Operators
and
Matrices
Volume 16, Number 4 (2022), 1005–1026 doi:10.7153/oam-2022-16-67

# ON THE $A_\alpha$ SPECTRAL RADIUS OF STRONGLY CONNECTED DIGRAPHS

WEIGE XI AND LIGONG WANG*

(*Communicated by S. McCullough*)

*Abstract.* Let $A(G)$ and $D(G)$ be the adjacency matrix and the diagonal matrix with outdegrees of vertices of a digraph $G$, respectively. In 2017, Nikiforov proposed to study the convex combinations of the adjacency matrix and diagonal matrix of the degrees of undirected graphs. In 2019, Liu et al. extended the definition to digraphs. For any real $\alpha \in [0,1]$, the matrix $A_\alpha(G)$ of a digraph $G$ is defined as $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$. The largest modulus of the eigenvalues of $A_\alpha(G)$ is called the $A_\alpha$ spectral radius of $G$, denoted by $\lambda_\alpha(G)$. This paper proves some extremal results about the $A_\alpha$ spectral radius $\lambda_\alpha(G)$ that generalize previous results about $\lambda_0(G)$ and $\lambda_{\frac{1}{2}}(G)$. We mainly characterize the extremal digraph with the maximum (or minimum) $A_\alpha$ spectral radius among all $\widetilde{\infty}$-digraphs and $\widetilde{\theta}$-digraphs on $n$ vertices. Furthermore, we determine the digraphs with the second and the third minimum $A_\alpha$ spectral radius among all strongly connected bicyclic digraphs. For $0 \leqslant \alpha \leqslant \frac{1}{2}$, we also determine the digraphs with the second, the third and the fourth minimum $A_\alpha$ spectral radius among all strongly connected digraphs on $n$ vertices. Finally, we characterize the digraph with the minimum $A_\alpha$ spectral radius among all strongly connected bipartite digraphs which contain a complete bipartite subdigraph.

## 1. Introduction

Let $G = (V(G), E(G))$ be a digraph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and arc set $E(G)$. If there is an arc from $v_i$ to $v_j$, we indicate this by writing $(v_i, v_j)$, call $v_j$ the head of $(v_i, v_j)$, and $v_i$ the tail of $(v_i, v_j)$, respectively. A digraph $G$ is called strongly connected if for every pair of vertices $v_i, v_j \in V(G)$, there exists a directed path from $v_i$ to $v_j$ and a directed path from $v_j$ to $v_i$. For any vertex $v_i$, let $N_i^+ = \{v_j \in V(G) \mid (v_i, v_j) \in E(G)\}$ denote the out-neighbors of $v_i$. Let $d_i^+ = |N_i^+|$ denote the outdegree of the vertex $v_i$ in the digraph $G$. Let $P_n$ and $C_n$ denote the directed path and the directed cycle on $n$ vertices, respectively. Let $\overleftrightarrow{K_n}$ denote the complete digraph on $n$ vertices in which for any two distinct vertices $v_i, v_j \in V(\overleftrightarrow{K_n})$, there are arcs $(v_i, v_j)$ and $(v_j, v_i) \in E(\overleftrightarrow{K_n})$. Suppose $P_k = v_1 v_2 \ldots v_k$, we call $v_1$ the initial vertex of the directed path $P_k$, and $v_k$ the terminal vertex of the directed path $P_k$. All digraphs considered in this paper are simple digraphs, i.e., without loops and multiple arcs.

*Mathematics subject classification* (2020): 05C50,15A18.
*Keywords and phrases*: Strongly connected digraph, signless Laplacian matrix, adjacency matrix, $A_\alpha$ spectral radius.
Supported by the National Natural Science Foundation of China (Nos. 11871398 and 12001434).
* Corresponding author.

Let $G = (V(G), E(G))$ be a digraph, if $V(G) = U \cup W$, $U \cap W = \emptyset$ and for any arc $(v_i, v_j) \in E(G)$, $v_i \in U$ and $v_j \in W$ or $v_i \in W$ and $v_j \in U$, then the digraph $G$ is called a bipartite digraph. Let $\overleftrightarrow{K_{p,q}}$ be a complete bipartite digraph obtained from a complete bipartite undirected graph $K_{p,q}$ by replacing each edge with a pair of oppositely directed arcs.

The $\infty$-digraph [11] is a digraph on $n$ vertices obtained from two directed cycles $C_{k+1}$ and $C_{l+1}$ by identifying a vertex of $C_{k+1}$ with a vertex of $C_{l+1}$, denoted by $\infty(k,l)$, $1 \leqslant k \leqslant l$ and $k+l+1 = n$ (see Figure 1 when $s = 2$). The $\theta$-digraph consists of three directed paths $P_{a+2}$, $P_{b+2}$, and $P_{c+2}$ such that the initial vertex of $P_{a+2}$ and $P_{b+2}$ is the terminal vertex of $P_{c+2}$, and the initial vertex of $P_{c+2}$ is the terminal vertex of $P_{a+2}$ and $P_{b+2}$, denoted by $\theta(a,b,c)$, where $a \leqslant b$ and $a+b+c+2 = n$ (see Figure 2 when $s = 2$ ).

A digraph $G$ is called a strongly connected bicyclic digraph if $G$ is strongly connected and $|E(G)| = |V(G)| + 1$. Note that each strongly connected bicyclic digraph is either a $\theta$-digraph or a $\infty$-digraph.

For a digraph $G$, let $A(G) = (a_{ij})_{n\times n}$ be the adjacency matrix of $G$, where $a_{ij} = 1$ whenever $(v_i, v_j) \in E(G)$, and $a_{ij} = 0$ otherwise. Let $D(G)$ be the diagonal matrix with outdegrees of vertices of $G$. The sum of $A(G)$ and $D(G)$ is called the signless Laplacian matrix $Q(G)$, which has been extensively studied since then. More detailed information about this research see [6, 9, 19, 20], and their references. Nikiforov [16] proposed to study the convex linear combinations of the adjacency matrix and diagonal matrix of degrees of undirected graphs, which give a unified theory of adjacency spectral and signless Laplacian spectral theories. Liu et al. [14] extended the definition to digraphs, they proposed to study the convex combinations $A_\alpha(G)$ of $A(G)$ and $D(G)$ of the digraph $G$, which is defined as

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G), \ \ 0 \leqslant \alpha \leqslant 1.$$

Obviously,

$$A(G) = A_0(G), \quad D(G) = A_1(G), \quad \text{and} \quad Q(G) = 2A_{\frac{1}{2}}(G).$$

Since $A_{\frac{1}{2}}(G)$ is essentially equivalent to $Q(G)$, in this paper we take $A_{\frac{1}{2}}(G)$ as an exact substitute for $Q(G)$. The spectral radius of $A_\alpha(G)$, i.e., the largest modulus of the eigenvalues of $A_\alpha(G)$, is called the $A_\alpha$ spectral radius of $G$, denoted by $\lambda_\alpha(G)$. The $A_\alpha$ spectral radius of undirected graphs has been studied in the literature, see [13, 15, 17, 18, 22]. Recently, Liu et al. [14] determined the unique digraph which attains the maximum (resp. minimum) $A_\alpha$ spectral radius among all strongly connected bicyclic digraphs. Xi et al. [21] characterized the digraphs which attain the maximum or minimum $A_\alpha$ spectral radius among all strongly connected digraphs with given girth, clique number, vertex connectivity and arc connectivity, respectively. Ganie and Baghipur [3] obtained some lower and upper bounds on the $A_\alpha$ spectral radius of digraphs and characterized the extremal digraphs attaining these bounds. We are interested in the $A_\alpha$ spectral radius of some other strongly connected digraphs.

If $\alpha = 1$, $A_1(G) = D(G)$ the diagonal matrix with outdegrees of vertices of the digraph $G$ which is not interesting. So we only consider the cases $0 \leqslant \alpha < 1$ in the rest of this paper. If $G$ is a strongly connected digraph, then it follows from the Perron Frobenius Theorem [5] that $\lambda_\alpha(G)$ is an eigenvalue of $A_\alpha(G)$, and there is a unique positive unit eigenvector corresponding to $\lambda_\alpha(G)$. The positive unit eigenvector corresponding to $\lambda_\alpha(G)$ is called the Perron vector of $A_\alpha(G)$.

Spectral graph theory is a fast growing branch of algebraic graph theory. The most studied problems are those of characterization of extremal graphs, such as determine the maximum or minimum spectral (signless Laplacian spectral) radius over various families of graphs. Recently, in [12], Lin et al. determined the digraphs with the minimum $A_0$ spectral radius among all strongly connected digraphs with given clique number and girth. In [8], Lin and Drury gave the extremal digraphs with the maximum $A_0$ spectral radius among all strongly connected digraphs with given arc connectivity. In [10], Lin and Shu characterized the digraph which has the maximum $A_0$ spectral radius among all strongly connected digraphs with given dichromatic number. In [6], Hong and You determined the digraph which achieves the minimum (or maximum) $A_{\frac{1}{2}}$ spectral radius among all strongly connected digraphs with some given parameters such as clique number, girth or vertex connectivity. In [20], Xi and Wang determined the extremal digraph with the maximum $A_{\frac{1}{2}}$ spectral radius among all strongly connected digraphs with given dichromatic number. The main goal of this paper is to extend some results on maximum or minimum $A_0$ spectral radius and $A_{\frac{1}{2}}$ spectral radius for all $\alpha \in [0,1)$.

The rest of the paper is structured as follows. In Section 2, we will determine the extremal digraphs which achieve the maximum and minimum $A_\alpha$ spectral radius among all $\widetilde{\infty}$-digraphs and $\widetilde{\theta}$-digraphs (their definitions can be found in Section 2). In Section 3, for $0 \leqslant \alpha \leqslant \frac{1}{2}$, we determine the digraphs which achieve the second, the third and the forth minimum $A_\alpha$ spectral radius of strongly connected digraphs on $n$ vertices. For general case, we propose a conjecture. In Section 4, we determine the extremal digraph which attains the minimum $A_\alpha$ spectral radius of strongly connected bipartite digraphs which contain a complete bipartite subdigraph. The results in our paper generalize some results in [2, 4, 7, 9, 14].

