal labeling method. In [5], Liu and Lu extended this method to the p-spectral radii of uniform hypergraphs for $p \neq m$. For general hypergraphs, W. Zhang et al in [10] gave the definition of weighted incidence matrix and corresponding α-normal labeling method for $p = m$ as follows.

DEFINITION 1. [10] The matrix $M = (M(v,e))_{|V(H)|\times|S(H)|}$ is called a weighted incidence matrix of a general hypergraph H, if the element $M(v,e) > 0$ if $v \in e$ and $e \in S(H)$, and $M(v,e) = 0$ otherwise.

DEFINITION 2. [10] A general hypergraph H with $rank(H) = m$ is called α-normal if there exists a weighted incidence matrix M satisfying

(1). $\sum_{e'\in S(H)_v} a_{e'} M(v,e') = 1$, for any $v \in V(H)$.
(2). $\prod_{i=1}^{m} M(v_{p_i}, e') = \alpha$, for any $e' = \{v_{p_1}, v_{p_2}, \ldots, v_{p_m}\} \in S(H)$.
(3). $M(v,e'_1) = M(v,e'_2)$ when e'_1 equals to e'_2 except the order.

Moreover, the incidence matrix M is called consistent if for any cycle $v_0e_1v_1e_2\cdots v_l$ $(v_l = v_0)$ and any e'_i extending from e_i,

$$\prod_{i=1}^{l} \frac{M(v_i,e'_i)}{M(v_{i-1},e'_i)} = 1. \tag{3.1}$$

In this case, we call H consistently α-normal.

LEMMA 3.1. [10] *Let H be a general connected hypergraph with $rank(H) = m$. Then the spectral radius is ρ if and only if H is consistently α-normal with $\alpha = \rho^{-m}$.*

Naturally, we have the following problem.

PROBLEM 3.2. *Can we generalize the α-normal labeling method to general hypergraphs for $p \neq m$?*

Now we consider this problem.

DEFINITION 3. For a general hypergraph H with rank m, if there exists a weighted incidence matrix M and weights $\{\omega(e)\}$ satisfying the following conditions.

(i) $\sum_{e\in S(H)} \frac{a_e\omega(e)}{m} = 1$;
(ii) $\sum_{e\in S(H)_v} a_e M(v,e) = 1$ for any $v \in V(H)$;
(iii) $\omega(e)^{p-m}\Pi_{v\in e}M(v,e) = \alpha$ for any $e \in S(H)$;
(iv) $M(v,e_1) = M(v,e_2)$ and $\omega(e_1) = \omega(e_2)$ for e_1 equals to e_2 up to rewriting,

then H is called α-weighted normal. Furthermore, if $\frac{\omega(\tilde{e}_i)}{M(v,\tilde{e}_i)} = \frac{\omega(\tilde{e}_j)}{M(v,\tilde{e}_j)}$ for any $v \in V(H), e_i, e_j \in E(H)_v$ and $\tilde{e}_i \in S(e_i), \tilde{e}_j \in S(e_j)$, then M and $\{\omega(e)\}$ are called consistent.

According to the process of the proof of Lemma 4.4 in [10], the following result holds.

LEMMA 3.3. *Let M be a weighted incidence matrix of a general hypergraph H with rank m, which satisfies conditions (ii) and (iv) in Definition 3, for any $x = (x_1,x_2,\dots,x_n)^T \in S^{n-1}_{p,+}$, it has*

$$\sum_{e\in S(H)}\sum_{v\in e} a_e M(v,e)x_v^p = m\sum_{v\in V(H)} x_v^p.$$

Proof. It is easy to see that

$$\sum_{e\in S(H)}\sum_{v\in e} a_e M(v,e)x_v^p = m\sum_{v\in V(H)}\sum_{e\in S(H)_v} a_e M(v,e)x_v^p = m\sum_{v\in V(H)}\Big[\sum_{e\in S(H)_v} a_e M(v,e)\Big]x_v^p$$
$$= m\sum_{v\in V(H)} x_v^p. \quad \square$$

Note that the proof given here is only for the sake of completeness, in fact, it is included in the process of proof for Lemma 4.4 in [10] (On page 113, line -5).

