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AN INVERSE STURM–LIOUVILLE PROBLEM FROM PARTS OF THREE SPECTRA

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Abstract. Certain parts of the spectra of three Robin boundary value problems are used to find the potential of the Sturm-Liouville equation on a finite interval. The inverse problem possesses a unique solution. Conditions are found necessary and sufficient for three sequences to be the corresponding parts of the three spectra. Different from previous research, this paper emphasizes the importance of Robin boundary conditions in the study of three spectral inverse problem.

1. Introduction

In this paper, we consider the Sturm-Liouville boundary value problems (BVPs), denoted by $L(q,h_i,H_j)(=: L_{i,j})$ for $i,j=1,2,$ consisting of the Sturm-Liouville equation

$$-y''+q(x)y=\lambda^2 y \tag{1.1}$$

on the interval $[0,\pi]$ with the Robin boundary conditions

$$\begin{cases} y'(0)-h_i y(0)=0, \\ y'(\pi)+H_j y(\pi)=0. \end{cases} \tag{1.2}$$

We assume throughout this paper that the potential $q\in L^2(0,\pi)$ is real-valued and $h_i,H_j\in\mathbb{R}$ in (1.2).

Denote the spectrums of $L_{i,j}$ by $\sigma(L_{i,j})$, which consists of simple real eigenvalues $\{\lambda_n^2(i,j)\}_{n=0}^{+\infty}$, and form the sequence

$$-\infty<\lambda_0^2(i,j)<\lambda_1^2(i,j)<\lambda_2^2(i,j)<\cdots<\lambda_n^2(i,j)<\cdots \tag{1.3}$$

for $i,j=1,2$. It is well known [4, 5] that the eigenvalues $\lambda_n^2(i,j)$ have the following asymptotics

$$\lambda_n^2(i,j)=n^2+a_{i,j}+\alpha_n(i,j), \tag{1.4}$$

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where $\{\alpha_n(i,j)\}_{n=0}^{\infty} \in l^2$ and

$$a_{i,j} = \frac{2}{\pi}\left(h_i + H_j + \frac{1}{2}\int_0^{\pi} q(t)dt\right). \tag{1.5}$$

Also it is known [11, p. 93] that

$$\lambda_n(i,j) = n + \frac{a_{i,j}}{2n} + \frac{d(n)}{n} + \frac{\tilde{\alpha}_n(i,j)}{n^2}, \quad \{\tilde{\alpha}_n(i,j)\}_{n=-\infty,n\neq 0}^{+\infty} \in l^2, \tag{1.6}$$

where

$$d(n) = \frac{1}{2\pi}\int_0^{\pi} q(t)\cos(2nt)dt. \tag{1.7}$$

Many interesting papers are concerned with inverse spectral problems for Sturm-Liouville equations. Their research motivation for this problem is both in pure mathematics and physical applications (see, for example, the books [5, 11] and references therein). Borg [2], Chudov [3], Levitan [9] and Marchenko [11] have shown that two spectra $\sigma(L_{1,2})$ and $\sigma(L_{2,2})$ uniquely determine the potential function q and the boundary condition parameters h_1, h_2 and H_2. Recently, Pivovarchik [13] proved that given the parts of the Dirichlet–Dirichlet, Neumann–Dirichlet, Dirichlet–Neumann and Neumann–Neumann spectra are used to find the potential in (1.1).

The aim of the present paper is to solve the analogous inverse problem involved in three BVPs defined by (1.1)–(1.2). More precisely, we are concerned with the inverse problem by using the spectrum of $L_{1,2}$ and the parts of $L_{2,2}$ and $L_{1,1}$ spectra in a suitable situation for a unique determination of the potential q and the coefficients h_1, h_2, H_1 and H_2 of the Robin boundary conditions. Throughout this paper, we always assume that

$$h_1 > h_2, \quad H_1 < H_2.$$

We state the main result of this paper as follows.

THEOREM 1.1. *Let Λ and Λ^c be subsets of natural numbers such that $\Lambda \cap \Lambda^c = \emptyset$, $\Lambda \cup \Lambda^c = \mathbb{N}$. Let three increasing sequences $\{\lambda_n^2\}_{n=0}^{+\infty}$, $\{\mu_j^2\}_{j\in\Lambda}$ and $\{\mu_j^2\}_{j\in\Lambda^c}$ be given, which obey the following conditions:*

$$-\infty < \mu_0^2 < \lambda_0^2 < \mu_1^2 < \lambda_1^2 < \mu_2^2 < \lambda_2^2 < \cdots < \mu_n^2 < \lambda_n^2 < \cdots, \tag{1.8}$$

$$\lambda_n^2 = n^2 + a_0 + \sigma_{1,n}, \tag{1.9}$$

$$\mu_n^2 = n^2 + a_0' + \sigma_{2,n}, \tag{1.10}$$

and

$$\mu_n^2 - \lambda_n^2 = a_0' - a_0 + \frac{\sigma_n}{n}, \tag{1.11}$$

where $\{\mu_n^2\}_{n=0}^{+\infty} = \{\mu_j^2\}_{j\in\Lambda} \cup \{\mu_j^2\}_{j\in\Lambda^c}$, $a_0 > a_0'$, $\{\sigma_n\}_{n=0}^{+\infty}$ and $\{\sigma_{j,n}\}_{n=0}^{+\infty} \in l^2$ for $j = 1,2$. Then there exists a unique quintuple $(q; h_1, h_2, H_1, H_2) \in L^2(0,\pi) \times \mathbb{R}^4$ such that

$$a_0 - a_0' = \frac{2}{\pi}(h_1 - h_2), \quad H_1 = H_2 + h_2 - h_1, \tag{1.12}$$

$a_0 = a_{1,2}$, $a_0' = a_{2,2}$ *defined as* (1.5)*, and*

$$\{\lambda_n^2\}_{n=0}^{+\infty} = \sigma(L_{1,2}), \quad \{\mu_j^2\}_{j\in\Lambda} \subset \sigma(L_{2,2}), \quad \{\mu_j^2\}_{j\in\Lambda^c} \subset \sigma(L_{1,1}). \tag{1.13}$$

Conversely, if a quintuple $(q;h_1,h_2,H_1,H_2) \in L^2(0,\pi)\times\mathbb{R}^4$ *satisfies* (1.12)–(1.13)*, then* (1.8)–(1.11) *all hold.*

REMARK 1.2. The above result provides an existence-uniqueness for inverse spectral problems, which can be regarded as a generalization of the results in [12, 13]. The similar result for the Dirichlet-Robin boundary conditions can be obtained, according to the line of the paper.

REMARK 1.3. The assumption that the potential is of $L^2(0,\pi)$ is necessary, which enables us to use interpolation in the Paley-Wiener class using the results of [14].