## 2. The $A_\alpha$ spectral radius of $\widetilde{\infty}$-digraphs and $\widetilde{\theta}$-digraphs

We have known the $\theta$-digraphs and $\infty$-digraphs. The generalized strongly connected $\widetilde{\infty}$-digraph is a digraph consisting of $s$ ($s \geqslant 2$) directed cycles with just a vertex in common (as shown in Figure 1), denoted by $\widetilde{\infty}(k_1,k_2,\ldots,k_s)$ such that $\sum_{i=1}^{s} k_i + 1 = n$. Without loss of generality, let $1 \leqslant k_i \leqslant k_{i+1}$ for $i = 1,2,\ldots,s-1$. The generalized strongly connected $\widetilde{\theta}$-digraph consists of $s+1$ ($s \geqslant 2$) directed paths $P_{k_1+2},\ldots,P_{k_s+2}$ and $P_{l_1+2}$ such that the initial vertex of $P_{k_1+2},\ldots,P_{k_s+2}$ is the terminal vertex of $P_{l_1+2}$, and the initial vertex of $P_{l_1+2}$ is the terminal vertex of $P_{k_1+2},\ldots,P_{k_s+2}$ (as shown in Figure 2), denoted by $\widetilde{\theta}(k_1,k_2,\ldots,k_s,l_1)$ such that $\sum_{i=1}^{s} k_i + l_1 + 2 = n$. Without loss of generality, let $0 \leqslant k_i \leqslant k_{i+1}$ for $i = 1,2,\ldots,s-1$. Note that any $\widetilde{\theta}(k_1,k_2,\ldots,k_s,l_1)$-digraph contains $s$ directed cycles.

Guo and Liu [4] characterized the digraph which attains the minimum and max-

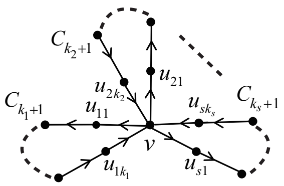


Figure 1: *The digraph* $\widetilde{\infty}(k_1,k_2,\ldots,k_s)$.

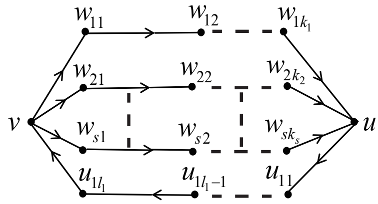


Figure 2: *The digraph* $\widetilde{\theta}(k_1,k_2,\ldots,k_s,l_1)$.

imum $A_0$ spectral radius among all $\widetilde{\theta}$-digraphs and $\widetilde{\infty}$-digraphs on $n$ vertices, respectively. Li et al. [9] determined that the digraph which attains the minimum and maximum $A_{\frac{1}{2}}$ spectral radius among all $\widetilde{\theta}$-digraphs and $\widetilde{\infty}$-digraphs on $n$ vertices, respectively. We generalize their results to $0 \leqslant \alpha < 1$. Moreover, Li and Zhou [7] characterized digraphs which achieve the second and the third minimum $A_{\frac{1}{2}}$ spectral radius among all strongly connected bipartite digraphs. We also generalize their results to $0 \leqslant \alpha < 1$.

LEMMA 2.1. ([5]) *Let $M$ be an $n\times n$ nonnegative irreducible matrix with spectral radius $\rho(M)$ and row sums $s_1,s_2,\ldots,s_n$. Then*

$$\min_{1\leqslant i\leqslant n} s_i \leqslant \rho(M) \leqslant \max_{1\leqslant i\leqslant n} s_i.$$

*Moreover, one of the equalities holds if and only if the row sums of $M$ are all equal.*

LEMMA 2.2. *For any $p,q\in\{1,2,\ldots,s\}$, if $2\leqslant k_p\leqslant k_q$, then we have*

$$\begin{aligned}&\lambda_\alpha(\widetilde{\infty}(k_1,k_2,\ldots,k_{p-1},k_p-1,k_{p+1},\ldots,k_{q-1},k_q+1,k_{q+1},\ldots,k_s))\\ &>\lambda_\alpha(\widetilde{\infty}(k_1,k_2,\ldots,k_{p-1},k_p,k_{p+1},\ldots,k_{q-1},k_q,k_{q+1},\ldots,k_s)).\end{aligned}$$

*Proof.* Let $G=\widetilde{\infty}(k_1,k_2,\ldots,k_{p-1},k_p,k_{p+1},\ldots,k_{q-1},k_q,k_{q+1},\ldots,k_s)$ be a digraph shown in Figure 1. Suppose $X=(x_v,x_{1,1},x_{1,2},\ldots,x_{1,k_1},x_{2,1},x_{2,2},\ldots,x_{2,k_2},\ldots,x_{s,1},x_{s,2},\ldots,x_{s,k_s})^T$ is the Perron vector of $A_\alpha(G)$, where $x_v$ corresponds to $v$, $x_{i,j}$ corresponds to $u_{ij}$ for $i=1,2,\ldots,s$ and $j=1,2,\ldots,k_i$, respectively. Since $A_\alpha(G)X=\lambda_\alpha(G)X$, one can easily see that

$$\begin{cases}\lambda_\alpha(G)x_{1,i_1}=\alpha x_{1,i_1}+(1-\alpha)x_{1,i_1+1}, & i_1=1,2,\ldots,k_1-1,\\ \lambda_\alpha(G)x_{2,i_2}=\alpha x_{2,i_2}+(1-\alpha)x_{2,i_2+1}, & i_2=1,2,\ldots,k_2-1,\\ \qquad\vdots & \\ \lambda_\alpha(G)x_{s,i_s}=\alpha x_{s,i_s}+(1-\alpha)x_{s,i_s+1}, & i_s=1,2,\ldots,k_s-1,\\ \lambda_\alpha(G)x_v=\alpha s x_v+(1-\alpha)(x_{1,1}+x_{2,1}+\cdots+x_{s,1}), & \\ \lambda_\alpha(G)x_{j,k_j}=\alpha x_{j,k_j}+(1-\alpha)x_v, & j=1,2,\ldots,s.\end{cases}$$

Then we have

$$x_{j,k_j} = \left(\frac{\lambda_\alpha(G)-\alpha}{1-\alpha}\right)^{k_j-1} x_{j,1}, \qquad j=1,2,\ldots,s.$$

Furthermore,

$$x_v = \left(\frac{\lambda_\alpha(G)-\alpha}{1-\alpha}\right)^{k_j} x_{j,1}, \qquad j=1,2,\ldots,s.$$

Thus, we have

$$\begin{aligned}\left(\frac{\lambda_\alpha(G)-\alpha s}{1-\alpha}\right)x_v &= \left(\frac{\lambda_\alpha(G)-\alpha}{1-\alpha}\right)^{-k_1} x_v + \left(\frac{\lambda_\alpha(G)-\alpha}{1-\alpha}\right)^{-k_2} x_v \\ &\quad + \cdots + \left(\frac{\lambda_\alpha(G)-\alpha}{1-\alpha}\right)^{-k_s} x_v.\end{aligned}$$

By the Perron-Frobenius Theorem, we have $x_v > 0$, therefore

$$\left(\frac{\lambda_\alpha(G)-s\alpha}{1-\alpha}\right)\left(\frac{\lambda_\alpha(G)-\alpha}{1-\alpha}\right)^{n-1} = \sum_{i=1}^{s}\left(\frac{\lambda_\alpha(G)-\alpha}{1-\alpha}\right)^{n-1-k_i}.$$

Let $G' = \widetilde{\theta}(k_1,k_2,\ldots,k_{p-1},k_p-1,k_{p+1},\ldots,k_{q-1},k_q+1,k_{q+1},\ldots,k_s)$. Similarly, we have

$$\begin{aligned}\left(\frac{\lambda_\alpha(G')-s\alpha}{1-\alpha}\right)\left(\frac{\lambda_\alpha(G')-\alpha}{1-\alpha}\right)^{n-1} &= \sum_{\substack{i=1, i\neq p\\ i\neq q}}^{s}\left(\frac{\lambda_\alpha(G')-\alpha}{1-\alpha}\right)^{n-1-k_i} \\ &\quad + \left(\frac{\lambda_\alpha(G')-\alpha}{1-\alpha}\right)^{n-k_p} + \left(\frac{\lambda_\alpha(G')-\alpha}{1-\alpha}\right)^{n-2-k_q}.\end{aligned}$$

Let $f(x) = \left(\frac{x-s\alpha}{1-\alpha}\right)\left(\frac{x-\alpha}{1-\alpha}\right)^{n-1} - \sum_{i=1}^{s}\left(\frac{x-\alpha}{1-\alpha}\right)^{n-1-k_i}$,

$$\begin{aligned}g(x) &= \left(\frac{x-s\alpha}{1-\alpha}\right)\left(\frac{x-\alpha}{1-\alpha}\right)^{n-1} - \sum_{\substack{i=1, i\neq p\\ i\neq q}}^{s}\left(\frac{x-\alpha}{1-\alpha}\right)^{n-1-k_i} \\ &\quad - \left(\frac{x-\alpha}{1-\alpha}\right)^{n-k_p} - \left(\frac{x-\alpha}{1-\alpha}\right)^{n-2-k_q}.\end{aligned}$$

It is easy to see that $\lambda_\alpha(G)$ is the largest real root of $f(x)=0$. Similarly, $\lambda_\alpha(G')$ is the largest real root of $g(x)=0$. Since for all $x>1$

$$f(x)-g(x) = \left(\left(\frac{x-\alpha}{1-\alpha}\right)-1\right)\left(\left(\frac{x-\alpha}{1-\alpha}\right)^{n-1-k_p} - \left(\frac{x-\alpha}{1-\alpha}\right)^{n-2-k_q}\right) > 0.$$

Since the minimum row sum of $A_\alpha(G')$ is 1, and the row sums of $A_\alpha(G')$ are not all equal, by Lemma 2.1, then we have $\lambda_\alpha(G') > 1$. Hence, we get

$$\begin{aligned}&\lambda_\alpha(\widetilde{\infty}(k_1,k_2,\dots,k_{p-1},k_p-1,k_{p+1},\dots,k_{q-1},k_q+1,k_{q+1},\dots,k_s))\\&> \lambda_\alpha(\widetilde{\infty}(k_1,k_2,\dots,k_{p-1},k_p,k_{p+1},\dots,k_{q-1},k_q,k_{q+1},\dots,k_s)),\end{aligned}$$

which prove the result. □

By Lemma 2.2, we immediately obtain the following theorem.