THEOREM 3.4. *Let H be a general hypergraph with rank m and $p \geqslant m$. Then the p-spectral radius of H is $\rho^{(p)}(H)$ if and only if H is consistently α-weighted normal with $\alpha = \frac{m^{p-m}}{(\rho^{(p)}(H))^p}$.*

Proof. Assume that H is consistently α-normal with weighted incidence matrix M and weights $\{\omega(e)\}$. For any $x = (x_1,x_2,\dots,x_n)^T \in S^{n-1}_{p,+}$,

$$P_H(x) = \sum_{f\in E(H)} a_f \sum_{\eta\in S(f)} x^\eta = \sum_{e\in S(H)} a_e \Pi_{v\in e} x_v$$
$$= \frac{m^{\frac{p-m}{p}}}{\alpha^{\frac{1}{p}}}\sum_{e\in S(H)}\left(\frac{a_e\omega(e)}{m}\right)^{\frac{p-m}{p}} a_e^{\frac{m}{p}}\Pi_{v\in e}\left(M(v,e)\right)^{\frac{1}{p}} x_v.$$

By the Hölder Inequality, we have

$$P_H(x) \leqslant \frac{m^{\frac{p-m}{p}}}{\alpha^{\frac{1}{p}}} \left(\sum_{e \in S(H)} \frac{a_e \omega(e)}{m} \right)^{\frac{p-m}{p}} \Big(\sum_{e \in S(H)} a_e \Pi_{v \in e} (M(v,e))^{\frac{1}{m}} x_v^{\frac{p}{m}} \Big)^{\frac{m}{p}} = L_1 \tag{3.2}$$

By the condition (i) in Definition 3, we have

$$L_1 = \frac{m^{\frac{p-m}{p}}}{\alpha^{\frac{1}{p}}} \left(\sum_{e \in S(H)} a_e \Pi_{v \in e} (M(v,e))^{\frac{1}{m}} x_v^{\frac{p}{m}} \right)^{\frac{m}{p}}$$

Further by the AM-GM inequality,

$$L_1 \leqslant \frac{m^{\frac{p-m}{p}}}{\alpha^{\frac{1}{p}}} \left(\sum_{e \in S(H)} a_e \sum_{v \in e} \frac{M(v,e) x_v^p}{m} \right)^{\frac{m}{p}} = L_2. \tag{3.3}$$

From Lemma 3.3, we have

$$L_2 = \frac{m^{\frac{p-m}{p}}}{\alpha^{\frac{1}{p}}} \left(\sum_{v \in V(H)} x_v^p \right)^{\frac{m}{p}} = \frac{m^{\frac{p-m}{p}}}{\alpha^{\frac{1}{p}}} \parallel x \parallel_p^m = \frac{m^{\frac{p-m}{p}}}{\alpha^{\frac{1}{p}}}.$$

By the Hölder Inequality, the equality holds in (3.2) if for any $e \in S(H)$, there is a constant c such that

$$\frac{\frac{a_e \omega(e)}{m}}{a_e \Pi_{v \in e} (M(v,e))^{\frac{1}{m}} x_v^{\frac{p}{m}}} = \frac{\omega(e)}{m \Pi_{v \in e} (M(v,e) x_v^p)^{\frac{1}{m}}} = c \tag{3.4}$$

Equality holds in (3.3) if for any $e = \{i_1, i_2, \ldots, i_m\} \in S(H)$,

$$M(i_1,e) x_{i_1}^p = M(i_2,e) x_{i_2}^p = \cdots = M(i_m,e) x_{i_m}^p. \tag{3.5}$$

Select $x_v^* = (\frac{\omega(e)}{mM(v,e)})^{\frac{1}{p}}$, for any $v \in e$, from the consistent condition, we know that x_v^* is independent of the choice of $e \in S(H)_v$. We can see that (3.4) and (3.5) hold. By Lemma 3.3, we have

$$m \sum_{v \in V(H)} (x_v^*)^p = \sum_{e \in S(H)} \sum_{v \in e} a_e M(v,e) (x_v^*)^p = \sum_{e \in S(H)} \sum_{v \in e} a_e \frac{\omega(e)}{m} = \sum_{e \in S(H)} a_e \omega(e).$$

Further by the condition (i) in Definition 3, we have

$$\sum_{v \in V(H)} (x_v^*)^p = \sum_{e \in S(H)} a_e \frac{\omega(e)}{m} = 1.$$

Then we have $\rho^{(p)}(H) = \frac{m^{\frac{p-m}{p}}}{\alpha^{\frac{1}{p}}}$.