The technique, used to prove the above theorem, is based on Levitan-Gasymov's theorem [6, Theorem 3.4.2] which gives a necessary and sufficient condition for two sequences of real numbers to be two spectra of BVPs $L_{1,2}$ and $L_{2,2}$, respectively. As concrete applications of it, for L^2 potential q in (1.1), we shall establish a variety of the theorem for treating our problem.

The paper is organized as follows. In Section 2, we shall give necessary preliminaries. Section 3 is to give the proof of Theorem 1.1.

2. Preliminary

In this section, we shall establish a variety of the existence-uniqueness theorem of Levitan-Gasymov [6] for inverse Sturm-Liouville problems for the case that $q \in L^2(0,\pi)$. This will be used later to prove our principal result, Theorem 1.1 above, of this paper.

Throughout this paper, we will denote by

(1) $\mathscr{L}^a$ the Paley–Wiener class of entire functions of exponential type $\leqslant a$ which belong to $L^2(-\infty,\infty)$ for real λ;

(2) $\mathscr{S}^a$ the sine type class of entire functions of exponential type $\leqslant a$ which satisfies that the zeros of $f(z)$ are separated, and there exist positive constants A, B and H such that

$$Ae^{a|y|} \leqslant |f(x+iy)| \leqslant Be^{a|y|}$$

whenever x and y are real and $|y| \geqslant H$.

From [14, Corollary 1], if $f \in \mathscr{S}^a$ and $\{z_k\}_{k\in\mathbb{Z}}$ are its zeros, then

$$\inf_{k\in\mathbb{Z}_0} |\dot{f}(z_k)| > 0.$$

Moreover, a function of sine type is bounded on the real axis and so must have infinitely many zeros. The zeros are all simple and lie in a strip parallel to the real axis.

Let $\varphi_i(x,\lambda)$ be the solutions of the Sturm-Liouville equation (1.1) satisfying the initial conditions

$$\varphi_i(0,\lambda)=1, \quad \varphi_i'(0,\lambda)=h_i \tag{2.1}$$

for $i=1,2$. According to [11, pp. 9–14],

$$\begin{aligned}\varphi_i(x,\lambda) &= \cos(\lambda x)+\int_0^x K_i(x,t)\cos(\lambda t)dt\\ &= \cos(\lambda x)+K_i(x,x)\frac{\sin(\lambda x)}{\lambda}-\frac{1}{\lambda}\int_0^x \frac{\partial K_i(x,t)}{\partial t}\sin(\lambda t)dt,\end{aligned}$$

where

$$K_i(x,t)=h_i+\tilde{K}(x,t)+\tilde{K}(x,-t)+h_i\int_t^x \left(\tilde{K}(x,s)-\tilde{K}(x,-s)\right)ds,$$

and $\tilde{K}(x,t)$ is the solution of the integral equation

$$\tilde{K}(x,t)=\frac{1}{2}\int_0^{\frac{x+t}{2}} q(s)ds+\int_0^{\frac{x+t}{2}} d\alpha\int_0^{\frac{x-t}{2}} q(\alpha+\beta)\tilde{K}(\alpha+\beta,\alpha-\beta)d\beta.$$

The solution $\tilde{K}(x,t)$ possesses partial derivatives of first order each belonging to $L^2(0,\pi)$ as a function of each of its variables when the other variable is fixed. Moreover, $\tilde{K}(x,0)=0$ and

$$\tilde{K}(x,x)=\frac{1}{2}\int_0^x q(t)dt.$$

Based on the three spectra of the BVPs $L_{1,1}$, $L_{1,2}$ and $L_{2,2}$, we have

LEMMA 2.1. *If $h_1>h_2$ and $H_2>H_1$, then, for all $n\in\mathbb{N}$, we have*

$$\max\{\lambda_n^2(1,1),\lambda_n^2(2,2)\}<\lambda_n^2(1,2)<\min\{\lambda_{n+1}^2(1,1),\lambda_{n+1}^2(2,2)\}. \tag{2.2}$$

Proof. It is known [6, Lemma 3.1.1] that the eigenvalue λ_n^2 of the BVP $L(q,h,H)$ is increasing at H $(h)\in\mathbb{R}$ for each fixed $n\in\mathbb{N}$. On the other hand, it is also known that the eigenvalues of $L(q,h_j,H_j)$ are interlaced with that of $L(q,h_1,H_2)$, where $j=1,2$. These facts imply that (2.2) holds. □

Before proving Theorem 2.3 below, we first recall an elementary result for the set $\{\lambda_n^2,\alpha_n\}_{n=0}^{+\infty}$ to be the spectral data for a certain boundary value problem $L(q,h_1,H_2)$ with $q\in L^2(0,\pi)$, see [4, Theorem 1.5.2].

LEMMA 2.2. *For a set $\{\lambda_n^2,\alpha_n\}_{n=0}^{+\infty}$ to be the spectral data for a certain boundary value problem $L(q,h_1,H_2)$ with $q\in L^2(0,\pi)$, it is necessary and sufficient that the following relations are satisfied*

$$\begin{aligned}&\lambda_n^2=n^2+a_0+\varepsilon_n, \quad \alpha_n=\frac{\pi}{2}+\frac{\kappa_n}{n},\\ &\alpha_n>0, \quad \lambda_n^2\neq\lambda_m^2\ (n\neq m),\end{aligned}$$

where $\{\varepsilon_n\}_{n=0}^{+\infty}$, $\{\kappa_n\}_{n=0}^{+\infty} \in l^2$, $a_0 = \frac{2}{\pi}\left(h_1 + H_2 + \frac{1}{2}\int_0^{\pi} q(t)dt\right)$ *and* α_n *is the normalizing constant corresponding to the eigenvalue* $\lambda_n^2(1,2)$, *which is defined by*

$$\alpha_n = \int_0^{\pi} \varphi_1^2(x,\lambda_n^2)dx. \tag{2.3}$$

Here $\varphi_1(x,\lambda_n^2)$ *is the eigenfunction corresponding to eigenvalue* λ_n^2, *which satisfies the initial-value conditions* (2.1) *for* $i = 1$.

The following theorem is the reformulation of Levitan-Gasymov's theorem [6, Theorem 3.4.2] that is used in the proof of Theorem 1.1.

THEOREM 2.3. *Let two sequences* $\{\lambda_n^2\}_{n=0}^{+\infty}$ *and* $\{\mu_n^2\}_{n=0}^{+\infty}$ *satisfy the following conditions that*

$$-\infty < \mu_0^2 < \lambda_0^2 < \mu_1^2 < \lambda_1^2 < \mu_2^2 < \lambda_2^2 < \cdots < \mu_n^2 < \lambda_n^2 < \cdots \tag{2.4}$$

and

$$\lambda_n^2 = n^2 + a_0 + \sigma_{1,n}, \tag{2.5}$$

$$\mu_n^2 = n^2 + a_0' + \sigma_{2,n} \tag{2.6}$$

with $a_0 > a_0'$, *where* $\{\sigma_{j,n}\}_{n=0}^{+\infty} \in l^2$ *for* $j = 1,2$. *Then necessary and sufficient condition for sequences* $\{\lambda_n^2\}_{n=0}^{+\infty}$ *and* $\{\mu_n^2\}_{n=0}^{+\infty}$ *to be the eigenvalues of two BVPs* $L(q,h_1,H_2)$ *and* $L(q,h_2,H_2)$, *respectively, is that*

$$\mu_n^2 - \lambda_n^2 = a_0' - a_0 + \frac{\sigma_n}{n} \tag{2.7}$$

with $\{\sigma_n\}_{n=0}^{+\infty} \in l^2$ *and* $h_2 - h_1 = \pi(a_0' - a_0)/2$, *where the tetrad* $(q;h_1,h_2,H_2) \in L^2(0,\pi) \times \mathbb{R}^3$.