THEOREM 2.3. *Among all $\widetilde{\infty}$-digraphs on $n$ vertices, the digraph $\widetilde{\infty}(1,1,1,\dots,n-s)$ is the unique digraph which attains the maximum $A_\alpha$ spectral radius, the digraph $\widetilde{\infty}(a_1,a_2,\dots,a_s)$ such that $a_i = \lfloor\frac{n-1}{s}\rfloor$ and $a_j = \lceil\frac{n-1}{s}\rceil$ for any $i \in \{1,2,\dots,s-(n-1-s\lfloor\frac{n-1}{s}\rfloor)\}$ and $j \in \{s-(n-1-s\lfloor\frac{n-1}{s}\rfloor)+1,\dots,s\}$, is the unique digraph which attains the minimum $A_\alpha$ spectral radius.*

LEMMA 2.4. *For any $p,q \in \{1,2,\dots,s\}$, if $1 \leqslant k_p \leqslant k_q$, then we have*

$$\begin{aligned}&\lambda_\alpha(\widetilde{\theta}(k_1,k_2,\dots,k_{p-1},k_p-1,k_{p+1},\dots,k_{q-1},k_q+1,k_{q+1},\dots,k_s,l_1))\\&> \lambda_\alpha(\widetilde{\theta}(k_1,k_2,\dots,k_{p-1},k_p,k_{p+1},\dots,k_{q-1},k_q,k_{q+1},\dots,k_s,l_1)).\end{aligned}$$

*Proof.* Let $G = \widetilde{\theta}(k_1,k_2,\dots,k_{p-1},k_p,k_{p+1},\dots,k_{q-1},k_q,k_{q+1},\dots,k_s,l_1)$ be a digraph shown in Figure 2. Similar to the proof of Lemma 2.2, we can know that $\lambda_\alpha(G)$ satisfies the follow equation

$$\left(\frac{\lambda_\alpha(G)-s\alpha}{1-\alpha}\right)\left(\frac{\lambda_\alpha(G)-\alpha}{1-\alpha}\right)^{n-1} = \sum_{i=1}^{s}\left(\frac{\lambda_\alpha(G)-\alpha}{1-\alpha}\right)^{n-2-l_1-k_i}.$$

Let $G' = \widetilde{\theta}(k_1,k_2,\dots,k_{p-1},k_p-1,k_{p+1},\dots,k_{q-1},k_q+1,k_{q+1},\dots,k_s,l_1)$. Similarly, we have

$$\begin{aligned}&\left(\frac{\lambda_\alpha(G')-s\alpha}{1-\alpha}\right)\left(\frac{\lambda_\alpha(G')-\alpha}{1-\alpha}\right)^{n-1}\\&= \sum_{\substack{i=1,i\neq p\\ i\neq q}}^{s}\left(\frac{\lambda_\alpha(G')-\alpha}{1-\alpha}\right)^{n-2-l_1-k_i} + \left(\frac{\lambda_\alpha(G')-\alpha}{1-\alpha}\right)^{n-1-l_1-k_p}\\&\quad+\left(\frac{\lambda_\alpha(G')-\alpha}{1-\alpha}\right)^{n-3-l_1-k_q}.\end{aligned}$$

Let $f(x) = \left(\frac{x-s\alpha}{1-\alpha}\right)\left(\frac{x-\alpha}{1-\alpha}\right)^{n-1} - \sum_{i=1}^{s}\left(\frac{x-\alpha}{1-\alpha}\right)^{n-2-l_1-k_i}$,

$$\begin{aligned}g(x) &= \left(\frac{x-s\alpha}{1-\alpha}\right)\left(\frac{x-\alpha}{1-\alpha}\right)^{n-1} - \sum_{\substack{i=1,i\neq p\\ i\neq q}}^{s}\left(\frac{x-\alpha}{1-\alpha}\right)^{n-2-l_1-k_i}\\&\quad-\left(\frac{x-\alpha}{1-\alpha}\right)^{n-1-l_1-k_p} - \left(\frac{x-\alpha}{1-\alpha}\right)^{n-3-l_1-k_q}.\end{aligned}$$

It is easy to see that $\lambda_\alpha(G)$ is the largest real root of $f(x)=0$. Similarly, $\lambda_\alpha(G')$ is the largest real root of $g(x)=0$. Since for all $x>1$

$$\begin{aligned} f(x)-g(x) &= \left(\frac{x-\alpha}{1-\alpha}\right)^{n-1-l_1-k_p} - \left(\frac{x-\alpha}{1-\alpha}\right)^{n-2-l_1-k_p} \\ &\quad + \left(\frac{x-\alpha}{1-\alpha}\right)^{n-3-l_1-k_q} - \left(\frac{x-\alpha}{1-\alpha}\right)^{n-2-l_1-k_q} \\ &= \left(\left(\frac{x-\alpha}{1-\alpha}\right)-1\right)\left(\left(\frac{x-\alpha}{1-\alpha}\right)^{n-2-l_1-k_p} - \left(\frac{x-\alpha}{1-\alpha}\right)^{n-3-l_1-k_q}\right) > 0. \end{aligned}$$

Since the minimum row sum of $A_\alpha(G')$ is 1, and the row sums of $A_\alpha(G')$ are not all equal, by Lemma 2.1, then we have $\lambda_\alpha(G')>1$. Hence, we get

$$\begin{aligned} &\lambda_\alpha(\widetilde{\theta}(k_1,k_2,\ldots,k_{p-1},k_p-1,k_{p+1},\ldots,k_{q-1},k_q+1,k_{q+1},\ldots,k_s,l_1)) \\ &> \lambda_\alpha(\widetilde{\theta}(k_1,k_2,\ldots,k_{p-1},k_p,k_{p+1},\ldots,k_{q-1},k_q,k_{q+1},\ldots,k_s,l_1)), \end{aligned}$$

which prove the result. $\square$

Similarly, we have the following lemma.

LEMMA 2.5. *If $l_1 \geqslant 1$, then for any $p \in \{1,2,\ldots,s\}$, we have*

$$\begin{aligned} &\lambda_\alpha(\widetilde{\theta}(k_1,k_2,\ldots,k_{p-1},k_p+1,k_{p+1},\ldots,k_s,l_1-1)) \\ &> \lambda_\alpha(\widetilde{\theta}(k_1,k_2,\ldots,k_{p-1},k_p,k_{p+1},\ldots,k_s,l_1)). \end{aligned}$$

By Lemmas 2.4 and 2.5, we immediately obtain the following theorem.

THEOREM 2.6. *Among all $\widetilde{\theta}$-digraphs on $n$ vertices, the digraph $\widetilde{\theta}(0,1,1,\ldots,n-s,0)$ is the unique digraph which attains the maximum $A_\alpha$ spectral radius, the digraph $\widetilde{\theta}(0,1,1,\ldots,1,n-s-1)$ is the unique digraph which attains the minimum $A_\alpha$ spectral radius.*

LEMMA 2.7. ([21]) *Let $0 \leqslant \alpha < 1$ and $G=(V(G),E(G))$ be a strongly connected digraph on $n$ vertices, $v_p, v_q$ be two distinct vertices of $V(G)$. Suppose that $v_1,v_2,\ldots,v_t \in N_{v_p}^- \setminus \{N_{v_q}^- \cup \{v_q\}\}$, where $1 \leqslant t \leqslant d_p^-$, and $X=(x_1,x_2,\ldots,x_n)^T$ be the unique positive unit eigenvector corresponding to the $A_\alpha$ spectral radius $\lambda_\alpha(G)$, where $x_i$ corresponds to the vertex $v_i$. Let $H = G-\{(v_i,v_p): i=1,2\ldots,t\}+\{(v_i,v_q): i=1,2\ldots,t\}$. If $x_q \geqslant x_p$, then $\lambda_\alpha(H) \geqslant \lambda_\alpha(G)$. Furthermore, if $H$ is strongly connected and $x_q > x_p$, then $\lambda_\alpha(H) > \lambda_\alpha(G)$.*

LEMMA 2.8. *For any $\widetilde{\theta}(k_1,k_2,\ldots,k_s,l_1)$-digraph, there exists $\widetilde{\infty}(k_2,k_3,\ldots,k_s,k_1+l_1+1)$ such that*

$$\lambda_\alpha(\widetilde{\theta}(k_1,k_2,\ldots,k_s,l_1)) < \lambda_\alpha(\widetilde{\infty}(k_2,k_3,\ldots,k_s,k_1+l_1+1)).$$

*Proof.* Let $\widetilde{\theta}(k_1,k_2,\ldots,k_s,l_1)$ be a digraph shown in Figure 2 and $X=(x_v,x_u,x_{11},x_{12},\ldots,x_{1k_1},x_{21},x_{22},\ldots,x_{2k_2},\ldots,x_{s1},x_{s2},\ldots,x_{sk_s},y_1,y_2,\ldots,y_{l_1})^T$ be the Perron vector of $A_\alpha(\widetilde{\theta}(k_1,k_2,\ldots,k_s,l_1))$, where $x_u$ and $x_v$ correspond to $u$ and $v$, respectively, and $x_{ij}$ correspond to $w_{ij}$ ($i=1,2,\ldots,s$; $j=1,2,\ldots,k_i$) and $y_j$ correspond to $u_{1j}$, ($j=1,2,\ldots,l_1$) respectively. It is not difficult to see that $\widetilde{\infty}(k_2,k_3,\ldots,k_s,k_1+l_1+1)\cong\widetilde{\theta}(k_1,k_2,\ldots,k_s,l_1)-\{(w_{2k_2},u),(w_{3k_3},u),\ldots,(w_{sk_s},u)\}+\{(w_{2k_2},v),(w_{3k_3},v),\ldots,(w_{sk_s},v)\}$. Similar to the proof of Lemma 2.2, we have

$$x_v=\left(\frac{\lambda_\alpha(\widetilde{\theta}(k_1,k_2,\ldots,k_s,l_1))-\alpha}{1-\alpha}\right)^{l_1+1}x_u.$$

Since $\lambda_\alpha((\widetilde{\theta}(k_1,k_2,\ldots,k_s,l_1))>1$, we have $x_v>x_u$. By Lemma 2.7, we have $\lambda_\alpha(\widetilde{\infty}(k_2,k_3,\ldots,k_s,k_1+l_1+1))>\lambda_\alpha(\widetilde{\theta}(k_1,k_2,\ldots,k_s,l_1))$. So we complete the proof. $\square$

LEMMA 2.9. *For any $\widetilde{\infty}(k_1,k_2,\ldots,k_s)$-digraph, there exists $\widetilde{\theta}(k_1,k_2,\ldots,k_{s-1},k_s-1,0)$ such that*

$$\lambda_\alpha(\widetilde{\theta}(k_1,k_2,\ldots,k_{s-1},k_s-1,0))<\lambda_\alpha(\widetilde{\infty}(k_1,k_2,\ldots,k_s)).$$

*Proof.* It is not difficult to see that $\widetilde{\infty}(k_1,k_2,\ldots,k_s)\cong\widetilde{\theta}(k_1,k_2,\ldots,k_{s-1},k_s-1,0)-\{(w_{1k_1},u),(w_{2k_2},u),\ldots,(w_{s-1k_{s-1}},u)\}+\{(w_{1k_1},v),(w_{2k_2},v),\ldots,(w_{s-1k_{s-1}},v)\}$. Similar as the proof of Lemma 2.8, we have $\lambda_\alpha(\widetilde{\infty}(k_1,k_2,\ldots,k_{s-1},k_s))>\lambda_\alpha(\widetilde{\theta}(k_1,k_2,\ldots,k_{s-1},k_s-1,0))$. So we complete the proof. $\square$

By Theorems 2.3 and 2.6, Lemmas 2.8 and 2.9, we immediately obtain the following theorem.

