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NOTES ON MAJORIZATIONS FOR SINGULAR VALUES

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Abstract. In this note, we mainly investigate the majorizations on the products and sums of matrices. Firstly, we present the following result: Let A_i, B_i and $X_i \in M_n(\mathscr{C})$ $(i=1,2,\cdots,m)$ with X_i $(i=1,2,\cdots,m)$ are invertible matrices, and let h be a nonnegative increasing continuous function on $[0,+\infty)$ with $h(0)=0$. If f, g are nonnegative continuous functions on $[0,+\infty)$ with $f(t)g(t)=t$ for $t\in[0,+\infty)$, then

$$\alpha\circ s\Big(\Big|\sum_{i=1}^{m}A_iX_i|X_i|^{-1}h(|X_i|)B_i\Big|^r\Big)$$
$$\prec_w \alpha\circ\Big\{\frac{1}{p}s\Big(\big(\sum_{i=1}^{m}A_if^2(h(|X_i^*|))A_i^*\big)^{\frac{pr}{2}}\Big)+\frac{1}{q}s\Big(\big(\sum_{i=1}^{m}B_i^*g^2(h(|X_i|))B_i\big)^{\frac{qr}{2}}\Big)\Big\},$$

where $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$, p, q, r and α_i $(i=1,2,\cdots,n)$ are positive real numbers with $\frac{1}{p}+\frac{1}{q}=1$. Then, some other weak majorizations are given. These obtained inequalities directly generalize the results obtained by Huang [H. Huang, On majorizations and singular values, Linear Multilinear Algebra, (2020), https://doi.org/10.1080/03081087.2020.1836117].

1. Introduction

Throughout, let $M_n(\mathscr{C})$ denote the space of all $n\times n$ complex matrices. $I_n(\in M_n(\mathscr{C}))$ is the identity matrix. For $A\in M_n(\mathscr{C})$, let $\lambda_j(A)(j=1,2,\cdots n)$, repeated according to multiplicities, be the eigenvalues of A with $|\lambda_1(A)|\geqslant|\lambda_2(A)|\geqslant\cdots\geqslant|\lambda_n(A)|$. Moreover, let $s_j(A)(j=1,2,\cdots n)$ be the singular values of A, i.e., the eigenvalues of the positive semidefinite matrix $|A|=(A^*A)^{\frac{1}{2}}$, where A^* is the conjugate transpose matrix of A. Let $|\lambda(A)|=(|\lambda_1(A)|,|\lambda_2(A)|,\cdots,|\lambda_n(A)|)^T$ denote the vector of the modulus of the eigenvalues of A and $s(A)=(s_1(A),s_2(A),\cdots,s_n(A))^T$ be the vector of the singular values of A. For $A=(a_{ij})$, $B=(b_{ij})\in M_n(\mathscr{C})$, the Schur product of A and B, denoted by $A\circ B$, is the matrix $(a_{ij}b_{ij})$. The same notation will be applied to vectors.

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The direct sum of matrices $A_i \in M_n(\mathscr{C})\,(i=1,2,\cdots,m)$, denoted by $\bigoplus_{i=1}^{m} A_i$, is the $m\times m$ block-diagonal matrix (on $\bigoplus_{i=1}^{m} M_n(\mathscr{C})$) defined by

$$\bigoplus_{i=1}^{m} A_i = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{bmatrix}.$$

When $m=2$, we write $A_1 \oplus A_2$ instead of $\bigoplus_{i=1}^{2} A_i$.

Let us recall the definitions of majorizations. For a real vector $\mathbf{x}=(x_1,x_2,\cdots,x_n)^T$, we rearrange its components as $x_{[1]} \geqslant x_{[2]} \geqslant x_{[n]}$. Let $\mathbf{x}=(x_1,x_2,\cdots,x_n)^T$ and $\mathbf{y}=(y_1,y_2,\cdots,y_n)^T$ be two real vectors. we call that $\mathbf{x}$ is weakly majorized by $\mathbf{y}$, denoted by $\mathbf{x} \prec_w \mathbf{y}$, if $\sum_{i=1}^{k} x_{[i]} \leqslant \sum_{i=1}^{k} y_{[i]}$ for $k=1,2,\cdots,n$. Moreover, we call that $\mathbf{x}$ is majorized by $\mathbf{y}$, denoted by $\mathbf{x} \prec \mathbf{y}$, if $\mathbf{x} \prec_w \mathbf{y}$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$. If $x_i \geqslant 0$ and $y_i \geqslant 0$ $(i=1,2,\cdots,n)$ and $\prod_{i=1}^{k} x_{[i]} \leqslant \prod_{i=1}^{k} y_{[i]}$ for $k=1,2,\cdots,n$, then we say that $\mathbf{x}$ is weakly log-majorized by $\mathbf{y}$, denoted by $\mathbf{x} \prec_{w\log} \mathbf{y}$. In addition, if $\mathbf{x} \prec_{w\log} \mathbf{y}$ and $\prod_{i=1}^{n} x_i = \prod_{i=1}^{n} y_i$, we say that $\mathbf{x}$ is log-majorized by $\mathbf{y}$, denoted by $\mathbf{x} \prec_{\log} \mathbf{y}$. It is well known that the weak log-majorization implies the weak majorization.