Proof. Necessity. We construct α_n by means of the following formula

$$\alpha_n := \frac{h_2 - h_1}{\mu_n^2 - \lambda_n^2} \prod_{k=0}^{+\infty}{}' \left(1 + \frac{\lambda_k^2 - \mu_k^2}{\mu_k^2 - \lambda_n^2}\right) \tag{2.8}$$

with $n \in \mathbb{N}$, where $\prod_{k=0}^{+\infty}{}'$ means that the factor $k = n$ is absent from the infinite product. We shall prove that, for all $n \in \mathbb{N}$,

$$\alpha_n = \frac{\pi}{2} + \frac{\kappa_n}{n} > 0 \tag{2.9}$$

with $\{\kappa_n\}_{n=0}^{+\infty} \in l^2$.

By virtue of (2.7) and the assumption of $h_2 - h_1 = \pi(a_0' - a_0)/2$, we have

$$\frac{h_2 - h_1}{\mu_n^2 - \lambda_n^2} = \frac{\pi}{2} + \frac{\tilde{\kappa}_n}{n}, \tag{2.10}$$

where $\{\tilde{\kappa}_n\}_{n=0}^{\infty} \in l^2$. Let

$$\Psi(\lambda_n) = \prod_{k=0}^{+\infty}{}' \left(1 + \frac{\lambda_k^2 - \mu_k^2}{\mu_k^2 - \lambda_n^2}\right). \tag{2.11}$$

Now we estimate the asymptotics of $\Psi(\lambda_n)$. Using (2.5)–(2.7), we arrive at

$$\begin{aligned} \left|\frac{\lambda_k^2 - \mu_k^2}{\mu_k^2 - \lambda_n^2}\right| &= \frac{|a_0' - a_0 - \frac{\sigma_k}{k}|}{|\mu_k + \lambda_n| \cdot |\mu_k - \lambda_n|} \\ &\leqslant \frac{|a_0' - a_0 - \frac{\sigma_k}{k}|}{\lambda_n |\mu_k - \lambda_n|} \\ &\leqslant \frac{C_0}{n} < 1 \end{aligned} \tag{2.12}$$

for sufficiently large n and $k \neq n$, where C_0 denotes a positive constant. Moreover, by the hypothesises of Theorem 2.3, we have

$$1 + \frac{\lambda_k^2 - \mu_k^2}{\mu_k^2 - \lambda_n^2} = \frac{\lambda_k^2 - \lambda_n^2}{\mu_k^2 - \lambda_n^2} > 0 \tag{2.13}$$

for all $k \in \mathbb{N}$ and $k \neq n$. Therefore, in terms of (2.12) and (2.13), the identity (2.11) can be rewritten as the following form (see also [6, p. 33])

$$\begin{aligned} \ln \Psi(\lambda_n) &= \sum_{k=0}^{+\infty}{}' \ln\left(1 + \frac{\lambda_k^2 - \mu_k^2}{\mu_k^2 - \lambda_n^2}\right) \\ &= -\sum_{k=0}^{+\infty}{}' \left[\sum_{p=1}^{+\infty} \frac{(-1)^p}{p} \left(\frac{\lambda_k^2 - \mu_k^2}{\mu_k^2 - \lambda_n^2}\right)^p\right] \\ &= \sum_{k=0}^{+\infty}{}' \frac{\lambda_k^2 - \mu_k^2}{\mu_k^2 - \lambda_n^2} + \sum_{k=0}^{+\infty}{}' \sum_{p=2}^{+\infty} \frac{(-1)^{p+1}}{p} \left(\frac{\lambda_k^2 - \mu_k^2}{\mu_k^2 - \lambda_n^2}\right)^p, \end{aligned} \tag{2.14}$$

where $\sum_{k=0}^{+\infty}{}'$ means that the factor $k = n$ is absent from the sum. Here, we have used an elementary result

$$\ln(1+x) = \sum_{k=1}^{+\infty} \frac{(-1)^{p+1}}{p} x^p \quad \text{for } |x| < 1.$$

According to [6, Lemma 2.2.1], we obtain

$$\sum_{k=0}^{+\infty}{}' \left|\frac{\lambda_k^2 - \mu_k^2}{\mu_k^2 - \lambda_n^2}\right|^p \leqslant M \frac{a^p}{n^p} \tag{2.15}$$

for $p \geqslant 2$, where $a = \max\{|\mu_k^2 - \lambda_k^2|,\ k \in \mathbb{N}\}$, and M denotes a positive constant independent of k and n. On the other hand, we have

$$\left|\sum_{k=0}^{+\infty}{}' \sum_{p=2}^{+\infty} \frac{(-1)^{p+1}}{p}\left(\frac{\lambda_k^2-\mu_k^2}{\mu_k^2-\lambda_n^2}\right)^p\right| = \left|\sum_{p=2}^{+\infty} \frac{(-1)^{p+1}}{p}\sum_{k=0}^{+\infty}{}' \left(\frac{\lambda_k^2-\mu_k^2}{\mu_k^2-\lambda_n^2}\right)^p\right|$$
$$\leqslant \sum_{p=2}^{+\infty}\frac{1}{p}\sum_{k=0}^{+\infty}{}' \left|\frac{\lambda_k^2-\mu_k^2}{\mu_k^2-\lambda_n^2}\right|^p. \tag{2.16}$$

Substituting (2.15) into the right-hand side of (2.16), we infer that

$$\left|\sum_{k=0}^{+\infty}{}' \sum_{p=2}^{+\infty} \frac{(-1)^{p}}{p}\left(\frac{\lambda_k^2-\mu_k^2}{\mu_k^2-\lambda_n^2}\right)^p\right| \leqslant \sum_{p=2}^{+\infty}\frac{M}{p}\frac{a^p}{n^p} \leqslant C\sum_{p=2}^{+\infty}\frac{a^p}{n^p}$$
$$= C\frac{a^2}{n^2}\sum_{p=0}^{+\infty}\frac{a^p}{n^p} = O\left(\frac{1}{n^2}\right), \tag{2.17}$$

where $C>0$ is a constant. Applying (2.17) to (2.14), we get

$$\ln\Psi(\lambda_n) = \sum_{k=0}^{+\infty}{}' \frac{\lambda_k^2-\mu_k^2}{\mu_k^2-\lambda_n^2} + O\left(\frac{1}{n^2}\right) \tag{2.18}$$

for sufficiently large n.