THEOREM 2.10. *Among all $\widetilde{\theta}$-digraphs and $\widetilde{\infty}$-digraphs on $n$ vertices, the digraph $\widetilde{\infty}(1,1,1,\ldots,n-s)$ is the unique digraph which attains the maximum $A_\alpha$ spectral radius, the digraph $\widetilde{\theta}(0,1,1,\ldots,1,n-s-1)$ is the unique digraph which attains the minimum $A_\alpha$ spectral radius.*

REMARK 2.11. If $s=2$, then the digraph $\widetilde{\infty}(k_1,k_2,\ldots,k_s)$ is $\infty(k_1,k_2)$, and the digraph $\widetilde{\theta}(k_1,k_2,\ldots,k_s,l_1)$ is $\theta(k_1,k_2,l_1)$. Liu et al. [14] proved that $\theta(0,1,n-3)$ and $\infty(1,n-2)$ are the digraphs which attain the minimum and maximum $A_\alpha$ spectral radii among all strongly connected bicyclic digraphs with order $n$, respectively. We generalize their result to $s\geqslant 2$.

We can know that each strongly connected bicyclic digraph is either a $\theta$-digraph or a $\infty$-digraph. In the following, we will determine which digraph has the second and the third minimum $A_\alpha$ spectral radius among all strongly connected bicyclic digraphs, respectively.

THEOREM 2.12. *Among all the strongly connected bicyclic digraphs with order $n \geqslant 5$, $\theta(1,1,n-4)$ and $\theta(0,2,n-4)$ are the unique digraph which achieve the second and the third minimum $A_\alpha$ spectral radius, respectively.*

*Proof.* Let $G$ be a strongly connected bicyclic digraph with order $n \geqslant 5$ and $G \neq \theta(0,1,n-3)$. Then $G$ is a $\theta$-digraph or a $\infty$-digraph. Suppose that $G$ is a $\theta$-digraph, then $G \neq \theta(0,1,n-3)$, and by Lemmas 2.4 and 2.5, we have $\lambda_\alpha(G) \geqslant \lambda_\alpha(\theta(0,2,n-4))$ with equality only if $G = \theta(0,2,n-4)$ or $\lambda_\alpha(G) \geqslant \lambda_\alpha(\theta(1,1,n-4))$ with equality only if $G = \theta(1,1,n-4)$. However, by Lemma 2.4, we have $\lambda_\alpha(\theta(0,2,n-4)) > \lambda_\alpha(\theta(1,1,n-4))$. Thus if $G$ is a $\theta$-digraph and $G \neq \theta(1,1,n-4)$, $\lambda_\alpha(G) \geqslant \lambda_\alpha(\theta(0,2,n-4)) > \lambda_\alpha(\theta(1,1,n-4))$ with equality only if $G = \theta(0,2,n-4)$. If $G$ is a $\infty$-digraph, then by Lemma 2.2, $\lambda_\alpha(G) \geqslant \lambda_\alpha(\infty(\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil))$. If $n$ is odd, $\frac{n-1}{2} \geqslant 2$, then by Lemmas 2.9, 2.4 and 2.5, we have $\lambda_\alpha(\infty(\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil)) = \lambda_\alpha(\infty(\frac{n-1}{2}, \frac{n-1}{2})) > \lambda_\alpha(\theta(\frac{n-3}{2}, \frac{n-1}{2}, 0)) > \lambda_\alpha(\theta(0,2,n-4))$. If $n$ is even, $\frac{n-2}{2} \geqslant 2$, then by Lemmas 2.9, 2.4 and 2.5, we have $\lambda_\alpha(\infty(\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil)) = \lambda_\alpha(\infty(\frac{n-2}{2}, \frac{n}{2})) > \lambda_\alpha(\theta(\frac{n-2}{2}, \frac{n-2}{2}, 0)) > \lambda_\alpha(\theta(0,2,n-4))$. Hence, if $G$ is a $\infty$-digraph, then we have

$$\lambda_\alpha(G) \geqslant \lambda_\alpha\left(\infty\left(\left\lfloor \frac{n-1}{2} \right\rfloor, \left\lceil \frac{n-1}{2} \right\rceil\right)\right) > \lambda_\alpha(\theta(0,2,n-4)).$$

Therefore, by the second part of Theorem 2.10, we get the result. $\square$

## 3. The second, the third and the forth minimum $A_\alpha$ spectral radius of strongly connected digraphs

In the followig, we determine the digraphs which achieve the second, the third and the forth minimum $A_\alpha$ spectral radius of strongly connected digraphs on $n$ vertices.

Recall that the spectral radius of a nonnegative irreducible matrix $B$ is larger than that of a principal submatrix of $B$ and it increases when an entry of $B$ increases [1]. Thus we have the following well known lemma.

LEMMA 3.1. *Let $G$ be a strongly connected digraph and $H$ be a proper subdigraph of $G$. Then $\lambda_\alpha(G) > \lambda_\alpha(H)$.*

COROLLARY 3.2. *Let $G$ be a strongly connected digraph. Then $1 \leqslant \lambda_\alpha(G) \leqslant n-1$, $\lambda_\alpha(G) = 1$ if and only if $G \cong C_n$, and $\lambda_\alpha(G) = n-1$ if and only if $G \cong \overleftrightarrow{K_n}$.*

LEMMA 3.3. ([21]) *Let $0 \leqslant \alpha < 1$ and $G$ ($\neq C_n$) be a strongly connected digraph with $V(G) = \{v_1, v_2, \cdots, v_n\}$, $(v_i, v_j) \in E(G)$ and $w \notin V(G)$, $G^w = (V(G^w), E(G^w))$ with $V(G^w) = V(G) \cup \{w\}$, $E(G^w) = E(G) - \{(v_i, v_j)\} + \{(v_i, w), (w, v_j)\}$. Then $\lambda_\alpha(G) \geqslant \lambda_\alpha(G^w)$.*

We follow the techniques in [7] to prove the following result.

THEOREM 3.4. *Let $0 \leqslant \alpha \leqslant \frac{1}{2}$ and $G$ be a strongly connected digraph of order $n \geqslant 5$ that is neither a bicyclic digraph nor $C_n$, Then $\lambda_\alpha(G) > \lambda_\alpha(\theta(0,2,n-4))$.*

*Proof.* Let $C$ be a shortest directed cycle in $G$. Obviously, $V(C) \neq V(G)$. There is a vertex $u \in V(G) \setminus V(C)$ such that there is a arc from $u$ to some vertex, say $v$ on $C$. Also, there is a directed path from some vertex on $C$ to $u$. Let $w$ be a vertex on $C$ such that the distance from $w$ to $u$ is as small as possible. Let $P$ be such a directed path. Then $P$ and $C$ have exactly one common vertex $w$. If $w = v$, then $G$ has a proper $\infty$-subdigraph. If $w \neq v$, then $G$ has a proper $\theta$-subdigraph.

*Case* 1. If $G$ has a proper $\infty$-subdigraph, say $\infty(k_1, l_1)$ with $k_1 + l_1 = n_1 - 1$ and $n_1 \leqslant n$, then by Lemma 3.1, the second part of Theorem 2.3, and Theorem 2.12, Lemma 3.3, we have

$$\begin{aligned}\lambda_\alpha(G) > \lambda_\alpha(\infty(k_1, l_1)) &\geqslant \lambda_\alpha\left(\infty\left(\left\lfloor \frac{n_1 - 1}{2} \right\rfloor, \left\lceil \frac{n_1 - 1}{2} \right\rceil\right)\right) \\ &> \lambda_\alpha(\theta(0, 2, n_1 - 4)) \\ &\geqslant \lambda_\alpha(\theta(0, 2, n - 4)).\end{aligned}$$

*Case* 2. If $G$ has a proper $\theta$-subdigraph, say $\theta(a_1, b_1, c_1)$ with $a_1 + b_1 + c_1 = n_2 - 2$ and $n_2 \leqslant n$.

*Subcase* 2.1. $n_2 \leqslant n - 1$. By Lemma 3.1, the second part of Theorem 2.6 and Lemma 3.3, we get

$$\lambda_\alpha(G) > \lambda_\alpha(\theta(a_1, b_1, c_1)) \geqslant \lambda_\alpha(\theta(0, 1, n_2 - 3)) \geqslant \lambda_\alpha(\theta(0, 1, n - 4)) \geqslant \lambda_\alpha(\theta(0, 2, n - 4)).$$

*Subcase* 2.2. $n_2 = n$ and $\theta(a_1, b_1, c_1) \neq \theta(0, 1, n - 3)$ and $\theta(a_1, b_1, c_1) \neq \theta(1, 1, n - 4)$. By Lemma 3.1, the second part of Theorem 2.6, and Theorem 2.12, we get

$$\lambda_\alpha(G) > \lambda_\alpha(\theta(a_1, b_1, c_1)) \geqslant \lambda_\alpha(\theta(0, 2, n - 4)).$$

*Subcase* 2.3. $n_2 = n$ and the $\theta$-subdigraph of $G$ can only be $\theta(0, 1, n - 3)$ or $\theta(1, 1, n - 4)$.