Recently, Huang [7, Theorem 2.5] obtained the following inequality: Let A_i, B_i and X_i, $(i=1,2,\cdots,k)$ be $n\times n$ matrices, and f_i, g_i $(i=1,2,\cdots,k)$ be nonnegative continuous functions on $[0,+\infty)$ with $f_i(t)g_i(t)=t$ for $t\in[0,+\infty)$. Then

$$\begin{aligned} & s\Big(\sum_{i=1}^{k} A_i X_i |X_i|^{m-1} B_i\Big) \\ & \quad \prec_w \frac{1}{2}\Big[\lambda\Big(\sum_{i=1}^{k} A_i f_i^2(|X_i^*|^m) A_i^*\Big) + \lambda\Big(\sum_{i=1}^{k} B_i^* g_i^2(|X_i|^m) B_i\Big)\Big] \\ & \quad \prec_w \frac{1}{2}\Big[\lambda\big(f_i^2(|X_i^*|^m)\big) + \lambda\big(g_i^2(|X_i|^m)\big)\Big] \circ \zeta, \end{aligned} \tag{1}$$

where ζ is any vector which weakly majorizes the following vectors:

$$\lambda\Big(\sum_{i=1}^{k} A_i A_i^*\Big), \quad \lambda\Big(\sum_{i=1}^{k} A_i^* A_i\Big), \quad \lambda\Big(\sum_{i=1}^{k} B_i B_i^*\Big), \quad \lambda\Big(\sum_{i=1}^{k} B_i^* B_i\Big).$$

Inequality (1) is a generalization of the following inequality obtained by Bapat [2, Theorem 2]:

$$\begin{aligned} &s\Big(\sum_{i=1}^{k} A_i X B_i\Big) \\ &\prec_w \frac{1}{2}\Big[\lambda\Big(\sum_{i=1}^{k} A_i|X^*|A_i^*\Big)+\lambda\Big(\sum_{i=1}^{k} B_i^*|X|B_i\Big)\Big] \\ &\prec_w s(X)\circ\zeta, \end{aligned} \tag{2}$$

where X, A_i, B_i $(i=1,2,\cdots,k)$ are $n\times n$ matrices and ζ is any vector which weakly majorizes the following vectors:

$$\lambda\Big(\sum_{i=1}^{k} A_iA_i^*\Big),\quad \lambda\Big(\sum_{i=1}^{k} A_i^*A_i\Big),\quad \lambda\Big(\sum_{i=1}^{k} B_iB_i^*\Big),\quad \lambda\Big(\sum_{i=1}^{k} B_i^*B_i\Big).$$

In the same paper, Huang [7, Corollary 2.7] also obtained

$$s(X|X|^{m-1}+Y|Y|^{m-1}) \prec_w \frac{1}{2}\Big[\lambda(|X^*|^m+|Y^*|^m)+\lambda(|X|^m+|Y|^m)\Big], \tag{3}$$

where X and Y are $n\times n$ matrices and m is a positive integer.

In this note, we mainly present some weak majorizations, which generalize inequalities (1), (2) and (3).

2. Main results

In this section, we mainly study some majorizations between the vectors of eigenvalues and singular values for the products and sums of matrices. To achieve our goal, we need the following lemmas. The first lemma is Proposition 1.3.2 in [4] and Lemma 4 is Corollary 1.3.7.

LEMMA 1. *Let A, $B\in M_n(\mathscr{C})$ with A and B are positive semidefinite matrices. Then the block matrix* $\begin{bmatrix} A & X \\ X^* & B\end{bmatrix}\geqslant 0$ *if and only if $X=A^{\frac{1}{2}}WB^{\frac{1}{2}}$ for some contraction W, i.e., $s_i(W)\leqslant 1$ for $i=1,2,\cdots,n$.*

The second lemma was obtained by Horn [10, Theorem 4.6].

LEMMA 2. *Let A, $B\in M_n(\mathscr{C})$. Then*

$$\prod_{j=1}^{k} s_j(AB)\leqslant \prod_{j=1}^{k} s_j(A)s_j(B),$$

for $k=1,2,\cdots n$.

Based on Lemmas 1 and 2, we have the following lemma.

LEMMA 3. *Let A, B and $X \in M_n(\mathscr{C})$ with A and B are positive semidefinite matrices. If the block matrix* $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \geqslant 0$, *then*

$$\alpha \circ s(|X|^r) \prec_w \alpha \circ \left(\frac{1}{p}s(A^{\frac{pr}{2}}) + \frac{1}{q}s(B^{\frac{qr}{2}})\right),$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, p, q, r *and* α_i $(i = 1, 2, \cdots, n)$ *are positive real numbers with* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Since the block matrix $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \geqslant 0$, then by Lemma 1, we get

$$X = A^{\frac{1}{2}} W B^{\frac{1}{2}}$$

for some contraction W. Lemma 2 implies

$$\begin{aligned}
\prod_{j=1}^{k} s_j(X) &= \prod_{j=1}^{k} s_j(A^{\frac{1}{2}} W B^{\frac{1}{2}}) \\
&\leqslant \prod_{j=1}^{k} s_j(A^{\frac{1}{2}}) s_j(W B^{\frac{1}{2}}) \\
&\leqslant \prod_{j=1}^{k} s_j(A^{\frac{1}{2}}) s_1(W) s_j(B^{\frac{1}{2}}) \\
&\leqslant \prod_{j=1}^{k} s_j(A^{\frac{1}{2}}) s_j(B^{\frac{1}{2}}),
\end{aligned}$$