We now consider the asymptotics of the sum appearing in the above formula. Using (2.5) and (2.7), we get

$$\sum_{k=0}^{+\infty}{}' \frac{\lambda_k^2-\mu_k^2}{\mu_k^2-\lambda_n^2} = \frac{\lambda_0^2-\mu_0^2}{\mu_0^2-\lambda_n^2} + \sum_{k=1}^{+\infty}{}' \frac{a_0-a_0'-\frac{\sigma_k}{k}}{\mu_k^2-\lambda_n^2}$$
$$= (a_0-a_0')\sum_{k=1}^{+\infty}{}' \frac{1}{\mu_k^2-\lambda_n^2} + \sum_{k=1}^{+\infty}{}' \frac{\sigma_k}{k(\mu_k^2-\lambda_n^2)} + O\left(\frac{1}{n^2}\right)$$
$$= (a_0-a_0')\sum_{k=1}^{+\infty}{}' \frac{1}{\mu_k^2-\lambda_n^2} + \frac{1}{\lambda_n^2}\sum_{k=1}^{+\infty}{}' \frac{\sigma_k}{k(\frac{\mu_k^2}{\lambda_n^2}-1)} + O\left(\frac{1}{n^2}\right) \tag{2.19}$$

for sufficiently large n. Clearly, the second series in the right-hand side of (2.19) converges. It follows from (2.5) and (2.19) that

$$\sum_{k=0}^{+\infty}{}' \frac{\lambda_k^2-\mu_k^2}{\mu_k^2-\lambda_n^2} = (a_0-a_0')\sum_{k=1}^{+\infty}{}' \frac{1}{\mu_k^2-\lambda_n^2} + O\left(\frac{1}{n^2}\right). \tag{2.20}$$

By (2.6), we see that

$$\begin{aligned}\sum_{k=1}^{+\infty}{}' \frac{1}{\mu_k^2-\lambda_n^2} &= \sum_{k=1}^{+\infty}{}' \frac{1}{k^2+a_0'+\sigma_{2,k}-\lambda_n^2}\\ &= \sum_{k=1}^{+\infty}{}' \frac{1}{(k^2-\lambda_n^2)(1+\frac{a_0'+\sigma_{2,k}}{k^2-\lambda_n^2})}\\ &= \sum_{k=1}^{+\infty}{}' \frac{1}{k^2-\lambda_n^2} - \sum_{k=1}^{+\infty}{}' \frac{1}{(k^2-\lambda_n^2)^2}\left(\frac{a_0'+\sigma_{2,k}}{1+\frac{a_0'+\sigma_{2,k}}{k^2-\lambda_n^2}}\right). \end{aligned} \tag{2.21}$$

Apparently,

$$\left|\frac{a_0'+\sigma_{2,k}}{1+\frac{a_0'+\sigma_{2,k}}{k^2-\lambda_n^2}}\right| \leqslant a_1$$

for all $k,n\in\mathbb{N}$ and $k\neq n$, where $a_1>0$ is a constant. Therefore,

$$\sum_{k=1}^{+\infty}{}' \frac{1}{(k^2-\lambda_n^2)^2}\left(1+\frac{a_0'+\sigma_{2,k}}{k^2-\lambda_n^2}\right) \leqslant a_1 \sum_{k=1}^{+\infty}{}' \frac{1}{(k^2-\lambda_n^2)^2}. \tag{2.22}$$

Using [6, Lemma 2.2.1] again, we arrive at

$$\sum_{k=1}^{+\infty}{}' \frac{1}{(k^2-\lambda_n^2)^2} = O\left(\frac{1}{n^2}\right).$$

This together with (2.21) and (2.22) yields that

$$\sum_{k=1}^{+\infty}{}' \frac{1}{\mu_k^2-\lambda_n^2} = \sum_{k=1}^{+\infty}{}' \frac{1}{k^2-\lambda_n^2} + O\left(\frac{1}{n^2}\right). \tag{2.23}$$

Now let's consider the asymptotics of the sum appearing in the right-hand side of (2.23). By (2.5), we see that

$$\lambda_n = n+\varepsilon_n, \quad \varepsilon_n = O\left(\frac{1}{n}\right).$$

According to [6, p. 36], we infer that

$$\begin{aligned}\sum_{k=1}^{+\infty}{}' \frac{1}{k^2-\lambda_n^2} &= \frac{1}{2\lambda_n^2} - \left(\frac{\pi}{2\lambda_n}\cot(\pi\lambda_n)+\frac{1}{n^2-\lambda_n^2}\right)\\ &= \frac{1}{2\lambda_n^2} - \left(\frac{\pi\cos\pi(n+\varepsilon_n)}{2(n+\varepsilon_n)\sin\pi(n+\varepsilon_n)} - \frac{1}{(2n+\varepsilon_n)\varepsilon_n}\right)\\ &= \frac{1}{2\lambda_n^2} - \frac{\pi\varepsilon_n(2n+\varepsilon_n)\cos(\pi\varepsilon_n)-2(n+\varepsilon_n)\sin(\pi\varepsilon_n)}{2\varepsilon_n(n+\varepsilon_n)(2n+\varepsilon_n)\sin(\pi\varepsilon_n)}\\ &= O\left(\frac{1}{n^2}\right). \end{aligned}$$

This together with (2.19), (2.20) and (2.23) yields that

$$\sum_{k=0}^{+\infty}{}' \frac{\lambda_k^2-\mu_k^2}{\mu_k^2-\lambda_n^2} = O\left(\frac{1}{n^2}\right) \tag{2.24}$$

for sufficiently large n. Applying (2.24) to (2.18), we find that

$$\ln\Psi(\lambda_n) = O\left(\frac{1}{n^2}\right),$$

i.e.,

$$\Psi(\lambda_n) = 1+O\left(\frac{1}{n^2}\right) \tag{2.25}$$

for sufficiently large n. Substituting (2.10) and (2.25) into the right-hand side of (2.8) we obtain that the numbers α_n have the following asymptotics

$$\alpha_n = \frac{\pi}{2}+\frac{\kappa_n}{n} \tag{2.26}$$

with $\{\kappa_n\}_{n=0}^{+\infty} \in l^2$.