By Theorem 2.3, we can set $x=(x_1,x_2,\ldots,x_n)^T\in S^{n-1}_{p,++}$ be an eigenvector to $\rho^{(p)}(H)$. Define a weighted incidence matrix M and weights $\{\omega(e)\}$ as following:

$$M(v,e)=\begin{cases}\frac{\Pi_{u\in e}x_u}{\rho^{(p)}(H)x_v^p}, & \text{if } v\in e;\\ 0, & \text{otherwise.}\end{cases}$$

$$\omega(e)=\frac{m\Pi_{u\in e}x_u}{\rho^{(p)}(H)}.$$

Obviously, (iv) and the condition of consistent (that is, $\frac{\omega(e)}{M(v,e)}=mx_v^p$ for $e\in E(H)_v$) in Definition 3 hold. Note that by (1.2), we have

$$\rho^{(p)}(H)=\sum_{e\in S(H)}a_e\Pi_{u\in e}x_u,$$

$$\rho^{(p)}(H)x_v^p=\sum_{e\in S(H)_v}a_e\Pi_{u\in e}x_u.$$

Then by direct calculation, we have the following results.

$$\sum_{e\in S(H)}\frac{a_e\omega(e)}{m}=\sum_{e\in S(H)}\left(\frac{a_e}{m}\cdot\frac{m\Pi_{u\in e}x_u}{\rho^{(p)}(H)}\right)=\frac{\sum_{e\in S(H)}a_e\Pi_{u\in e}x_u}{\rho^{(p)}(H)}=1;$$

$$\sum_{e\in S(H)_v}a_eM(v,e)=\sum_{e\in S(H)_v}a_e\frac{\Pi_{u\in e}x_u}{\rho^{(p)}(H)x_v^p}=\sum_{e\in S(H)_v}a_e\frac{\Pi_{u\in e}x_u}{\rho^{(p)}(H)x_v^p}=1;$$

$$\omega(e)^{p-m}\Pi_{v\in e}M(v,e)=\left(\frac{m\Pi_{u\in e}x_u}{\rho^{(p)}(H)}\right)^{p-m}\cdot\Pi_{v\in e}\frac{\Pi_{u\in e}x_u}{\rho^{(p)}(H)x_v^p}=\frac{m^{p-m}}{(\rho^{(p)}(H))^p}=\alpha.$$

That is, all of the conditions in Definition 3 hold. $\square$

DEFINITION 4. For a general hypergraph H with rank m, if there exists a weighted incidence matrix M and weights $\{\omega(e)\}$ satisfying the following conditions.

(i) $\sum_{e\in S(H)}\frac{a_e\omega(e)}{m}\leqslant 1$;
(ii) $\sum_{e\in S(H)_v}a_eM(v,e)\leqslant 1$ for any $v\in V(H)$;
(iii) $\omega(e)^{p-m}\Pi_{v\in e}M(v,e)\geqslant\alpha$ for any $e\in S(H)$;
(iv) $M(v,e_1)=M(v,e_2)$ and $\omega(e_1)=\omega(e_2)$ for e_1 equals to e_2 up to rewriting,

then H is called α-weighted subnormal. H is called strictly α-weighted subnormal if it is α-weighted subnormal but not α-weighted normal.