which gives

$$\prod_{j=1}^{k} s_j(|X|^r) \leqslant \prod_{j=1}^{k} s(A^{\frac{r}{2}}) s_j(B^{\frac{r}{2}}),$$

where $k = 1, 2, \cdots, n$. By the Young inequality $ab \leqslant \frac{1}{p}a^p + \frac{1}{q}b^q$ for positive real numbers a, b, p and q with $\frac{1}{p} + \frac{1}{q} = 1$, we obtain

$$\prod_{j=1}^{k} s_j(A^{\frac{r}{2}}) s_j(B^{\frac{r}{2}}) \leqslant \prod_{j=1}^{k} \left(\frac{1}{p} s_j(A^{\frac{pr}{2}}) + \frac{1}{q} s_j(B^{\frac{qr}{2}})\right),$$

which implies

$$\prod_{j=1}^{k} \alpha_j s_j(A^{\frac{r}{2}}) s_j(B^{\frac{r}{2}}) \leqslant \prod_{j=1}^{k} \alpha_j \left(\frac{1}{p} s_j(A^{\frac{pr}{2}}) + \frac{1}{q} s_j(B^{\frac{qr}{2}})\right),$$

for positive real numbers α_i $(i = 1, 2, \cdots, n)$. The above inequalities entail

$$\prod_{j=1}^{k} \alpha_j s_j(|X|^r) \leqslant \prod_{j=1}^{k} \alpha_j \left(\frac{1}{p} s_j(A^{\frac{pr}{2}}) + \frac{1}{q} s_j(B^{\frac{qr}{2}})\right).$$

Since weak log-majorization implies weak majorization, by the above inequality, we get

$$\sum_{j=1}^{k} \alpha_j s_j(|X|^r) \leqslant \sum_{j=1}^{k} \alpha_j \Big(\frac{1}{p} s_j(A^{\frac{pr}{2}}) + \frac{1}{q} s_j(B^{\frac{qr}{2}})\Big),$$

equivalently,

$$\alpha \circ s(|X|^r) \prec_w \alpha \circ \Big(\frac{1}{p} s(A^{\frac{pr}{2}}) + \frac{1}{q} s(B^{\frac{qr}{2}})\Big),$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$.

We obtain the desired inequality. $\square$

LEMMA 4. *If $A \in M_n(\mathscr{C})$, then the block matrix $\begin{bmatrix} |A| & A^* \\ A & |A^*| \end{bmatrix}$ in $M_2(M_n(\mathscr{C}))$ is a positive semidefinite matrix.*

The fifth lemma was given by Kittaneh [8].

LEMMA 5. *Let A, B and $X \in M_n(\mathscr{C})$ with A, B are positive semidefinite matrices and $BX = XA$ and f, g be two nonnegative continuous functions on $[0,+\infty)$ with $f(t)g(t) = t$ for $t \in [0,+\infty)$. If $\begin{bmatrix} A & X^* \\ X & B \end{bmatrix} \geqslant 0$, then so is $\begin{bmatrix} f^2(A) & X^* \\ X & g^2(B) \end{bmatrix} \geqslant 0$.*

The next lemma was due to Bapat and Sunder [3].

LEMMA 6. *If $X \geqslant 0$ and $A_i \in M_n(\mathscr{C})$ $(i = 1, 2, \cdots, k)$, then*

$$\lambda\Big(\sum_{i=1}^{k} A_i X A_i^*\Big) \prec_w \lambda(X) \circ \zeta,$$

where ζ is any vector which majorizes both $\lambda\big(\sum_{i=1}^{k} A_i A_i^*\big)$ and $\lambda\big(\sum_{i=1}^{k} A_i^* A_i\big)$.

The following lemma is a special case of the Weyl's Majorant Theorem [5, Theorem II 3.6].

LEMMA 7. *Let $A \in M_n(\mathscr{C})$. Then*

$$|\lambda(A)| \prec_{\log} s(A).$$

The Young inequality for singular values obtained by Ando [1] can be stated as follows.

LEMMA 8. *Let A, $B \in M_n(\mathscr{C})$. Then*

$$s_i(AB^*) \leqslant s_i\Big(\frac{1}{p}|A|^p + \frac{1}{q}|B|^q\Big),$$

for $i = 1, 2, \cdots, n$, where p and q are positive real numbers with $\frac{1}{p} + \frac{1}{q} = 1$.

Kittaneh and Lin [9, Proposition 2.6] showed the following lemma.

LEMMA 9. *Let A, B and $X \in M_n(\mathscr{C})$ with A and B are positive semidefinite matrices. If the block matrix* $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \geqslant 0$*, then*

$$|\lambda(X^N)| \prec_{w\log} \lambda\left(A^{\frac{N}{2}} B^{\frac{N}{2}}\right),$$

for any positive integer N.