We shall show that $\alpha_n > 0$ for all $n \in \mathbb{N}$. Let $z=\lambda^2$. By (2.4), the zeros and poles of the function

$$m_1(\lambda^2) = m_1(z) = -\frac{1}{h_2-h_1}\prod_{k=0}^{+\infty}\frac{\mu_k^2-z}{\lambda_k^2-z} \tag{2.27}$$

are alternating. Clearly,

$$m_1(z) \to -\frac{1}{h_2-h_1}$$

as $|z| \to +\infty$. From (2.8) and (2.27), we get

$$\frac{1}{\alpha_n} = \mathrm{Res}_{z=\lambda_n^2} m_1(z).$$

Then according to [1, 7], we have

$$m_1(\lambda^2) = m_1(z) = -\frac{1}{h_2-h_1}+\sum_{n=0}^{+\infty}\frac{1}{\alpha_n(\lambda^2-\lambda_n^2)}, \tag{2.28}$$

where all the α_n are of the same sign. Due to $\alpha_n > 0$ for sufficiently large n (cf. (2.26)), and hence we see that $\alpha_n > 0$ for all $n \in \mathbb{N}$. This together with (2.26) shows that (2.9) holds.

By (2.5), (2.9) and Lemma 2.2, the set $\{\lambda_n^2, \alpha_n\}_{n=0}^{+\infty}$ determines the potential $q \in L^2(0,\pi)$ and the coefficients h_1 and H_2 of the Robin boundary conditions uniquely, such that the set $\{\lambda_n^2\}_{n=0}^{+\infty}$ is the spectrum of $L(q,h_1,H_2)$ and α_n is the normalizing constant corresponding to the eigenvalue λ_n^2, which is defined by (2.3).

As in the proof of [6, Theorem 2.3.1], we define the number h_2 by the equality

$$h_2 = h_1 + \frac{\pi(a_0' - a_0)}{2}. \tag{2.29}$$

We shall prove that $\{\mu_n^2\}_{n=0}^{+\infty} = \sigma(L_{2,2})$. Let the set $\{\tilde{\mu}_n^2\}_{n=0}^{+\infty}$ be the spectrum of $L(q, h_2, H_2)$. We shall show that $\mu_n^2 = \tilde{\mu}_n^2$ for all $n \in \mathbb{N}$. Recall that $\varphi_i(x,\lambda)$, $i = 1,2$, denote the solutions of (1.1) subject to the initial conditions (2.1). Clearly, the poles and zeros of the function

$$m(\lambda) = -\frac{\varphi_2'(\pi,\lambda) + H_2\varphi_2(\pi,\lambda)}{\varphi_1'(\pi,\lambda) + H_2\varphi_1(\pi,\lambda)}$$

coincide with the numbers $\{\lambda_n^2\}_{n=0}^{+\infty}$ and $\{\tilde{\mu}_n^2\}_{n=0}^{+\infty}$, respectively. According to Lemma 2.1, the sequences $\{\lambda_n^2\}_{n=0}^{+\infty}$ and $\{\tilde{\mu}_n^2\}_{n=0}^{+\infty}$ are alternating, and

$$m(\lambda) \to -1$$

as $|\lambda| \to +\infty$. On the other side, by eq. (2.1.13) in [6] (see also [6, p. 39]), we infer that

$$\frac{1}{\alpha_n} = \frac{1}{h_2 - h_1} \mathrm{Res}_{\lambda = \lambda_n^2} m(\lambda).$$

By [1, 7], we can see that

$$\frac{1}{h_2 - h_1} m(\lambda) = -\frac{1}{h_2 - h_1} + \sum_{n=0}^{+\infty} \frac{1}{\alpha_n(\lambda^2 - \lambda_n^2)}.$$

This together with (2.28) shows that

$$m_1(\lambda^2) = m_1(z) = \frac{1}{h_2 - h_1} m(\lambda). \tag{2.30}$$

The relation (2.30) implies that $\mu_n^2 = \tilde{\mu}_n^2$ for all $n \in \mathbb{N}$. Basing on the above discussions, we conclude that the set $\{\mu_n^2\}_{n=0}^{+\infty}$ is the spectrum of $L(q, h_2, H_2)$.

Sufficiency. If there exists $(q; h_1, h_2, H_2) \in L^2(0,\pi) \times \mathbb{R}^3$ such that $\{\lambda_n^2\}_{n=0}^{+\infty} = \sigma(L(q; h_1, H_2))$ and $\{\mu_n^2\}_{n=0}^{+\infty} = \sigma(L(q; h_2, H_2))$, then by (1.6)–(1.7) (see also [11, p. 93]), we have

$$\lambda_n = n + \frac{d(1,2)}{n} + \frac{d(n)}{n} + \frac{\delta_{1,n}}{n^2}, \tag{2.31}$$

$$\mu_n = n + \frac{d(2,2)}{n} + \frac{d(n)}{n} + \frac{\delta_{2,n}}{n^2}, \tag{2.32}$$

where $\{\delta_{i,n}\}_{n=0}^{+\infty} \in l^2$ and

$$d(i,2) = \frac{1}{\pi}\left(h_i + H_2 + \frac{1}{2}\int_0^\pi q(t)dt\right), \tag{2.33}$$

$$d(n) = \frac{1}{2\pi}\int_0^\pi q(t)\cos(2nt)dt \tag{2.34}$$

for $i = 1,2$. Using (2.31)–(2.34), we easily see that (2.5)–(2.7) hold.

On the other hand, by Lemma 2.1 and the assumption of $h_2 > h_1$, we arrive at (2.4). This completes the proof of Theorem 2.3. $\square$

3. The proof of Theorem 1.1

In this section, based on Theorem 2.3, we give the proof of Theorem 1.1. Recall that $\varphi_i(x,\lambda)$ are the solutions of the Sturm-Liouville equation (1.1) which satisfy the initial-value conditions (2.1) for $i=1,2$. According to [5], the characteristic functions $f_{i,j}(\lambda) := \varphi_i'(\pi,\lambda)+H_j\varphi_i(\pi,\lambda)$ of BVPs $L_{i.j}$ behave asymptotically as follows

$$f_{i,j}(\lambda) = -\lambda\sin(\lambda\pi)+\frac{\pi}{2}a_{i,j}\cos(\lambda\pi)+\alpha_{i,j}(\lambda), \tag{3.1}$$

where $a_{i,j}$ are defined by (1.5) and $\alpha_{i,j}(\lambda)\in\mathscr{L}^\pi$ for $i,j=1,2$. It is well known [4, p. 6] that the Wronskian $\varphi_1(x,\lambda)\varphi_2'(x,\lambda)-\varphi_2(x,\lambda)\varphi_1'(x,\lambda)$ does not depend on $x\in[0,\pi]$. It follows from (2.1) that

$$\begin{aligned} h_2-h_1 &= \begin{vmatrix} \varphi_1(\pi,\lambda) & \varphi_2(\pi,\lambda) \\ \varphi_1'(\pi,\lambda) & \varphi_2'(\pi,\lambda) \end{vmatrix} \\ &= \frac{1}{H_1-H_2}\begin{vmatrix} \varphi_1'(\pi,\lambda)+H_1\varphi_1(\pi,\lambda) & \varphi_1'(\pi,\lambda)+H_2\varphi_1(\pi,\lambda) \\ \varphi_2'(\pi,\lambda)+H_1\varphi_2(\pi,\lambda) & \varphi_2'(\pi,\lambda)+H_2\varphi_2(\pi,\lambda) \end{vmatrix}. \end{aligned} \tag{3.2}$$

This together with (1.12) and (3.1) yields

$$\begin{vmatrix} f_{1,1}(\lambda) & f_{1,2}(\lambda) \\ f_{2,1}(\lambda) & f_{2,2}(\lambda) \end{vmatrix} = \frac{\pi^2}{4}(a_0-a_0')^2. \tag{3.3}$$

The above identity is crucial in the proof of Theorem 1.1.