THEOREM 3.5. *Let H be a general hypergraph with rank m. If H is α-weighted subnormal, then the p-spectral radius $\rho^{(p)}(H)$ of H satisfies $\rho^{(p)}(H)\leqslant\frac{m^{1-\frac{m}{p}}}{\alpha^{\frac{1}{p}}}$.*

Proof. For any $x=(x_1,x_2,\ldots,x_n)^T\in S^{n-1}_{p,+}$,

$$P_H(x)=\sum_{e\in S(H)}a_e\Pi_{v\in e}x_v\leqslant\frac{m^{\frac{p-m}{p}}}{\alpha^{\frac{1}{p}}}\sum_{e\in S(H)}\left(\frac{a_ew(e)}{m}\right)^{\frac{p-m}{p}}a_e^{\frac{m}{p}}\Pi_{v\in e}M^{\frac{1}{p}}(v,e)x_v=L_1.$$

By the AM-GM inequality,

$$L_1 \leqslant \frac{m^{\frac{p-m}{p}}}{\alpha^{\frac{1}{p}}} \left(\sum_{e \in S(H)} a_e \sum_{v \in e} \frac{M(v,e) x_v^p}{m} \right)^{\frac{m}{p}} = L_2.$$

From Lemma 3.3 and (ii) in Definition 4, we have

$$L_2 = \frac{m^{\frac{p-m}{p}}}{\alpha^{\frac{1}{p}}} \left(\sum_{v \in V(H)} [\sum_{e \in S(H)_v} a_e M(v,e)] x_v^p \right)^{\frac{m}{p}} \leqslant \frac{m^{\frac{p-m}{p}}}{\alpha^{\frac{1}{p}}} \parallel x \parallel_p^m = \frac{m^{\frac{p-m}{p}}}{\alpha^{\frac{1}{p}}}.$$

This completes the proof. $\square$

DEFINITION 5. For a general hypergraph H with rank m, if there exists a weighted incidence matrix M and weights $\{\omega(e)\}$ satisfying the following conditions.

(i) $\sum_{e \in S(H)} \frac{a_e \omega(e)}{m} \geqslant 1$;
(ii) $\sum_{e \in S(H)_v} a_e M(v,e) \geqslant 1$;
(iii) $\omega(e)^{p-m} \Pi_{v \in e} M(v,e) \leqslant \alpha$, for any $e \in S(H)$;
(iv) $M(v,e_1) = M(v,e_2)$ and $\omega(e_1) = \omega(e_2)$ for e_1 equals to e_2 up to rewriting,

then H is called α-weighted supernormal. H is called strictly α-weighted supernormal if it is α-weighted supernormal but not α-weighted normal.

THEOREM 3.6. *Let H be a general hypergraph with rank m. If H is consistently α-weighted supernormal, then the p-spectral radius $\rho^{(p)}(H)$ of H satisfies $\rho^{(p)}(H) \geqslant \frac{m^{1-\frac{m}{p}}}{\alpha^{\frac{1}{p}}}$.*

Proof. By the condition (iii) in definition 5, we have $\frac{\alpha}{\omega(e)^{p-m} \Pi_{v \in e} M(v,e)} \geqslant 1$. Define a vector $x = (x_1, x_2, \ldots, x_n)^T \in S_{p,+}^{n-1}$, where $x_v^p = \frac{\omega(e)}{mM(v,e)}$, $v \in e \in S(H)_v$. The consistent condition guarantees that x_v is independent of choice of e. Then

$$\begin{aligned} P_H(x) &= \sum_{e \in S(H)} a_e \Pi_{v \in e} x_v \\ &= \left(\frac{\alpha}{\omega(e)^{p-m} \Pi_{v \in e} M(v,e)} \right)^{\frac{1}{p}} \cdot \left(\frac{\alpha}{\omega(e)^{p-m} \Pi_{v \in e} M(v,e)} \right)^{-\frac{1}{p}} \cdot \sum_{e \in S(H)} a_e \Pi_{v \in e} x_v \\ &\geqslant \frac{m^{\frac{p-m}{p}}}{\alpha^{\frac{1}{p}}} \sum_{e \in S(H)} \left(\frac{a_e w(e)}{m} \right)^{\frac{p-m}{p}} \Pi_{v \in e} (a_e M(v,e))^{\frac{1}{p}} x_v = L_3. \end{aligned}$$