Based on Lemmas 7, 8 and 9, we have:

LEMMA 10. *Let A, B and $X \in M_n(\mathscr{C})$ with A and B are positive semidefinite matrices. If the block matrix* $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \geqslant 0$*, then*

$$\alpha \circ |\lambda(X^N)| \prec_w \alpha \circ s\left(\frac{1}{p} A^{\frac{Np}{2}} + \frac{1}{q} B^{\frac{Nq}{2}}\right),$$

for any positive integer N and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, where p, q and α_i $(i = 1, 2, \cdots n)$ are positive real numbers with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By Lemmas 9, 7 and 8, we have $|\lambda(X^N)| \prec_{w\log} s\left(\frac{1}{p} A^{\frac{Np}{2}} + \frac{1}{q} B^{\frac{Nq}{2}}\right)$, which implies $\prod_{i=1}^{k} \alpha_i |\lambda_i(X^N)| \leqslant \prod_{i=1}^{k} \alpha_i s_i\left(\frac{1}{p} A^{\frac{Np}{2}} + \frac{1}{q} B^{\frac{Nq}{2}}\right)$, for $k = 1, 2, \cdots, n$. Since weak log-majorization implies weak majorization, we get the desired inequality. □

Next, we present the main results.

THEOREM 1. *Let A, B and $X \in M_n(\mathscr{C})$ with X is an invertible matrix, and let h be a nonnegative increasing continuous function on $[0, +\infty)$ with $h(0) = 0$. If f, g are nonnegative continuous functions on $[0, +\infty)$ with $f(t)g(t) = t$ for $t \in [0, +\infty)$, then*

$$\begin{aligned} &\alpha \circ s\left(\left|AX|X|^{-1} h(|X|) B\right|^r\right) \\ &\quad \prec_w \alpha \circ \left(\frac{1}{p} s\left(\left(A f^2(h(|X^*|)) A^*\right)^{\frac{pr}{2}}\right) + \frac{1}{q} s\left(\left(B^* g^2(h(|X|)) B\right)^{\frac{qr}{2}}\right)\right), \end{aligned}$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, p, q, r and α_i $(i = 1, 2, \cdots, n)$ are positive real numbers with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let $X = U|X|$ be the polar decomposition of X, then by Lemma 4,

$$0 \leqslant \begin{bmatrix} |X^*| & X \\ X^* & |X| \end{bmatrix} = \begin{bmatrix} U & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} |X| & |X| \\ |X| & |X| \end{bmatrix} \begin{bmatrix} U^* & 0 \\ 0 & I_n \end{bmatrix}$$

and

$$\begin{bmatrix} 2|X| & 0 \\ 0 & 0 \end{bmatrix} = \left(\tfrac{1}{\sqrt{2}} \begin{bmatrix} I_n & I_n \\ I_n & -I_n \end{bmatrix} \right) \begin{bmatrix} |X| & |X| \\ |X| & |X| \end{bmatrix} \left(\tfrac{1}{\sqrt{2}} \begin{bmatrix} I_n & I_n \\ I_n & -I_n \end{bmatrix} \right).$$

Thus

$$\frac{1}{2} \begin{bmatrix} |X^*| & X \\ X^* & |X| \end{bmatrix} = \begin{bmatrix} U & 0 \\ 0 & I_n \end{bmatrix} \left(\tfrac{1}{\sqrt{2}} \begin{bmatrix} I_n & I_n \\ I_n & -I_n \end{bmatrix} \right) \begin{bmatrix} |X| & 0 \\ 0 & 0 \end{bmatrix} \times \left(\tfrac{1}{\sqrt{2}} \begin{bmatrix} I_n & I_n \\ I_n & -I_n \end{bmatrix} \right) \begin{bmatrix} U^* & 0 \\ 0 & I_n \end{bmatrix}.$$

Therefore

$$0 \leqslant 2h \left(\tfrac{1}{2} \begin{bmatrix} |X^*| & X \\ X^* & |X| \end{bmatrix} \right) = \begin{bmatrix} h(|X^*|) & Uh(|X|) \\ h(|X|)U^* & h(|X|) \end{bmatrix}.$$

Since $h(|X|)(U^*h(|X^*|)) = h(|X|)U^*(Uh(|X|)U^*) = h(|X|)(h(|X|)U^*)$, then Lemma 5 gives

$$\begin{bmatrix} f^2(h(|X^*|)) & Uh(|X|) \\ h(|X|)U^* & g^2(h(|X|)) \end{bmatrix} \geqslant 0.$$

Therefore

$$\begin{bmatrix} A & 0 \\ 0 & B^* \end{bmatrix} \begin{bmatrix} f^2(h(|X^*|)) & Uh(|X|) \\ h(|X|)U^* & g^2(h(|X|)) \end{bmatrix} \begin{bmatrix} A^* & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} Af^2(h(|X^*|))A^* & AUh(|X|)B \\ B^*h(|X|)U^*A^* & B^*g^2(h(|X|))B \end{bmatrix} \geqslant 0. \tag{4}$$

By inequality (4) and Lemma 3, we get

$$\alpha \circ s(|AUh(|X|)B|^r) \prec_w \alpha \circ \left(\frac{1}{p} s\big((Af^2(h(|X^*|))A^*)^{\frac{pr}{2}} \big) + \frac{1}{q} s\big((B^*g^2(h(|X|))B)^{\frac{qr}{2}} \big) \right).$$

Since $U = X|X|^{-1}$, then the desired result follows from the above inequality. This completes the proof. □

For majorization on the products and sums of matrices, we have the following theorem.