We are now in a position to prove Theorem 1.1.

Proof. Let two sequences $\{\lambda_n^2\}_{n=0}^{+\infty}$ and $\{\mu_n^2\}_{n=0}^{+\infty}$ satisfy (2.4)–(2.7). By Theorem 2.3, there exists a unique tetrad $(\hat{q};\hat{h}_1,\hat{h}_2,\hat{H}_2)\in L^2(0,\pi)\times\mathbb{R}^3$, such that the set $\{\lambda_n^2\}_{n=0}^{+\infty}$ is the spectrum of BVP $L(\hat{q},\hat{h}_1,\hat{H}_2)(=:\hat{L}_{1,2})$ and $\{\mu_n^2\}_{n=0}^{+\infty}$ is the spectrum of BVP $L(\hat{q},\hat{h}_2,\hat{H}_2)(=:\hat{L}_{2,2})$.

For easy comprehension of the readers, we denote by

$$\lambda_n^2 = \gamma_n^2(1,2), \qquad \mu_n^2 = \gamma_n^2(2,2).$$

In the following, we shall prove the theorem throughout three steps.

Step 1. *Find* $(q;h_1,h_2,H_2)\in L^2(0,\pi)\times\mathbb{R}^3$.

Firstly, we need to find the spectrum of $\sigma(\hat{L}_{1,1})$ by $\hat{q}(x)$ and $\hat{h}_1,\hat{h}_2,\hat{H}_2$. We assume that $0\notin\{\gamma_k^2(2,2)\}_{k=0}^{+\infty}$, otherwise, we can shift the spectral parameter $\lambda^2\to\lambda^2+c$ with $c>0$. By using [4, Theorem 1.1.4], we have

$$\hat{f}_{2,2}(\lambda) := \pi(\gamma_0^2(2,2)-\lambda^2)\prod_{k=1}^{+\infty}\frac{(\gamma_k^2(2,2)-\lambda^2)}{k^2},$$

which is the characteristic function of BVP $\hat{L}_{2,2}$. Let $\hat{\varphi}_2(\pi,\lambda)$ denote the solution of Eq. (1.1), with replacing q by $\hat{q}$, subject to the initial conditions (2.1), with replacing

h_i by $\hat{h}_2$. Taking [5] into account, we also have

$$\begin{aligned}\hat{f}_{2,2}(\lambda) =&\hat{\varphi}_2'(\pi,\lambda)+\hat{H}_2\hat{\varphi}_2(\pi,\lambda)\\ =&-\lambda\sin(\lambda\pi)+(\frac{\pi}{2}a_0')\cos(\lambda\pi)+\hat{\psi}_{2,2}(\lambda),\end{aligned}$$

where $\hat{\psi}_{2,2}(\lambda)\in\mathscr{L}^{\pi}$ and

$$a_0'=\frac{2}{\pi}\left(\hat{h}_2+\hat{H}_2+\frac{1}{2}\int_0^{\pi}\hat{q}(t)dt\right).$$

Similarly, according to [4, Theorem 1.1.4], we have

$$\hat{f}_{1,2}(\lambda)=\pi(\gamma_0^2(1,2)-\lambda^2)\prod_{k=1}^{+\infty}\frac{(\gamma_k^2(1,2)-\lambda^2)}{k^2}, \tag{3.4}$$

and

$$\begin{aligned}\hat{f}_{1,2}(\lambda) =& \hat{\varphi}_1'(\pi,\lambda)+\hat{H}_2\hat{\varphi}_{1,2}(\pi,\lambda)\\ =&-\lambda\sin(\lambda\pi)+(\frac{\pi}{2}a_0)\cos(\lambda\pi)+\hat{\psi}_{1,2}(\lambda),\end{aligned} \tag{3.5}$$

where $\hat{\psi}_{1,2}(\lambda)\in\mathscr{L}^{\pi}$ and

$$a_0=\frac{2}{\pi}\left(\hat{h}_1+\hat{H}_2+\frac{1}{2}\int_0^{\pi}\hat{q}(t)dt\right).$$

By (3.2)–(3.3), under the condition that $\hat{H}_1=\hat{h}_2-\hat{h}_1+\hat{H}_2$, we have

$$\begin{vmatrix}\hat{f}_{1,1}(\lambda) & \hat{f}_{1,2}(\lambda)\\ \hat{f}_{2,1}(\lambda) & \hat{f}_{2,2}(\lambda)\end{vmatrix}=\frac{\pi^2}{4}(a_0-a_0')^2, \tag{3.6}$$

where $\hat{f}_{i,j}(\lambda)$ are the characteristic functions of BVPs $\hat{L}_{i,j}$ for $i,j=1,2$. By (3.1), we have

$$\hat{f}_{1,1}(\lambda)=-\lambda\sin(\lambda\pi)+(\frac{\pi}{2}a_0')\cos(\lambda\pi)+\hat{\psi}_{1,1}(\lambda), \tag{3.7}$$

where $\hat{\psi}_{1,1}(\lambda)\in\mathscr{L}^{\pi}$.

Let us consider an interpolation problem in the Paley-Wiener space $\mathscr{L}^{\pi}$. We choose $\{\gamma_k^2(1,2)\}_{k=0}^{+\infty}$ as the nodes of interpolation for finding the function $\hat{\psi}_{1,1}(\lambda)$ in (3.7), and the values at the nodes, by (3.6), we find

$$\hat{\psi}_{1,1}(\gamma_k(1,2))=\gamma_k(1,2)\sin(\gamma_k(1,2)\pi)-(a_0'\pi/2)\cos(\gamma_k(1,2)\pi)-\frac{\pi^2(a_0-a_0')^2}{4\hat{f}_{2,2}(\gamma_k(1,2))}. \tag{3.8}$$

Now, we estimate the asymptotics of $\hat{\psi}_{1,1}(\gamma_k(1,2))$. Using the asymptotics of the eigenvalues (see (1.9)), we get

$$\gamma_k(1,2)=k+\frac{a_0}{2k}+\frac{\alpha_{1,k}}{k},$$

it follows that

$$\begin{cases} \sin(\gamma_k(1,2)\pi) = (-1)^k \dfrac{a_0}{k} + \dfrac{\delta_{1,k}}{k}, \\ \cos(\gamma_k(1,2)\pi) = (-1)^k + \dfrac{\delta_{2,k}}{k}, \end{cases} \tag{3.9}$$

where $\{\alpha_{1,k}\}_{k=-\infty,k\neq 0}^{+\infty}, \{\delta_{j,k}\}_{k=-\infty,k\neq 0}^{+\infty} \in l^2$ for $j=1,2$. From (3.8)–(3.9), we get

$$\{\hat{\psi}_{1,1}(\gamma_k(1,2))\}_{k=-\infty,k\neq 0}^{+\infty} \in l^2. \tag{3.10}$$