Let $A_e = \left(\frac{a_e w(e)}{m}\right)^{\frac{p-m}{p}}$, $B_e = \Pi_{v\in e}\left(a_e M(v,e)\right)^{\frac{1}{p}} x_v$, it has $\frac{A_e^{\frac{p}{p-m}}}{B_e^{\frac{p}{m}}} = 1$. By the condition of equality holding for the Hölder Inequality, it has

$$\begin{aligned}
L_3 &= \frac{m^{\frac{p-m}{p}}}{\alpha^{\frac{1}{p}}}\left(\sum_{e\in S(H)} \frac{a_e w(e)}{m}\right)^{\frac{p-m}{p}} \left(\sum_{e\in S(H)} a_e \Pi_{v\in e}(M(v,e))^{\frac{1}{m}} x_v^{\frac{p}{m}}\right)^{\frac{m}{p}} \\
&\geqslant \frac{m^{\frac{p-m}{p}}}{\alpha^{\frac{1}{p}}}\left(\sum_{e\in S(H)} a_e \Pi_{v\in e}(M(v,e))^{\frac{1}{m}} x_v^{\frac{p}{m}}\right)^{\frac{m}{p}} \\
&= \frac{m^{\frac{p-m}{p}}}{\alpha^{\frac{1}{p}}}\left(\sum_{e\in S(H)} \sum_{v\in e} \frac{a_e M(v,e) x_v^p}{m}\right)^{\frac{m}{p}} \\
&\geqslant \frac{m^{\frac{p-m}{p}}}{\alpha^{\frac{1}{p}}} \parallel x \parallel_p^m = \frac{m^{\frac{p-m}{p}}}{\alpha^{\frac{1}{p}}}.
\end{aligned}$$

This completes the proof. □

REMARK. Let H be an m-uniform hypergraph and $C_k = v_0 e_1 v_1 e_2 v_2 \cdots v_{k-1} e_k v_k$ $(= v_0)$ be a cycle in H. If there is a weighted incidence matrix M and weights $\{\omega(e)\}$ such that H is consistently α-weighted normal, we have

$$\sum_{e\in S(H)} \frac{a_e \omega(e)}{m} = m! \sum_{e\in E(H)} \frac{\frac{1}{(m-1)!}\omega(e)}{m} = \sum_{e\in E(H)} \omega(e);$$

$$\sum_{e\in S(H)_v} a_e M(v,e) = (m-1)! \sum_{e\in E(H)_v} \frac{1}{(m-1)!} M(v,e) = \sum_{e\in E(H)_v} M(v,e).$$

Note that $v_0 \in e_1 \cap e_k$ and $v_i \in e_i \cap e_{i+1}$ for $i \in [k-1]$, by the condition of consistent of H, we have

$$\begin{aligned}
&\frac{\omega(e_1)}{mM(v_0,e_1)} = \frac{\omega(e_k)}{mM(v_0,e_k)}; \quad \frac{\omega(e_2)}{mM(v_1,e_2)} = \frac{\omega(e_1)}{mM(v_1,e_1)}; \\
&\frac{\omega(e_3)}{mM(v_2,e_3)} = \frac{\omega(e_2)}{mM(v_2,e_2)}; \quad \cdots; \quad \frac{\omega(e_k)}{mM(v_{k-1},e_k)} = \frac{\omega(e_{k-1})}{mM(v_{k-1},e_{k-1})},
\end{aligned}$$

then

$$\begin{aligned}
&\frac{M(v_0,e_k)}{M(v_0,e_1)} = \frac{\omega(e_k)}{\omega(e_1)}; \quad \frac{M(v_1,e_1)}{M(v_1,e_2)} = \frac{\omega(e_1)}{\omega(e_2)}; \\
&\frac{M(v_2,e_2)}{M(v_2,e_3)} = \frac{\omega(e_2)}{\omega(e_3)}; \quad \cdots; \quad \frac{M(v_{k-1},e_{k-1})}{M(v_{k-1},e_k)} = \frac{\omega(e_{k-1})}{\omega(e_k)},
\end{aligned}$$

Hence we have

$$\prod_{i=1}^{k} \frac{M(v_i,e_i)}{M(v_{i-1},e_i)} = 1.$$

From these we can see that the definitions of consistent and α-normal labeling method in [5, 10] (that is Definition 2) are a special case of Definition 3 as given in this paper.