THEOREM 2. *Let A_i, B_i and $X_i \in M_n(\mathscr{C})$ $(i = 1, 2, \cdots, m)$ with X_i $(i = 1, 2, \cdots, m)$ are invertible matrices, and let h be a nonnegative increasing continuous function on $[0, +\infty)$ with $h(0) = 0$. If f, g are nonnegative continuous functions on $[0, +\infty)$ with $f(t)g(t) = t$ for $t \in [0, +\infty)$, then*

$$\alpha \circ s\Big(\Big| \sum_{i=1}^{m} A_i X_i |X_i|^{-1} h(|X_i|) B_i \Big|^r \Big) \prec_w \alpha \circ \Big\{ \frac{1}{p} s\Big(\big(\sum_{i=1}^{m} A_i f^2(h(|X_i^*|)) A_i^* \big)^{\frac{pr}{2}} \Big) + \frac{1}{q} s\Big(\big(\sum_{i=1}^{m} B_i^* g^2(h(|X_i|)) B_i \big)^{\frac{qr}{2}} \Big) \Big\}, \tag{5}$$

where $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$*,* p*,* q*,* r *and* α_i $(i=1,2,\cdots,n)$ *are positive real numbers with* $\frac{1}{p}+\frac{1}{q}=1$*.*

Proof. Let

$$A=\begin{bmatrix} A_1 & A_2 & \cdots & A_m \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B=\begin{bmatrix} B_1 & 0 & \cdots & 0 \\ B_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_m & 0 & \cdots & 0 \end{bmatrix}, \quad \text{and} \quad X=\bigoplus_{i=1}^{m} X_i.$$

Then

$$AX|X|^{-1}h(|X|)B=\Big(\sum_{i=1}^{m} A_iX_i|X_i|^{-1}h(|X_i|)B_i\Big)\oplus 0,$$

$$Af^2(h(|X^*|))A^* * \Big(\sum_{i=1}^{m} A_if^2(h(|X_i^*|))A_i^*\Big)\oplus 0$$

and

$$B^*g^2(h(|X|))B=\Big(\sum_{i=1}^{m} B_i^*g^2(h(|X_i|))B_i\Big)\oplus 0,$$

we get the desired inequality by Theorem 1. □

REMARK 1. For positive integer k, taking $h(x)=x^k$ and noting that $A_iX_i|X_i|^{-1}h(|X_i|)\cdot B_i=A_iX_i|X_i|^{k-1}B_i$, $A_if^2(h(|X_i^*|))A_i^*=A_if^2(|X_i^*|^k)A_i^*$ and $B_i^*g^2(h(|X_i|))B_i=B_i^*g^2(|X_i|^k)B_i$, then by Theorem 2, we get

$$\alpha\circ s\Big(\Big|\sum_{i=1}^{m} A_iX_i|X_i|^{k-1}B_i\Big|^r\Big) \prec_w \alpha\circ\Big\{\frac{1}{p}s\Big(\big(\sum_{i=1}^{m} A_if^2(|X_i^*|^k)A_i^*\big)^{\frac{pr}{2}}\Big) + \frac{1}{q}s\Big(\big(\sum_{i=1}^{m} B_i^*g^2(|X_i|^k)B_i\big)^{\frac{qr}{2}}\Big)\Big\}.$$

Since the class of invertible matrices is dense in the class of matrices, then by the continuous argument, replacing X_i by $X_i+\varepsilon I_n$ $(i=1,2,\cdots,m,\varepsilon>0)$ and as $\varepsilon\to 0$, the above inequality also holds for any matrix X_i $(i=1,2,\cdots,m)$, where $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$, p, q, r, α_i $(i=1,2,\cdots,n)$ are positive real numbers with $\frac{1}{p}+\frac{1}{q}=1$. Setting $p=q=2$ in the above inequality, we obtain

$$\alpha\circ s\Big(\Big|\sum_{i=1}^{m} A_iX_i|X_i|^{k-1}B_i\Big|^r\Big) \prec_w \alpha\circ\frac{1}{2}\Big\{s\Big(\big(\sum_{i=1}^{m} A_if^2(|X_i^*|^k)A_i^*\big)^{r}\Big) + s\Big(\big(\sum_{i=1}^{m} B_i^*g^2(|X_i|^k)B_i\big)^{r}\Big)\Big\},$$

where $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$, r, α_i $(i=1,2,\cdots,n)$ are positive real numbers. Putting $\alpha_1=\alpha_2=\cdots=\alpha_n=r=1$ in the above inequality, we have the first inequality in (1).