*Subcase* 2.3.1. $G$ has a $\theta$-subdigraph $\theta(0, 1, n - 3)$. Let $wv$, $wuv$ and $vu_1u_2 \ldots u_{n-3}w$ be the basic directed paths of the $\theta$-subdigraph $\theta(0, 1, n - 3)$. We consider the possible arc(s) in $G$ except the arcs in $\theta(0, 1, n - 3)$ as follows.

(1) $(v, w) \notin E(G)$, otherwise, $G$ has a $\theta$-subdigraph $\theta(0, n - 3, 0)$, a contradiction.

(2) $(v, u) \notin E(G)$ and $(u, w) \notin E(G)$, otherwise, $G$ has a $\theta$-subdigraph $\theta(0, n - 2, 0)$, a contradiction.

(3) $(u, u_k) \notin E(G)$ and $(u_{n-k-2}, u) \notin E(G)$ for $2 \leqslant k \leqslant n - 3$, otherwise, $G$ has a $\theta$-subdigraph $\theta(0, k, n - k - 2)$, a contradiction.

(4) $(w, u_k) \notin E(G)$ and $(u_{n-k-2}, v) \notin E(G)$ for $1 \leqslant k \leqslant n - 3$, otherwise, $G$ has a $\theta$-subdigraph $\theta(0, k + 1, n - k - 3)$, a contradiction.

(5) $(u_k, w) \notin E(G)$ and $(v, u_{n-k-2}) \notin E(G)$ for $1 \leqslant k \leqslant n - 4$, otherwise, $G$ has a $\theta$-subdigraph $\theta(0, 1, k)$, a contradiction.

(6) $(u_l, u_k) \notin E(G)$ for $1 \leqslant k < l \leqslant n - 3$, otherwise, $G$ has a $\theta$-subdigraph $\theta(0, n - l + k - 1, l - k - 1)$, a contradiction.

(7) $(u_k, u_l) \notin E(G)$ for $1 \leqslant k < l - 1 \leqslant n - 4$, otherwise, $G$ has a $\theta$-subdigraph $\theta(0, 1, n - 2 + k - l)$, a contradiction.

(8) $\{(u,u_1),(u_{n-3},u)\} \notin E(G)$, otherwise, $G$ has a $\theta$-subdigraph $\theta(0,1,n-4)$, a contradiction.

From (1)–(8), we find that besides these arcs in $\theta(0,1,n-3)$, $G$ contains one additional arc $(u,u_1)$ or $(u_{n-3},u)$. Thus $G$ is isomorphic to the digraph $G'$ obtained from $\theta(0,1,n-3)$ by adding the arcs $(u,u_1)$, as shown in the Figure 3.

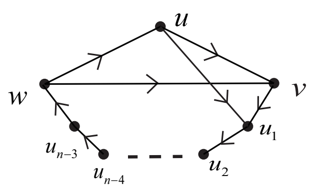


Figure 3: *The digraph* $G'$.

Similar to the proofs of Lemmas 2.2 and 2.4, we have $\lambda_\alpha(G')$ is the largest real root of $p(x) = (\frac{x-2\alpha}{1-\alpha})^2(\frac{x-\alpha}{1-\alpha})^{n-2} - \frac{2x-3\alpha}{1-\alpha} - 1 = 0$. From the proof of Lemma 2.4, we know that $\lambda_\alpha(\theta(0,2,n-4))$ is the largest real root of $q(x) = \frac{x-2\alpha}{1-\alpha}(\frac{x-\alpha}{1-\alpha})^{n-1} - (\frac{x-\alpha}{1-\alpha})^2 - 1 = 0$. Note that

$$q(x)-p(x) = \frac{x-2\alpha}{1-\alpha}\left(\frac{x-\alpha}{1-\alpha}\right)^{n-2}\frac{\alpha}{1-\alpha} - \left(\frac{x-\alpha}{1-\alpha}\right)^2 + \frac{2x-3\alpha}{1-\alpha}.$$

For $0 \leqslant \alpha \leqslant \frac{1}{2}$,

$$\begin{aligned} q(x)-p(x) &> \frac{x-2\alpha}{1-\alpha}\left(\frac{x-\alpha}{1-\alpha}\right)\frac{\alpha}{1-\alpha} - \frac{x^2-2x+3\alpha-2\alpha^2}{(1-\alpha)^2} \\ &= \frac{\alpha x^2-3\alpha^2 x+2\alpha^3}{(1-\alpha)^3} - \frac{(x^2-2x+3\alpha-2\alpha^2)(1-\alpha)}{(1-\alpha)^3} \\ &= \frac{(2\alpha-1)x^2+(2-2\alpha-3\alpha^2)x-3\alpha+5\alpha^2}{(1-\alpha)^3}. \end{aligned}$$

Taking $g(x) = (2\alpha-1)x^2+(2-2\alpha-3\alpha^2)x-3\alpha+5\alpha^2$. If $\alpha = \frac{1}{2}$, then $g(x) = \frac{1}{4}(x-1)$. Thus $g(x) > 0$ for all $x > 1$. Then $q(x)-p(x) > 0$ for all $x > 1$. However, by Lemma 2.1, we have $\lambda_\alpha(G') > 1$, Then, we get $\lambda_\alpha(G) = \lambda_\alpha(G') > \lambda_\alpha(\theta(0,2,n-4))$. If $0 \leqslant \alpha < \frac{1}{2}$, then $2\alpha-1 < 0$, and $g(x)'' < 0$ for $1 < x < 2$. Hence $g(x) > \min\{g(1),g(2)\} = \min\{1-3\alpha+2\alpha^2, \alpha-\alpha^2\} \geqslant 0$ for $0 \leqslant \alpha \leqslant \frac{1}{2}$. Hence $q(x)-p(x) > 0$ for all $1 < x < 2$. However, by Lemma 2.1, we have $1 < \lambda_\alpha(G') < 2$. Then, we have $\lambda_\alpha(G) = \lambda_\alpha(G') > \lambda_\alpha(\theta(0,2,n-4))$.

*Subcase 2.3.2.* $G$ has a $\theta$-subdigraph $\theta(1,1,n-4)$. Let $uwv$, $uw_1v$ and $vw'_1w'_2\dots w'_{n-4}u$ be the basic directed paths of the $\theta$-subdigraph $\theta(1,1,n-4)$. We consider the possible arc(s) in $G$ except the arcs in $\theta(1,1,n-4)$ as follows.

(1) $(w,u) \notin E(G)$ and $(v,w) \notin E(G)$, otherwise, $G$ has a $\theta$-subdigraph $\theta(0,n-3,0)$, a contradiction.

(2) $(w_1,u)\notin E(G)$ and $(v,w_1)\notin E(G)$, otherwise, $G$ has a $\theta$-subdigraph $\theta(0,n-3,0)$, a contradiction.

(3) $(v,u)\notin E(G)$, otherwise, $G$ has a $\theta$-subdigraph $\theta(0,n-4,1)$, a contradiction.

(4) $(u,v)\notin E(G)$, otherwise, $G$ has a $\theta$-subdigraph $\theta(0,1,n-4)$, a contradiction.

(5) $(w,w'_k)\notin E(G)$ and $(w'_{n-k-3},w)\notin E(G)$ for $1\leqslant k\leqslant n-4$, otherwise, $G$ has a $\theta$-subdigraph $\theta(0,k,n-k-3)$, a contradiction.

(6) $(w_1,w'_k)\notin E(G)$ and $(w'_{n-k-3},w_1)\notin E(G)$ for $1\leqslant k\leqslant n-4$, otherwise, $G$ has a $\theta$-subdigraph $\theta(0,k,n-k-3)$, a contradiction.

(7) $(v,w'_k)\notin E(G)$ for $2\leqslant k\leqslant n-4$, otherwise, $G$ has a $\theta$-subdigraph $\theta(0,k-1,n-k-2)$, a contradiction.

(8) $(w'_k,v)\notin E(G)$ for $1\leqslant k\leqslant n-4$, otherwise, $G$ has a $\theta$-subdigraph $\theta(0,n-k-2,k-1)$, a contradiction.

(9) $(u,w'_k)\notin E(G)$ for $1\leqslant k\leqslant n-4$, otherwise, $G$ has a $\theta$-subdigraph $\theta(0,k+1,n-k-4)$, a contradiction.

(10) $(w'_k,u)\notin E(G)$ for $1\leqslant k\leqslant n-5$, otherwise, $G$ has a $\theta$-subdigraph $\theta(0,n-k-4,k+1)$, a contradiction.

(11) $(w'_l,w'_k)\notin E(G)$ for $1\leqslant k<l\leqslant n-4$, otherwise, $G$ has a $\theta$-subdigraph $\theta(0,n-l+k-2,l-k-1)$, a contradiction.

(12) $(w'_k,w'_l)\notin E(G)$ for $1\leqslant k<l-1\leqslant n-5$, otherwise, $G$ has a $\theta$-subdigraph $\theta(1,1,n-3+k-l)$, a contradiction.

(13) $\{(w,w_1),(w_1,w)\}\notin E(G)$, otherwise, $G$ has a $\theta$-subdigraph $\theta(0,n-2,0)$, a contradiction.

From (1)–(13), we find that besides these arcs in $\theta(1,1,n-4)$, $G$ only contains one additional arc $(w,w_1)$ or $(w_1,w)$. Thus $G$ is isomorphic to the digraph $G_1$ or $G_2$, where $G_1$ and $G_2$ as shown in the Figure 4.