4. Some applications of the α-labeling method for the p-spectral radius

In this section we will give some applications of the α-labeling method for the p-spectral radius.

THEOREM 4.1. *Let $H_1, H_2, \dots, H_k$ be the components of a general hypergraph H with rank m and order n. If H has no isolated vertex, then*

(i) *If $1 \leqslant p \leqslant m$, $\rho^{(p)}(H) = \max_{1\leqslant i\leqslant k} \rho^{(p)}(H_i)$.*

(ii) *If $p > m$, $\rho^{(p)}(H) = (\sum_{i=1}^{k}\left(\rho^{(p)}(H_i)\right)^{\frac{p}{p-m}})^{\frac{p-m}{p}}$.*

Proof. (i) Let $\rho^{(p)}(H_i) = P_{H_i}(y^{(i)})$ with $y^{(i)} \in R_+^{|V(H_i)|}$, and $x \in R_+^n$ which satisfies

$$x_v = \begin{cases} y_v^{(i)}, & \text{if } v \in V(H_i); \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $\rho^{(p)}(H) \geqslant P_H(x) = P_{H_i}(y) = \rho^{(p)}(H_i)$. This implies $\rho^{(p)}(H) \geqslant \max_{1\leqslant i\leqslant k} \rho^{(p)}(H_i)$.

Let $\rho^{(p)}(H) = P_H(u)$ with $u \in S_{p,+}^{n-1}$. By Lemma 2.1 and $1 \leqslant p \leqslant m$, we have

$$\begin{aligned}
\rho^{(p)}(H) = P_H(u) &= \sum_{i=1}^{k} P_{H_i}(u|_{V(H_i)}) \\
&\leqslant \sum_{i=1}^{k} \rho^{(p)}(H_i) \parallel u|_{V(H_i)} \parallel_p^m \\
&\leqslant \max_{1\leqslant i\leqslant k} \rho^{(p)}(H_i) \sum_{i=1}^{k} \parallel u|_{V(H_i)} \parallel_p^m \\
&\leqslant \max_{1\leqslant i\leqslant k} \rho^{(p)}(H_i) \sum_{i=1}^{k} \parallel u|_{V(H_i)} \parallel_p^p = \max_{1\leqslant i\leqslant k} \rho^{(p)}(H_i).
\end{aligned}$$

(ii) Let H_i be consistently α_i-normal with weighted incidence matrix $M_i = (M_i(v,e))_{|V(H_i)|\times|S(H_i)|}$ and $\{\omega_i(e)\}$ (that is, the weight of $e \in E(H_i)$), where $\alpha_i = \frac{m^{p-m}}{(\rho^{(p)}(H_i))^p}$ for $i \in [k]$, that is

$$\begin{cases}
\sum_{e\in S(H_i)} \frac{a_e \omega_i(e)}{m} = 1, & \\
\sum_{e\in S(H_i)_v} a_e M_i(v,e) = 1 & \text{for any } v \in V(H_i), \\
\omega_i(e)^{p-m} \Pi_{v\in e} M_i(v,e) = \alpha_i & \text{for any } e \in S(H_i), \\
M_i(v,e_1) = M_i(v,e_2) & \text{for } e_1 \text{ equals to } e_2 \text{ up to rewriting.}
\end{cases}$$

Let $C=\sum_{i=1}^{k}\frac{1}{\alpha_i^{\frac{1}{p-m}}}=\frac{1}{m}\left(\sum_{i=1}^{k}(\rho^{(p)}(H_i))^{\frac{p}{p-m}}\right)$. Now we construct a weighted incidence matrix $M=(M(v,e))_{|V(H)|\times|S(H)|}$ and $\{\omega(e)\}$ for H as follows:

$$M(v,e)=\begin{cases} M_i(v,e) & \text{for if } v\in V(H_i),\ e\in S(H_i),\\ 0 & \text{otherwise,}\end{cases}$$

$$\omega(e)=\frac{\omega_i(e)}{C\alpha_i^{\frac{1}{p-m}}},\ \text{ if } e\in S(H_i).$$