REMARK 2. Taking $h(x) = x$ and $p = q = 2$ in Theorem 2, we obtain

$$\alpha \circ s\Big(\Big|\sum_{i=1}^{m} A_i X B_i\Big|^r\Big) \prec_w \alpha \circ \frac{1}{2}\Big\{s\Big(\big(\sum_{i=1}^{m} A_i f^2(|X_i^*|)A_i^*\big)^r\Big) + s\Big(\big(\sum_{i=1}^{m} B_i^* g^2(|X_i|)B_i\big)^r\Big)\Big\},$$

which implies

$$\alpha \circ s\Big(\Big|\sum_{i=1}^{m} A_i X B_i\Big|^r\Big) \prec_w \alpha \circ \frac{1}{2}\Big\{s\Big(\big(\sum_{i=1}^{m} A_i |X_i^*| A_i^*\big)^r\Big) + s\Big(\big(\sum_{i=1}^{m} B_i^* |X_i| B_i\big)^r\Big)\Big\}.$$

Since the class of invertible matrices is dense in the class of matrices, then by the continuous argument, the above inequality also holds for any matrix X, where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, r, α_i $(i = 1, 2, \cdots, n)$ are positive real numbers. Putting $\alpha_1 = \alpha_2 = \cdots = \alpha_n = r = 1$ in the above inequality, we get inequality (2).

REMARK 3. Let X_1 and $X_2 \in M_n(\mathscr{C})$ and $r > 0$. Putting $A_1 = A_2 = B_1 = B_2 = I_n$ and $p = q = 2$, inequality (5) gives

$$\alpha \circ s\Big(\Big|X_1|X_1|^{-1}h(|X_1|) + X_2|X_2|^{-1}h(|X_2|)\Big|^r\Big)$$
$$\prec_w \alpha \circ \frac{1}{2}\Big\{s\Big(f^2(h(|X_1^*|)) + f^2(h(|X_2^*|))\Big)^r + s\Big(g^2(h(|X_1|)) + g^2(h(|X_2|)\Big)^r\Big\}.$$

Since the class of invertible matrices is dense in the class of matrices, then by the continuous argument, the above inequality also holds for any matrix X_i $(i = 1, 2)$. Setting $f(t) = g(t) = \sqrt{t}$ and $h(x) = x^k$ for some positive integer k in the above inequality, we get

$$\alpha \circ s\Big(\Big|X_1|X_1|^{k-1} + X_2|X_2|^{k-1}\Big|^r\Big)$$
$$\prec_w \alpha \circ \frac{1}{2}\Big\{s\Big(|X_1^*|^k + |X_2^*|^k\Big)^r + s\Big(|X_1|^k + |X_2|^k\Big)^r\Big\}.$$

When $r = 1$ and $\alpha = (1, 1, \cdots, 1)$, we obtain inequality (3).

In [7, Corollary 2.6], Huang also got that: Let X, A_i, B_i $(i = 1, 2, \cdots, k)$, then for positive integer m, we have

$$s\Big(\sum_{i=1}^{k} A_i X|X|^{m-1} B_i\Big)$$
$$\prec_w \frac{1}{2}\Big[\lambda\Big(\sum_{i=1}^{k} A_i |X|^m A_i^*\Big) + \lambda\Big(\sum_{i=1}^{k} B_i |X|^m B_i^*\Big)\Big]$$
$$\prec_w \frac{1}{2}\big[\lambda(|X^*|^m) + \lambda(|X|^m)\big] \circ \zeta$$
$$\prec_w s(X|X|^{m-1}) \circ \zeta,$$

where ζ is any vector which majorizes the following vectors

$$\lambda\Big(\sum_{i=1}^{k} A_iA_i^*\Big), \quad \lambda\Big(\sum_{i=1}^{k} A_i^*A_i\Big), \quad \lambda\Big(\sum_{i=1}^{k} B_iB_i^*\Big), \quad \lambda\Big(\sum_{i=1}^{k} B_i^*B_i\Big).$$

Since $\lambda(|X^*|) = s(|X^*|) = s(|X|) = \lambda(|X|)$ and $s(X|X|^{m-1}) = s(U|X||X|^{m-1}) = s(U|X|^m) = s(|X|^m)$, then $\lambda(|X^*|^m) = \lambda(|X|^m) = s(|X|^m) = s(X|X|^{m-1})$, therefore, the last weak majorization in [7, Corollary 2.6] should be equal to each other, where $X = U|X|$ is the polar decomposition of X.

In fact, we have the following result.

THEOREM 3. *Let A_i, B_i $(i=1,2,\cdots,m)$ and $X \in M_n(\mathscr{C})$ with X is an invertible matrix, and let h be a nonnegative increasing continuous function on $[0,+\infty)$ with $h(0)=0$. If f, g are nonnegative continuous functions on $[0,+\infty)$ with $f(t)g(t)=t$ for $t \in [0,+\infty)$, then*

$$s\Big(\Big|\sum_{i=1}^{m} A_iX|X|^{-1}h(|X|)B_i\Big|^r\Big) \prec_w \frac{1}{p}\lambda^{\frac{pr}{2}}\big(f^2(h(|X^*|))\big)\circ\zeta^{\frac{pr}{2}} + \frac{1}{q}\lambda^{\frac{qr}{2}}\big(g^2(h(|X|))\big)\circ\zeta^{\frac{qr}{2}},$$

where p, q and r are positive real numbers with $\frac{1}{p}+\frac{1}{q}=1$ and $r \geqslant 2\max\{\frac{1}{p},\frac{1}{q}\}$, and ζ is any vector which majorizes the following vectors

$$\lambda\Big(\sum_{i=1}^{m} A_iA_i^*\Big), \quad \lambda\Big(\sum_{i=1}^{m} A_i^*A_i\Big), \quad \lambda\Big(\sum_{i=1}^{m} B_iB_i^*\Big), \quad \lambda\Big(\sum_{i=1}^{m} B_i^*B_i\Big).$$