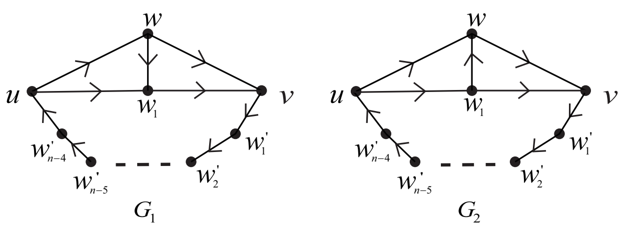


Figure 4: *The digraphs $G_1$ and $G_2$.*

If $G$ is isomorphic to the digraph $G_1$, one can easily get that $\lambda_\alpha(G_1)$ is the largest real root of the equation $(\frac{x-2\alpha}{1-\alpha})^2(\frac{x-\alpha}{1-\alpha})^{n-2}-\frac{2x-3\alpha}{1-\alpha}-1=0$. From the proof of subcase 2.3.1, we have $\lambda_\alpha(G_1)=\lambda_\alpha(G')>\lambda_\alpha(\theta(0,2,n-4))$. Thus we have $\lambda_\alpha(G)=\lambda_\alpha(G_1)=\lambda_\alpha(G')>\lambda_\alpha(\theta(0,2,n-4))$.

If $G$ is isomorphic to the digraph $G_2$, note that $G_2$ isomorphic to the digraph $G'$ as shown in subcase 2.3.1. Thus we have $\lambda_\alpha(G)=\lambda_\alpha(G_2)=\lambda_\alpha(G')>\lambda_\alpha(\theta(0,2,n-4))$.

Combining the above two cases, we have $\lambda_\alpha(G)>\lambda_\alpha(\theta(0,2,n-4))$, if $G$ is a strongly connected digraph of order $n\geqslant 5$ that is neither a bicyclic digraph nor $C_n$. $\square$

By Corollary 3.2, we know that $C_n$ is the unique strongly connected digraph with the minimum $A_\alpha$ spectral radius among all the strongly connected digraphs of order $n$. Therefore, from Theorems 2.10, 2.12 and 3.4, we have the following theorem.

THEOREM 3.5. *Among all the strongly connected digraphs with order $n \geqslant 5$ and $0 \leqslant \alpha \leqslant \frac{1}{2}$, $\theta(0,1,n-3)$, $\theta(1,1,n-4)$ and $\theta(0,2,n-4)$ are the digraphs which achieve the second, the third and the fourth minimum $A_\alpha$ spectral radius, respectively.*

REMARK 3.6. If $\alpha = 0$, Li and Zhou [7] proved that $\theta(0,1,n-3)$, $\theta(1,1,n-4)$ and $\theta(0,2,n-4)$ are the unique digraphs which achieve the second, the third and the fourth minimum $A_0$ spectral radius among all strongly connected digraphs, respectively. If $\alpha = \frac{1}{2}$, Hong and You [6] determined that $\theta(0,1,n-3)$, $\theta(1,1,n-4)$ and $\theta(0,2,n-4)$ also attain the second, the third and the fourth minimum $A_{\frac{1}{2}}$ spectral radius among all strongly connected digraphs, respectively.

For general $0 \leqslant \alpha < 1$, we propose the following conjecture based on numerical examples.

CONJECTURE 3.7. Among all the strongly connected digraphs with order $n \geqslant 5$ and $0 \leqslant \alpha < 1$, $\theta(0,1,n-3)$, $\theta(1,1,n-4)$ and $\theta(0,2,n-4)$ are the digraphs which achieve the second, the third and the fourth minimum $A_\alpha$ spectral radius, respectively.

## 4. The $A_\alpha$ spectral radius of strongly connected bipartite digraphs which contain a complete bipartite subdigraph

Let $\overleftrightarrow{K_{p,q}}$ be a complete bipartite digraph with $V(\overleftrightarrow{K_{p,q}}) = V_p \cup V_q$ and $|V_p| = p$, $|V_q| = q$. Let $\mathscr{G}_{n,p,q}$ denote the set of strongly connected bipartite digraphs on $n$ vertices which contain a complete bipartite subdigraph $\overleftrightarrow{K_{p,q}}$. As we all know, if $p+q=n$, then $\mathscr{G}_{n,p,q} = \{\overleftrightarrow{K_{p,q}}\}$. It is easy to know that $\lambda_\alpha(\overleftrightarrow{K_{p,q}}) = \frac{\alpha(p+q)+\sqrt{(\alpha(p+q))^2-8\alpha pq+4pq}}{2}$. Thus we only consider the cases when $p+q \leqslant n-1$ and $p \geqslant q \geqslant 2$. In the rest of this section, we just discuss under this assumption.

Chen et al. [2] proved that if $n \equiv p+q \ (mod\ 2)$ then $B^5_{n,p,q}$ or $B^6_{n,p,q}$ is the unique bipartite digraph with the minimum $A_0$ spectral radius among all digraphs in $\mathscr{G}_{n,p,q}$, otherwise, if $n \not\equiv p+q \ (mod\ 2)$ then $B^1_{n,p,q}$ is the unique bipartite digraph with the minimum $A_0$ spectral radius among all digraphs in $\mathscr{G}_{n,p,q}$. We generalize their results to $0 \leqslant \alpha < 1$.

Let $B^1_{n,p,q}$ be a digraph obtained by adding a directed path $P_{n-p-q+2} = v_1 v_{p+q+1} v_{p+q+2} \dots v_n v_p$ to a complete bipartite digraph $\overleftrightarrow{K_{p,q}}$ such that $V(\overleftrightarrow{K_{p,q}}) \cap V(P_{n-p-q+2}) = \{v_1, v_p\}$ as shown in Figure 5(a), where $V(B^1_{n,p,q}) = \{v_1, v_2, \dots, v_n\}$. Clearly, if $n-p-q$ is odd, then $B^1_{n,p,q} \in \mathscr{G}_{n,p,q}$.

Let $B^2_{n,p,q}$ be a digraph obtained by adding a directed path $P_{n-p-q+2} = v_{p+1} v_{p+q+1} v_{p+q+2} \dots v_n v_{p+q}$ to a complete bipartite digraph $\overleftrightarrow{K_{p,q}}$ such that $V(\overleftrightarrow{K_{p,q}}) \cap V(P_{n-p-q+2})$

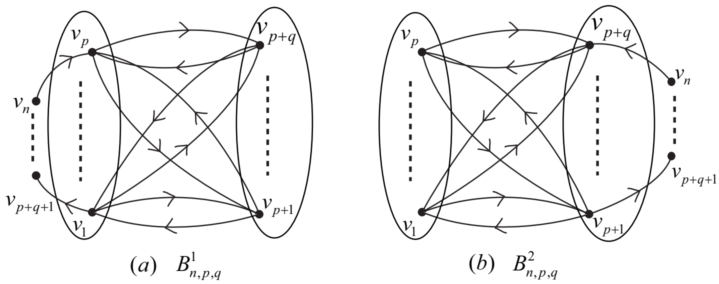


Figure 5: $B^1_{n,p,q}$ *and* $B^2_{n,p,q}$.

$= \{v_{p+1}, v_{p+q}\}$ as shown in Figure 5(b), where $V(B^2_{n,p,q}) = \{v_1, v_2, \ldots, v_n\}$. Clearly, if $n-p-q$ is odd, then $B^2_{n,p,q} \in \mathscr{G}_{n,p,q}$.

Let $B^5_{n,p,q}$ be a digraph obtained by adding a directed path $P_{n-p-q+2} = v_1 v_{p+q+1} v_{p+q+2} \ldots v_n v_{p+1}$ to a complete bipartite digraph $\overleftrightarrow{K_{p,q}}$ such that $V(\overleftrightarrow{K_{p,q}}) \cap V(P_{n-p-q+2}) = \{v_1, v_{p+1}\}$ as shown in Figure 6(a), where $V(B^5_{n,p,q}) = \{v_1, v_2, \ldots, v_n\}$. Clearly, if $n-p-q$ is even, then $B^5_{n,p,q} \in \mathscr{G}_{n,p,q}$.

Let $B^6_{n,p,q}$ be a digraph obtained by adding a directed path $P_{n-p-q+2} = v_{p+1} v_{p+q+1} v_{p+q+2} \ \ldots v_n v_1$ to a complete bipartite digraph $\overleftrightarrow{K_{p,q}}$ such that $V(\overleftrightarrow{K_{p,q}}) \cap V(P_{n-p-q+2}) = \{v_1, v_{p+1}\}$ as shown in Figure 6(b), where $V(B^6_{n,p,q}) = \{v_1, v_2, \ldots, v_n\}$. Clearly, if $n-p-q$ is even, then $B^6_{n,p,q} \in \mathscr{G}_{n,p,q}$.

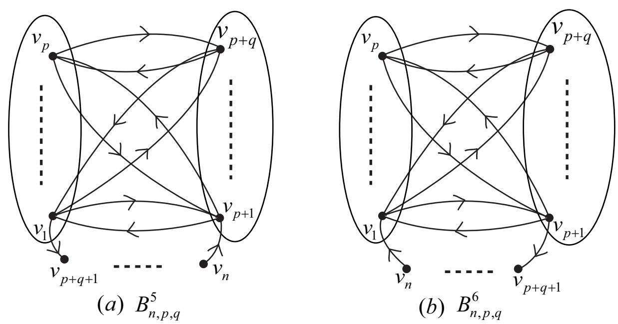


Figure 6: $B^5_{n,p,q}$ *and* $B^6_{n,p,q}$.

LEMMA 4.1. ([14]) *Let $0 \leqslant \alpha < 1$ and $G$ be a strongly connected digraph. Then $\lambda_\alpha(G) > \alpha\Delta^+$, where $\Delta^+$ denotes the maximum outdegree of $G$.*

LEMMA 4.2. *Let $G$ be a strongly connected digraph containing two vertices $v_i, v_j$ such that $(v_i, v_j) \notin E(G)$ and $(v_j, v_i) \notin E(G)$, and let $X$ be the Perron vector of $A_\alpha(G)$, where $x_i$ corresponds to the vertex $v_i$. If $N_i^+ \subseteq N_j^+$, then $x_j \geqslant x_i$. Moreover, if $N_i^+ \subsetneq N_j^+$, then $x_j > x_i$, if $N_i^+ = N_j^+$, then $x_j = x_i$.*

*Proof.* From $A_\alpha(G)X = \lambda_\alpha(G)X$, we have

$$\lambda_\alpha(G)x_i = \alpha d_i^+ x_i + (1-\alpha)\sum_{v_k \in N_i^+} x_k,$$

$$\lambda_\alpha(G)x_j = \alpha d_j^+ x_j + (1-\alpha)\sum_{v_k \in N_j^+} x_k.$$

Since $(v_i, v_j) \notin E(G)$ and $(v_j, v_i) \notin E(G)$, and $N_i^+ \subseteq N_j^+$, we have $d_i^+ \leqslant d_j^+$. Furthermore, we get $(\lambda_\alpha(G) - \alpha d_j^+)x_j \geqslant (\lambda_\alpha(G) - \alpha d_i^+)x_i$. By Lemma 4.1, $\lambda_\alpha(G) > \alpha\Delta^+$. So $x_j \geqslant x_i$.