For any $v\in V(H)$, assume that $v\in V(H_i)$, then

$$\sum_{e\in S(H)_v} a_e M(v,e)=\sum_{e\in S(H_i)_v} a_e M_i(v,e)=1.$$

For any $e\in S(H)$, assume that $e\in S(H_i)$, then

$$\omega(e)^{p-m}\Pi_{v\in e}M(v,e)=\left(\frac{\omega_i(e)}{C\alpha_i^{\frac{1}{p-m}}}\right)^{p-m}\Pi_{v\in e}M_i(v,e)=\frac{1}{C^{p-m}},$$

$$\sum_{e\in S(H)}\frac{a_e\omega(e)}{m}=\sum_{i=1}^{k}\sum_{e\in S(H_i)}\frac{a_e\omega_i(e)}{Cm\alpha_i^{\frac{1}{p-m}}}=\frac{1}{C}\sum_{i=1}^{k}\frac{1}{\alpha_i^{\frac{1}{p-m}}}=1,$$

$$\frac{\omega(e)}{M(v,e)}=\frac{\omega_i(e)}{M_i(v,e)C\alpha_i^{\frac{1}{p-m}}}\ \text{ for } v\in e\in S(H_i)_v.$$

Therefore H is consistently α-weighted normal with $\alpha=C^{-(p-m)}$, by Theorem 3.4, we obtain our desired result. □

THEOREM 4.2. *Let H be a general hypergraph H with rank m and d_u be the degree of vertex u. If $p>m$, then $\rho^{(p)}(H)\leqslant\left(m\sum_{e\in S(H)}\Pi_{u\in e}d(u)^{\frac{1}{p-m}}\right)^{\frac{p-m}{p}}$.*

Proof. Let $C=\sum_{e\in S(H)}\Pi_{u\in e}d(u)^{\frac{1}{p-m}}$. Define a weighted incidence matrix $M=(M(u,e))_{|V(H)|\times|S(H)|}$ and $\{\omega(e)\}$ for H as follows.

$$M(u,e)=\begin{cases}\frac{1}{d(u)} & \text{for if } u\in e\in S(H),\\ 0 & \text{otherwise,}\end{cases}$$

$$\omega(e)=\frac{1}{C}\Pi_{u\in e}d(u)^{\frac{1}{p-m}}.$$

For any $u\in V(H)$,

$$\sum_{e\in S(H)_u}a_eM(u,e)=\sum_{e\in S(H)_u}a_e\frac{1}{d(u)}=1$$

For any $e \in S(H)$,

$$\begin{aligned}
\omega(e)^{p-m}\Pi_{u\in e}M(u,e) &= \frac{1}{C^{p-m}}\Pi_{u\in e}d(u)\Pi_{u\in e}\frac{1}{d(u)} \\
&= \frac{1}{C^{p-m}}, \\
\sum_{e\in S(H)}\frac{a_e\omega(e)}{m} &= \sum_{e\in S(H)}\frac{a_e\Pi_{u\in e}d(u)^{\frac{1}{p-m}}}{Cm} \\
&= \sum_{e\in S(H)}\frac{a_e\Pi_{u\in e}d(u)^{\frac{1}{p-m}}}{\left(\sum_{e\in S(H)}\Pi_{u\in e}d(u)^{\frac{1}{p-m}}\right)m} \\
&\leqslant \frac{\max_{e\in S(H)}a_e}{m}\sum_{e\in S(H)}\frac{\Pi_{u\in e}d(u)^{\frac{1}{p-m}}}{\left(\sum_{e\in S(H)}\Pi_{u\in e}d(u)^{\frac{1}{p-m}}\right)} \\
&= \frac{\max_{e\in S(H)}a_e}{m} < 1.
\end{aligned}$$

Hence H is $\frac{1}{C^{p-m}}$-weighted subnormal, by Theorem 3.5, we obtain our desired result. $\square$

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Junhao Zhang
Faculty of Mathematics and Statistics
South Central University for Nationalities
Wuhan 430074, P.R. China
e-mail: 987682282@qq.com

Zhongxun Zhu
Faculty of Mathematics and Statistics
South Central University for Nationalities
Wuhan 430074, P.R. China
e-mail: zzxun73@163.com