Let

$$\omega(\lambda) = \frac{\lambda \hat{f}_{1,2}(\lambda)}{\lambda^2 - \gamma_0^2(1,2)}.$$

Then, by (3.5), the function $\omega(\lambda)$ is of sine-type. Therefore, taking (3.4) and (3.10) into account and utilizing Theorem A in [8], we infer

$$\hat{\psi}_{1,1}(\lambda) = \omega(\lambda) \sum_{k=-\infty,k\neq 0}^{+\infty} \frac{\hat{\psi}_{1,1}(\gamma_k(1,2))}{\left.\frac{d\omega(\lambda)}{d\lambda}\right|_{\lambda=\gamma_k(1,2)} (\lambda - \gamma_k(1,2))}. \tag{3.11}$$

According to [8], we mention that the obtained $\hat{\psi}_{1,1}(\lambda)$ is the unique solution of the following interpolation problem: given the nodes $\{\gamma_k(1,2)\}_{k=-\infty,k\neq 0}^{+\infty}$ and the values $\{\hat{\psi}_{1,1}(\gamma_k(1,2))\}_{k=-\infty,k\neq 0}^{+\infty}$ at these nodes, find $\hat{\psi}_{1,1}(\lambda)$.

We substitute (3.11) into (3.7) and find $\hat{f}_{1,1}(\lambda)$. We denote by $\{\gamma_k(1,1)\}_{k=-\infty}^{+\infty}$ the zeros of $\hat{f}_{1,1}(\lambda)$. Then

$$\hat{f}_{1,1}(\lambda) = \pi(\gamma_0^2(1,1) - \lambda^2) \prod_{k=0}^{+\infty} \frac{(\gamma_k^2(1,1) - \lambda^2)}{k^2}, \tag{3.12}$$

and

$$\gamma_k^2(1,1) = k^2 + a_0' + \hat{\alpha}_k, \tag{3.13}$$

where $\{\hat{\alpha}_k\}_{k=0}^{+\infty} \in l^2$. By (1.6)–(1.7) and taking Theorem 2.3 into account, we have

$$\begin{cases} \gamma_k^2(1,1) - \gamma_k^2(1,2) = a_0' - a_0 + \dfrac{\hat{\beta}_{1,k}}{k}, \\ \gamma_k^2(2,2) - \gamma_k^2(1,2) = a_0' - a_0 + \dfrac{\hat{\beta}_{2,k}}{k}, \end{cases} \tag{3.14}$$

where $\{\hat{\beta}_{j,k}\}_{k=0}^{+\infty} \in l^2$ for $j=1,2$. Moreover, by Lemma 2.1, we also obtain the interlacing property as follows

$$\max\{\gamma_k^2(1,1), \gamma_k^2(2,2)\} < \gamma_k^2(1,2) < \min\{\gamma_{k+1}^2(1,1), \gamma_{k+1}^2(2,2)\} < \gamma_{k+1}^2(1,2),$$

that is, each of the intervals $(\gamma_k^2(1,2), \gamma_{k+1}^2(1,2))$ exactly contains $\gamma_{k+1}^2(1,1)$ and $\gamma_{k+1}^2(2,2)$. Therefore, we define two sequences $\{\lambda_k^2(1,2)\}_{k=0}^{+\infty}$ and $\{\lambda_k^2(2,2)\}_{k=0}^{+\infty}$ as follows

$$\lambda_k^2(1,2) = \gamma_k^2(1,2) = \lambda_k^2, \ \ k \in \mathbb{N},$$

$$\lambda_k^2(2,2) = \gamma_k^2(2,2) = \mu_k^2, \ \ k \in \Lambda,$$
$$\lambda_k^2(2,2) = \gamma_k^2(1,1), \ \ k \in \Lambda^c.$$

From (1.9), (1.10) and (3.13)–(3.14), it is easy to see that two sequences $\{\lambda_k^2(1,2)\}_{k=0}^{+\infty}$ and $\{\lambda_k^2(2,2)\}_{k=0}^{+\infty}$ satisfy the conditions

$$\lambda_k^2(1,2) = k^2 + a_0 + \sigma_{k,1},$$
$$\lambda_k^2(2,2) = k^2 + a_0' + \sigma_{k,2},$$

and

$$\lambda_k^2(1,2) - \lambda_k^2(2,2) = a_0 - a_0' + \frac{\sigma_{k,3}}{k},$$

where $\{\sigma_{k,j}\}_{k=0}^{+\infty}$ belong to l^2 for $j = 1,2,3$. Thus, two sequences $\{\lambda_k^2(1,2)\}_{k=0}^{+\infty}$ and $\{\lambda_k^2(2,2)\}_{k=0}^{+\infty}$ satisfy the conditions of Theorem 2.3. Therefore, there exists a unique tetrad $(q;h_1,h_2,H_2) \in L^2(0,\pi)\times\mathbb{R}^3$ which generates two Robin BVPs $L_{1,2}$ and $L_{2,2}$ with the spectra $\{\lambda_k^2(1,2)\}_{k=0}^{+\infty}$ and $\{\lambda_k^2(2,2)\}_{k=0}^{+\infty}$, respectively. This shows that

$$\{\lambda_k^2\}_{k=0}^{+\infty} = \sigma(L_{1,2}), \qquad \{\mu_j^2\}_{j\in\Lambda} \subset \sigma(L_{2,2}). \tag{3.15}$$

Step 2. *Prove* $\{\mu_n^2\}_{n\in\Lambda^c} \subset \sigma(L_{1,1})$.