Since $v_j \notin N_i^+$, $v_i \notin N_j^+$, if $N_i^+ \subsetneq N_j^+$, then $d_i^+ < d_j^+$ and $(\lambda_\alpha(G) - \alpha d_j^+)x_j > (\lambda_\alpha(G) - \alpha d_i^+)x_i$, which implies $x_j > x_i$, and if $N_i^+ = N_j^+$, then $d_i^+ = d_j^+$ and $(\lambda_\alpha(G) - \alpha d_j^+)x_j = (\lambda_\alpha(G) - \alpha d_i^+)x_i$, which implies $x_i = x_j$. $\square$

THEOREM 4.3. *For digraphs $B^1_{n,p,q}$ and $B^2_{n,p,q}$, as shown in Figure 5,*

$$\lambda_\alpha(B^2_{n,p,q}) \geqslant \lambda_\alpha(B^1_{n,p,q}),$$

*with equality if and only if $p = q$.*

*Proof.* If $p = q$, then $B^2_{n,p,q} \cong B^1_{n,p,q}$. Hence $\lambda_\alpha(B^1_{n,p,q}) = \lambda_\alpha(B^2_{n,p,q})$. Otherwise $p > q$, let $G = B^1_{n,p,q}$ and $X = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector corresponding to $\lambda_\alpha(G)$, where $x_i$ corresponds to the vertex $v_i$. By Lemma 4.2, $x_2 = x_3 = \cdots = x_p \triangleq x_p$ and $x_{p+1} = x_{p+2} = \cdots = x_{p+q} \triangleq x_{p+1}$. From $A_\alpha(G)X = \lambda_\alpha(G)X$, we have

$$\begin{cases} \lambda_\alpha(G)x_1 = \alpha(q+1)x_1 + (1-\alpha)x_{p+q+1} + (1-\alpha)qx_{p+1}, \\ \lambda_\alpha(G)x_p = \alpha q x_p + (1-\alpha)qx_{p+1}, \\ \lambda_\alpha(G)x_{p+1} = \alpha p x_{p+1} + (1-\alpha)x_1 + (1-\alpha)(p-1)x_p, \\ \lambda_\alpha(G)x_i = \alpha x_i + (1-\alpha)x_{i+1}, & i = p+q+1, \ldots, n-1, \\ \lambda_\alpha(G)x_n = \alpha x_n + (1-\alpha)x_p. \end{cases}$$

Then

$$x_n = \left(\frac{\lambda_\alpha(G) - \alpha}{1-\alpha}\right)^{n-p-q-1} x_{p+q+1},$$

$$x_p = \left(\frac{\lambda_\alpha(G) - \alpha}{1-\alpha}\right) x_n = \left(\frac{\lambda_\alpha(G) - \alpha}{1-\alpha}\right)^{n-p-q} x_{p+q+1}.$$

Therefore,

$$\begin{aligned}(\lambda_\alpha(G) - \alpha q)(\lambda_\alpha(G) - \alpha p)x_p &= (1-\alpha)\cdot q \cdot (\lambda_\alpha(G) - \alpha p)x_{p+1} \\ &= (1-\alpha)^2 \cdot qx_1 + (1-\alpha)^2 \cdot q \cdot (p-1)x_p.\end{aligned}$$

Furthermore

$$\begin{aligned}
&(\lambda_\alpha(G)-\alpha q)\cdot(\lambda_\alpha(G)-\alpha p)\cdot(\lambda_\alpha(G)-\alpha(q+1))x_p\\
&=(1-\alpha)^2\cdot q\cdot(\lambda_\alpha(G)-\alpha(q+1))x_1+(1-\alpha)^2\cdot q\cdot(p-1)(\lambda_\alpha(G)-\alpha(q+1))x_p\\
&=(1-\alpha)^2\cdot q\cdot((1-\alpha)x_{p+q+1}+(1-\alpha)\cdot qx_{p+1})\\
&\quad+(1-\alpha)^2\cdot q\cdot(p-1)(\lambda_\alpha(G)-\alpha(q+1))x_p\\
&=(1-\alpha)^3\cdot q\cdot\frac{1}{\left(\frac{\lambda_\alpha(G)-\alpha}{1-\alpha}\right)^{n-p-q}}x_p+(1-\alpha)^2\cdot q\cdot(\lambda_\alpha(G)-\alpha q)x_p\\
&\quad+(1-\alpha)^2\cdot q\cdot(p-1)\cdot(\lambda_\alpha(G)-\alpha(q+1))x_p.
\end{aligned}$$

Note that $x_p>0$. Hence

$$\begin{aligned}
&\left(\frac{\lambda_\alpha(G)-\alpha}{1-\alpha}\right)^{n-p-q}[(\lambda_\alpha(G)-\alpha q)\cdot(\lambda_\alpha(G)-\alpha p)\cdot(\lambda_\alpha(G)-\alpha(q+1))\\
&\quad-(1-\alpha)^2\cdot q\cdot(\lambda_\alpha(G)-\alpha q)-(1-\alpha)^2\cdot q\cdot(p-1)(\lambda_\alpha(G)-\alpha(q+1))]\\
&\quad-(1-\alpha)^3q=0.
\end{aligned}$$

Let $f(x)=\left(\frac{x-\alpha}{1-\alpha}\right)^{n-p-q}[x^3-(\alpha p+2\alpha q+\alpha)x^2+(\alpha^2q^2+\alpha^2pq+2\alpha pq+\alpha^2q+\alpha^2p-pq)x-2\alpha^2q^2p-2\alpha^2pq+\alpha q^2p+\alpha pq+2\alpha^2q-\alpha^3q-\alpha q]-(1-\alpha)^3q$. It is not difficult to see that $\lambda_\alpha(B^1_{n,p,q})$ is the largest real root of $f(x)=0$. Similarly, let $g(x)=\left(\frac{x-\alpha}{1-\alpha}\right)^{n-p-q}[x^3-(\alpha q+2\alpha p+\alpha)x^2+(\alpha^2p^2+\alpha^2pq+2\alpha pq+\alpha^2p+\alpha^2q-pq)x-2\alpha^2p^2q-2\alpha^2pq+\alpha p^2q+\alpha pq+2\alpha^2p-\alpha^3p-\alpha p]-(1-\alpha)^3p$, then $\lambda_\alpha(B^2_{n,p,q})$ is the largest real root of $g(x)=0$. Thus

$$\begin{aligned}
&f(x)-g(x)\\
&=\left(\frac{x-\alpha}{1-\alpha}\right)^{n-p-q}[\alpha(p-q)x^2-\alpha^2(p-q)(p+q)x+\alpha(p-q)\\
&\quad-\alpha pq(p-q)+2\alpha^2pq(p-q)-2\alpha^2(p-q)+\alpha^3(p-q)]+(1-\alpha)^3(p-q)\\
&=\left(\frac{x-\alpha}{1-\alpha}\right)^{n-p-q}\cdot\alpha\cdot(p-q)\cdot[x^2-\alpha(p+q)x+1-pq+2\alpha pq-2\alpha+\alpha^2]\\
&\quad+(1-\alpha)^3(p-q).
\end{aligned}$$

Since $\lambda_\alpha(\overleftrightarrow{K_{p,q}})$ is the the largest real root of the equation $x^2-\alpha(p+q)x-pq+2\alpha pq=0$, $x^2-\alpha(p+q)x-pq+2\alpha pq>0$ for all $x>\lambda_\alpha(\overleftrightarrow{K_{p,q}})$. Thus $x^2-\alpha(p+q)x+1-pq+2\alpha pq-2\alpha+\alpha^2=x^2-\alpha(p+q)x-pq+2\alpha pq+(1-\alpha)^2>0$ for all $x>\lambda_\alpha(\overleftrightarrow{K_{p,q}})$. Since $p>q$, $f(x)-g(x)>0$ for all $x>\lambda_\alpha(\overleftrightarrow{K_{p,q}})>1$. By Lemma 3.1, we have $\lambda_\alpha(B^2_{n,p,q})>\lambda_\alpha(\overleftrightarrow{K_{p,q}})>1$. Hence $f(x)-g(x)>0$ for all $x\geqslant\lambda_\alpha(B^2_{n,p,q})$. Then $\lambda_\alpha(B^2_{n,p,q})>\lambda_\alpha(B^1_{n,p,q})$.

Therefore, $\lambda_\alpha(B^2_{n,p,q})\geqslant\lambda_\alpha(B^1_{n,p,q})$ with equality if and only if $p=q$. $\square$

THEOREM 4.4. *For digraphs $B^5_{n,p,q}$ and $B^6_{n,p,q}$, as shown in Figure 6,*

$$\lambda_\alpha(B^6_{n,p,q}) \geqslant \lambda_\alpha(B^5_{n,p,q}),$$

*with equality if and only if $p = q$ or $\alpha = 0$.*

*Proof.* If $p = q$, then $B^5_{n,p,q} \cong B^6_{n,p,q}$. Hence $\lambda_\alpha(B^5_{n,p,q}) = \lambda_\alpha(B^6_{n,p,q})$. Otherwise $p > q$, let $G = B^5_{n,p,q}$. Similar to the proof of Theorem 4.3, we can know that $\lambda_\alpha(G)$ satisfies the follow equation

$$\begin{aligned}
&\left(\frac{\lambda_\alpha(G)-\alpha}{1-\alpha}\right)^{n-p-q}[\lambda_\alpha^3(G)-(\alpha p+2\alpha q+\alpha)\lambda_\alpha^2(G)\\
&+(\alpha^2q^2+\alpha^2pq+2\alpha pq+\alpha^2q+\alpha^2p-pq)\lambda_\alpha(G)-2\alpha^2q^2p-2\alpha^2pq\\
&+\alpha q^2p+\alpha pq+2\alpha^2q-\alpha^3q-\alpha q]-(1-\alpha)^2(\lambda_\alpha(G)-\alpha q)=0.
\end{aligned}$$