Proof. By Lemma 6, we have

$$s\Big(\sum_{i=1}^{m} A_if^2(h(|X^*|))A_i^*\Big) \prec_w \lambda\big(f^2(h(|X^*|))\big)\circ\zeta_1, \tag{6}$$

where ζ_1 majorizes both $\lambda(\sum_{i=1}^{m} A_iA_i^*)$ and $\lambda(\sum_{i=1}^{m} A_i^*A_i)$. Similarly, by Lemma 6, we have

$$s\Big(\sum_{i=1}^{m} B_i^*g^2(h(|X|))B_i\Big) \prec_w \lambda\big(g^2(h(|X|))\big)\circ\zeta_2, \tag{7}$$

where ζ_2 majorizes both $\lambda\big(\sum_{i=1}^{m} B_iB_i^*\big)$ and $\lambda\big(\sum_{i=1}^{m} B_i^*B_i\big)$. Since the functions $x^{\frac{pr}{2}}$ and $x^{\frac{qr}{2}}$ are increasing convex functions on $[0,+\infty)$, then by Theorem 3.21 in [10], inequalities (6) and (7) give

$$s^{\frac{pr}{2}}\Big(\sum_{i=1}^{m} A_if^2(h(|X^*|))A_i^*\Big) \prec_w \lambda^{\frac{pr}{2}}\big(f^2(h(|X^*|))\big)\circ\zeta_1^{\frac{pr}{2}}, \tag{8}$$

and

$$s^{\frac{qr}{2}}\Big(\sum_{i=1}^{m} B_i^* g^2(h(|X|))B_i\Big) \prec_w \lambda^{\frac{qr}{2}}\big(g^2(h(|X|))\big)\circ \zeta_2^{\frac{qr}{2}}, \tag{9}$$

where ζ_1 majorizes both $\lambda(\sum\limits_{i=1}^{m} A_iA_i^*)$ and $\lambda(\sum\limits_{i=1}^{m} A_i^*A_i)$ and ζ_2 majorizes both $\lambda(\sum\limits_{i=1}^{m} B_iB_i^*)$ and $\lambda(\sum\limits_{i=1}^{m} B_i^*B_i)$, respectively. Here, $\zeta_1^r = (\zeta_{11}^r, \zeta_{12}^r, \cdots, \zeta_{1n}^r)$ if $\zeta_1 = (\zeta_{11}, \zeta_{12}, \cdots, \zeta_{1n})$.

On the other hand, since ζ is any vector which majorizes the following vectors

$$\lambda\big(\sum_{i=1}^{m} A_iA_i^*\big),\quad \lambda\big(\sum_{i=1}^{m} A_i^*A_i\big),\quad \lambda\big(\sum_{i=1}^{m} B_iB_i^*\big),\quad \lambda\big(\sum_{i=1}^{m} B_i^*B_i\big).$$

By inequalities (8) and (9), we have

$$s^{\frac{pr}{2}}\Big(\sum_{i=1}^{m} A_i f^2(h(|X^*|))A_i^*\Big) \prec_w \lambda^{\frac{pr}{2}}\big(f^2(h(|X^*|))\big)\circ \zeta^{\frac{pr}{2}}, \tag{10}$$

and

$$s^{\frac{qr}{2}}\Big(\sum_{i=1}^{m} B_i^* g^2(h(|X|))B_i\Big) \prec_w \lambda^{\frac{qr}{2}}\big(g^2(h(|X|))\big)\circ \zeta^{\frac{qr}{2}}. \tag{11}$$

Inequalities (5) $(\alpha = (1,1,\cdots,1))$, (10) and (11) entail

$$s\Big(\Big|\sum_{i=1}^{m} A_iX|X|^{-1}h(|X|)B_i\Big|^r\Big) \prec_w \frac{1}{p}\lambda^{\frac{pr}{2}}\big(f^2(h(|X^*|))\big)\circ\zeta^{\frac{pr}{2}} + \frac{1}{q}\lambda^{\frac{qr}{2}}\big(g^2(h(|X|))\big)\circ\zeta^{\frac{qr}{2}}.$$

This completes the proof. □

REMARK 4. Putting $p = q = 2$, then by Theorem 3, we have

$$s\Big(\Big|\sum_{i=1}^{m} A_iX|X|^{-1}h(|X|)B_i\Big|^r\Big) \prec_w \frac{1}{2}\Big[\lambda^r\big(f^2(h(|X^*|))\big) + \lambda^r\big(g^2(h(|X|))\big)\Big]\circ\zeta^r,$$

for $r \geqslant 1$. Moreover, taking $f(t) = g(t) = \sqrt{t}$ and $h(x) = x^k$ for some positive integer k in the above inequality, we obtain

$$s\Big(\Big|\sum_{i=1}^{m} A_iX|X|^{k-1}B_i\Big|^r\Big) \prec_w \frac{1}{2}\big[s^r\big(|X^*|^k\big) + s^r\big(|X|^k\big)\big]\circ\zeta^r.$$

Since $s(|X^*|) = s(|X|) = s(X)$, the above inequality implies

$$s\Big(\Big|\sum_{i=1}^{m} A_iX|X|^{k-1}B_i\Big|^r\Big) \prec_w s^{rk}(X)\circ\zeta^r,$$

Since the class of invertible matrices is dense in the class of matrix, then by the continuous argument, the above inequality also holds for any matrix X. Therefore, the above inequality implies inequality (2) by setting $r = k = 1$.