We check that the set $\{\mu_k^2\}_{k\in\Lambda^c}$ is a part of spectra of $L_{1,1}$. From [4, Theorem 1.1.4], we have

$$f_{1,2}(\lambda) = \pi(\lambda_0^2(1,2)-\lambda^2)\prod_{k=1}^{+\infty}\frac{(\lambda_k^2(1,2)-\lambda^2)}{k^2}, \tag{3.16}$$

$$f_{2,2}(\lambda) = \pi(\lambda_0^2(2,2)-\lambda^2)\prod_{k=1}^{+\infty}\frac{(\lambda_k^2(2,2)-\lambda^2)}{k^2}. \tag{3.17}$$

By (3.3) and (3.6) we have

$$\begin{vmatrix} f_{1,1}(\lambda) & f_{1,2}(\lambda) \\ f_{2,1}(\lambda) & f_{2,2}(\lambda) \end{vmatrix} = \frac{\pi^2}{4}(a_0-a_0')^2 = \begin{vmatrix} \hat f_{1,1}(\lambda) & \hat f_{1,2}(\lambda) \\ \hat f_{2,1}(\lambda) & \hat f_{2,2}(\lambda) \end{vmatrix}. \tag{3.18}$$

Using (3.4) and (3.16) we obtain $f_{1,2}(\lambda) \equiv \hat f_{1,2}(\lambda)$. This together with (3.18) yields

$$(f_{1,1}f_{2,2} - \hat f_{1,1}\hat f_{2,2})(\lambda) = f_{1,2}(\lambda)(f_{2,1}-\hat f_{2,1})(\lambda). \tag{3.19}$$

From the asymptotic expressions of the $f_{2,1}(\lambda)$ and $\hat f_{2,1}(\lambda)$, we have

$$(\hat f_{2,1} - f_{2,1})(\lambda) = \varphi_0(\lambda),$$

where $\varphi_0(\lambda) \in \mathscr{L}^\pi$.

On the other hand, we also have

$$f_{2,2}(\gamma_k^2(1,1)) = \hat f_{1,1}(\gamma_k^2(1,1)) = 0, \qquad k \in \Lambda^c,$$
$$f_{2,2}(\mu_k^2) = \hat f_{2,2}(\mu_k^2) = 0, \qquad k \in \Lambda.$$

This, together with (3.16), (3.17) and (3.19), implies that $f_{2,1}(\lambda_k^2(2,2)) = \hat{f}_{2,1}(\lambda_k^2(2,2))$ for $k \in \mathbb{N}$. It follows from [14] that $\hat{f}_{2,1}(\lambda) = f_{2,1}(\lambda)$ for all $\lambda \in \mathbb{C}$. Therefore, from (3.19) we get

$$f_{1,1}(\mu_k^2) = 0 \qquad \text{for all } k \in \Lambda^c,$$

since $\hat{f}_{2,2}(\mu_k^2) = 0$. This shows that $\{\mu_k^2\}_{k\in\Lambda^c} \subset \sigma(L_{1,1})$.

By the above discussions, we have found a quintuple $(q;h_1,h_2,H_1,H_2) \in L^2(0,\pi) \times \mathbb{R}^4$ such that (1.12)–(1.13) remain true.

Step 3. *Uniqueness.* If there exists another quintuple $(\tilde{q};\tilde{h}_1,\tilde{h}_2,\tilde{H}_1,\tilde{H}_2) \in L^2(0,\pi) \times \mathbb{R}^4$ such that (1.12) holds with replacing h_i by $\tilde{h}_i$, H_j by $\tilde{H}_j$ for $i,j = 1,2$ and

$$\{\lambda_n^2\}_{n=0}^{+\infty} = \sigma(\tilde{L}_{1,2}), \quad \{\mu_j^2\}_{j\in\Lambda} \subset \sigma(\tilde{L}_{2,2}), \quad \{\mu_j^2\}_{j\in\Lambda^c} \subset \sigma(\tilde{L}_{1,1}). \tag{3.20}$$

By (1.12), we easily conclude that $\tilde{h}_1 > \tilde{h}_2$ and $\tilde{H}_1 < \tilde{H}_2$. Using (3.2), we have

$$\begin{vmatrix} f_{1,1}(\lambda) & f_{1,2}(\lambda) \\ f_{2,1}(\lambda) & f_{2,2}(\lambda) \end{vmatrix} = \frac{\pi^2}{4}(a_0 - a_0')^2 = \begin{vmatrix} \tilde{f}_{1,1}(\lambda) & \tilde{f}_{1,2}(\lambda) \\ \tilde{f}_{2,1}(\lambda) & \tilde{f}_{2,2}(\lambda) \end{vmatrix}. \tag{3.21}$$

Note that $f_{1,2}(\lambda) \equiv \tilde{f}_{1,2}(\lambda)$, which together with (3.21) yields

$$(f_{1,1}f_{2,2} - \tilde{f}_{1,1}\tilde{f}_{2,2})(\lambda) = f_{1,2}(\lambda)(f_{2,1} - \tilde{f}_{2,1})(\lambda) \tag{3.22}$$

for all $\lambda \in \mathbb{C}$. Since $f_{2,2}(\mu_n^2) = \tilde{f}_{2,2}(\mu_n^2) = 0$ for $n \in \Lambda$ and $f_{1,1}(\mu_n^2) = \tilde{f}_{1,1}(\mu_n^2) = 0$ for $n \in \Lambda^c$, it follows that $f_{2,1}(\mu_n^2) = \tilde{f}_{2,1}(\mu_n^2)$ for all $n \in \mathbb{N}$. According to [14], we obtain $f_{2,1}(\lambda) = \tilde{f}_{2,1}(\lambda)$ for all $\lambda \in \mathbb{C}$. Therefore, taking into account (3.22) we have

$$(f_{1,1}f_{2,2} - \tilde{f}_{1,1}\tilde{f}_{2,2})(\lambda) \equiv 0,$$

i.e.,

$$f_{1,1}(\lambda)f_{2,2}(\lambda) \equiv \tilde{f}_{1,1}(\lambda)\tilde{f}_{2,2}(\lambda). \tag{3.23}$$

We denote by $\{\tilde{\lambda}_n(i,j)\}_{n=0}^{+\infty}$ the zeros of $\tilde{f}_{i,j}(\lambda)$ for $i,j = 1,2$. By (3.20) and (3.23), we infer that

$$\{\lambda_n^2(1,1)\}_{n\in\Lambda} \cup \{\lambda_n^2(2,2)\}_{n\in\Lambda^c} = \{\tilde{\lambda}_n^2(1,1)\}_{n\in\Lambda} \cup \{\tilde{\lambda}_n^2(2,2)\}_{n\in\Lambda^c}.$$

Applying Lemma 2.1 to the above equation, we find that

$$\{\lambda_n^2(1,1)\}_{n\in\Lambda} = \{\tilde{\lambda}_n^2(1,1)\}_{n\in\Lambda}, \ \{\lambda_n^2(2,2)\}_{n\in\Lambda^c} = \{\tilde{\lambda}_n^2(2,2)\}_{n\in\Lambda^c}.$$

This together with (3.20) shows that $f_{1,1}(\lambda) \equiv \tilde{f}_{1,1}(\lambda)$ and $f_{2,2}(\lambda) \equiv \tilde{f}_{2,2}(\lambda)$. To sum up, we conclude that $f_{i,j}(\lambda) = \tilde{f}_{i,j}(\lambda)$ for $i,j = 1,2$. From [10, Corollary 3.3], it decides the uniqueness of the solution of our inverse problem.

Conversely, if the tetrad $(q;h_1,h_2,H_2) \in L^2(0,\pi) \times \mathbb{R}^3$ satisfies (1.12)–(1.13), then from Lemma 2.1 and Theorem 2.3 we easily check that (1.8)–(1.11) all hold. This completes the proof. □

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