Utilizing Lemma 10, we have the following majorizations.

THEOREM 4. *Let A, B and $X \in M_n(\mathscr{C})$ with X is an invertible matrix, and let h be a nonnegative increasing continuous function on $[0,+\infty)$ with $h(0)=0$. If f, g are nonnegative continuous functions on $[0,+\infty)$ with $f(t)g(t)=t$ for $t \in [0,+\infty)$, then*

$$\alpha \circ \left|\lambda\left(\left(AX|X|^{-1}h(|X|)B\right)^{N}\right)\right| \prec_w \alpha \circ s\left(\frac{1}{p}\left(Af^2(h(|X^*|))A^*\right)^{\frac{Np}{2}} + \frac{1}{q}\left(B^*g^2(h(|X|))B\right)^{\frac{Nq}{2}}\right),$$

for any positive integer N, where $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$, α_i, p and q are positive real numbers with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By inequality (4), we have

$$\begin{bmatrix} Af^2(h(|X^*|))A^* & AUh(|X|)B \\ B^*h(|X|)U^*A & B^*g^2(h(|X|))B \end{bmatrix} \geqslant 0,$$

then by Lemma 10, we get inequality

$$\alpha \circ \left|\lambda\left(\left(AUh(|X|)B\right)^{N}\right)\right| \prec_w \alpha \circ s\left(\frac{1}{p}\left(Af^2(h(|X^*|))A^*\right)^{\frac{Np}{2}} + \frac{1}{q}\left(B^*g^2(h(|X|))B\right)^{\frac{Nq}{2}}\right).$$

Replacing U by $X|X|^{-1}$, we obtain the desired inequality. This completes the proof. □

Similar to Theorem 2, based on Theorem 4, we also have:

THEOREM 5. *Let A_i, B_i and $X_i \in M_n(\mathscr{C})$ $(i=1,2,\cdots,m)$ with X_i $(i=1,2,\cdots,m)$ are invertible matrices, and let h be a nonnegative increasing continuous function on $[0,+\infty)$ with $h(0)=0$. If f, g are nonnegative continuous functions on $[0,+\infty)$ with $f(t)g(t)=t$ for $t \in [0,+\infty)$, then*

$$\alpha \circ \left|\lambda\left(\left(\sum_{i=1}^{m} A_iX_i|X_i|^{-1}h(|X_i|)B_i\right)^{N}\right)\right| \prec_w \alpha \circ s\left(\frac{1}{p}\left(\sum_{i=1}^{m} A_if^2(h(|X_i^*|))A_i^*\right)^{\frac{Np}{2}} + \frac{1}{q}\left(\sum_{i=1}^{m} B_i^*g^2(h(|X_i|))B_i\right)^{\frac{Nq}{2}}\right), \tag{12}$$

for any positive integer N, where $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$, α_i, p and q are positive real numbers with $\frac{1}{p}+\frac{1}{q}=1$.

REMARK 5. Putting $N=1$, $p=q=2$ and $h(x)=x$ in Theorem 5, we get

$$\alpha \circ \left|\lambda\left(\sum_{i=1}^{m} A_iX_iB_i\right)\right| \prec_w \alpha \circ \frac{1}{2}s\left(\sum_{i=1}^{m} A_if^2(|X_i^*|)A_i^* + \sum_{i=1}^{m} B_i^*g^2(|X_i|)B_i\right).$$

Taking $f(t) = g(t) = \sqrt{t}$ in the above inequality, we have

$$\alpha \circ \left|\lambda\left(\sum_{i=1}^{m} A_i X_i B_i\right)\right| \prec_w \alpha \circ \frac{1}{2} s\left(\sum_{i=1}^{m} A_i |X_i^*| A_i^* + \sum_{i=1}^{m} B_i^* |X_i| B_i\right).$$

Since $|\lambda(X)| \prec_w s(X)$ and $s(X+Y) \prec_w s(X)+s(Y)$ for X, $Y \in M_n(\mathscr{C})$, then the inequality

$$s\left(\sum_{i=1}^{m} A_i X B_i\right) \prec_w \frac{1}{2}\left[\lambda\left(\sum_{i=1}^{m} A_i |X| A_i^*\right) + \lambda\left(\sum_{i=1}^{m} B_i^* |X^*| B_i\right)\right]$$

is not uniformly better than

$$\left|\lambda\left(\sum_{i=1}^{m} A_i X_i B_i\right)\right| \prec_w \frac{1}{2} s\left(\sum_{i=1}^{m} A_i |X_i^*| A_i^* + \sum_{i=1}^{m} B_i^* |X_i| B_i\right).$$

Thus, inequality (12) is another type of majorizations.

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