Let $f(x) = \left(\frac{x-\alpha}{1-\alpha}\right)^{n-p-q}[x^3-(\alpha p+2\alpha q+\alpha)x^2+(\alpha^2q^2+\alpha^2pq+2\alpha pq+\alpha^2q+\alpha^2p-pq)x-2\alpha^2q^2p-2\alpha^2pq+\alpha q^2p+\alpha pq+2\alpha^2q-\alpha^3q-\alpha q]-(1-\alpha)^2(x-\alpha q)$. Then $\lambda_\alpha(B^5_{n,p,q})$ is the largest real root of $f(x) = 0$. Similarly, let $g(x) = \left(\frac{x-\alpha}{1-\alpha}\right)^{n-p-q}[x^3-(\alpha q+2\alpha p+\alpha)x^2+(\alpha^2p^2+\alpha^2pq+2\alpha pq+\alpha^2p+\alpha^2q-pq)x-2\alpha^2p^2q-2\alpha^2pq+\alpha p^2q+\alpha pq+2\alpha^2p-\alpha^3p-\alpha p]-(1-\alpha)^2(x-\alpha p)$, then $\lambda_\alpha(B^6_{n,p,q})$ is the largest real root of $g(x) = 0$. Thus

$$\begin{aligned}
f(x)-g(x) = &\left(\frac{x-\alpha}{1-\alpha}\right)^{n-p-q}\cdot\alpha\cdot(p-q)\cdot[x^2-\alpha(p+q)x-pq+2\alpha pq+(1-\alpha)^2]\\
&-(1-\alpha)^2\alpha(p-q).
\end{aligned}$$

For $\alpha = 0$, $f(x) = g(x)$, then $\lambda_\alpha(B^5_{n,p,q}) = \lambda_\alpha(B^6_{n,p,q})$.

For $0 < \alpha < 1$. Since $\lambda_\alpha(\overleftrightarrow{K_{p,q}})$ is the largest real root of the equation $x^2-\alpha(p+q)x-pq+2\alpha pq = 0$, $x^2-\alpha(p+q)x-pq+2\alpha pq > 0$ for all $x > \lambda_\alpha(\overleftrightarrow{K_{p,q}})$. Thus $x^2-\alpha(p+q)x-pq+2\alpha pq+(1-\alpha)^2 > (1-\alpha)^2$ for all $x > \lambda_\alpha(\overleftrightarrow{K_{p,q}})$. Since $p > q$, $f(x)-g(x) > \left(\frac{x-\alpha}{1-\alpha}\right)^{n-p-q}(1-\alpha)^2\alpha(p-q)-(1-\alpha)^2\alpha(p-q) > 0$ for all $x > \lambda_\alpha(\overleftrightarrow{K_{p,q}}) > 1$. By Lemma 3.1, we have $\lambda_\alpha(B^6_{n,p,q}) > \lambda_\alpha(\overleftrightarrow{K_{p,q}}) > 1$. Hence $f(x)-g(x) > 0$ for all $x \geqslant \lambda_\alpha(B^6_{n,p,q})$. Then $\lambda_\alpha(B^6_{n,p,q}) > \lambda_\alpha(B^5_{n,p,q})$.

Therefore, $\lambda_\alpha(B^6_{n,p,q}) \geqslant \lambda_\alpha(B^5_{n,p,q})$ with equality if and only if $p = q$ or $\alpha = 0$. □

THEOREM 4.5. *Let $B^3_{n,p,q} = B^1_{n,p,q} - \{(v_n, v_p)\} + \{(v_n, v_1)\}$. Then*

$$\lambda_\alpha(B^3_{n,p,q}) > \lambda_\alpha(B^1_{n,p,q}).$$

*Proof.* Clearly $B^3_{n,p,q}$ is strongly connected. Let $X = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector corresponding to $\lambda_\alpha(B^1_{n,p,q})$, where $x_i$ corresponds to the vertex $v_i$. By Lemma 4.2, we get $x_1 > x_p$. Thus $\lambda_\alpha(B^3_{n,p,q}) > \lambda_\alpha(B^1_{n,p,q})$ by Lemma 2.7. □

THEOREM 4.6. *Let $B^4_{n,p,q} = B^2_{n,p,q} - \{(v_n, v_{p+q})\} + \{(v_n, v_{p+1})\}$. Then*

$$\lambda_\alpha(B^4_{n,p,q}) > \lambda_\alpha(B^2_{n,p,q}).$$

*Proof.* Clearly $B^4_{n,p,q}$ is strongly connected. Let $X = (x_1, x_2, \dots, x_n)^T$ be the Perron vector corresponding to $\lambda_\alpha(B^2_{n,p,q})$, where $x_i$ corresponds to the vertex $v_i$. By Lemma 4.2, we get $x_{p+1} > x_{p+q}$. Thus $\lambda_\alpha(B^4_{n,p,q}) > \lambda_\alpha(B^2_{n,p,q})$ by Lemma 2.7. $\square$

THEOREM 4.7. *For digraphs $B^1_{n,p,q}$ and $B^5_{n,p,q}$, as shown in Figures 5 and 6,*

$$\lambda_\alpha(B^5_{n-1,p,q}) > \lambda_\alpha(B^1_{n,p,q}).$$

*Proof.* Since $B^5_{n,p,q} = B^1_{n,p,q} - \{(v_n, v_p)\} + \{(v_n, v_{p+1})\}$ and $B^5_{n,p,q}$ is strongly connected. Let $X = (x_1, x_2, \dots, x_n)^T$ be the Perron vector corresponding to $\lambda_\alpha(B^1_{n,p,q})$, where $x_i$ corresponds to the vertex $v_i$. In the following, we will prove $x_{p+1} > x_p$.

By Lemma 4.2, $x_2 = x_3 = \cdots = x_p \triangleq x_p$, $x_{p+1} = x_{p+2} = \cdots = x_{p+q} \triangleq x_{p+1}$ and $x_1 > x_p$. Therefore

$$\begin{aligned}\lambda_\alpha(B^1_{n,p,q})x_{p+1} &= \alpha p x_{p+1} + (1-\alpha)x_1 + (1-\alpha)(p-1)x_p \\ &> \alpha p x_{p+1} + (1-\alpha)x_p + (1-\alpha)(p-1)x_p \\ &= \alpha p(x_{p+1} - x_p) + p x_p,\end{aligned}$$

$$\lambda_\alpha(B^1_{n,p,q})x_p = \alpha q x_p + (1-\alpha) q x_{p+1}.$$

Hence

$$\lambda_\alpha(B^1_{n,p,q})(x_{p+1} - x_p) > (\alpha p + \alpha q - q)(x_{p+1} - x_p) + (p-q)x_p.$$

Furthermore

$$(\lambda_\alpha(B^1_{n,p,q} - (\alpha p + \alpha q - q))(x_{p+1} - x_p) > (p-q)x_p \geqslant 0.$$

However, by Lemma 4.1, we get $\lambda_\alpha(B^1_{n,p,q}) > \alpha\Delta^+ \geqslant \alpha p > \alpha p + \alpha q - q$. Thus $x_{p+1} > x_p$. Therefore $\lambda_\alpha(B^5_{n,p,q}) > \lambda_\alpha(B^1_{n,p,q})$. By Lemma 3.3, we have $\lambda_\alpha(B^5_{n-1,p,q}) \geqslant \lambda_\alpha(B^5_{n,p,q})$. Then $\lambda_\alpha(B^5_{n-1,p,q}) > \lambda_\alpha(B^1_{n,p,q})$. $\square$

THEOREM 4.8. *For digraphs $B^1_{n,p,q}$ and $B^5_{n,p,q}$, as shown in Figures 5 and 6,*

$$\lambda_\alpha(B^1_{n-1,p,q}) \geqslant \lambda_\alpha(B^5_{n,p,q}).$$

*Proof.* Let $B^{5*}_{n,p,q} = B^5_{n,p,q} - \{(v_{n-1}, v_n)\} + \{(v_{n-1}, v_p)\}$. Let $X = (x_1, x_2, \dots, x_n)^T$ be the Perron vector corresponding to $\lambda_\alpha(B^5_{n,p,q})$, where $x_i$ corresponds to the vertex $v_i$. By Lemma 4.2, we get $x_p > x_n$. Then $\lambda_\alpha(B^{5*}_{n,p,q}) \geqslant \lambda_\alpha(B^5_{n,p,q})$. Since the in-degree of $v_n$ is 0 in $B^{5*}_{n,p,q}$, $B^{5*}_{n,p,q}$ is not strongly connected which contains $B^1_{n-1,p,q}$

as a induced subdigraph, we have $\lambda_\alpha(B^{5*}_{n,p,q}) = \lambda_\alpha(B^1_{n-1,p,q})$. Thus $\lambda_\alpha(B^1_{n-1,p,q}) \geqslant \lambda_\alpha(B^5_{n,p,q})$. $\square$

In the following, we give the main results of this section.

THEOREM 4.9. *Let $p \geqslant q \geqslant 2$, $p+q \leqslant n-1$, $n \equiv p+q \pmod 2$ and $G \in \mathscr{G}_{n,p,q}$. Then*

*(i) For $\alpha = 0$, $\lambda_\alpha(G) \geqslant \lambda_\alpha(B^5_{n,p,q}) = \lambda_\alpha(B^6_{n,p,q})$ and the equality holds if and only if $G \cong B^5_{n,p,q}$ or $G \cong B^6_{n,p,q}$.*

*(ii) For $0 < \alpha < 1$, $\lambda_\alpha(G) \geqslant \lambda_\alpha(B^5_{n,p,q})$ and the equality holds if and only if $G \cong B^5_{n,p,q}$.*

*Proof.* Since $G \in \mathscr{G}_{n,p,q}$, $\overleftrightarrow{K_{p,q}}$ is a proper subdigraph of $G$. Since $G$ is strongly connected, it is possible to obtain a digraph $H$ from $G$ by deleting vertices and arcs in a way such that one has a subdigraph $\overleftrightarrow{K_{p,q}}$. Therefore

(1) $H \cong B^1_{p+q+k,p,q}$, $(k \equiv 1 \pmod 2,\ k \geqslant 1)$ or
(2) $H \cong B^2_{p+q+k,p,q}$, $(k \equiv 1 \pmod 2,\ k \geqslant 1)$ or
(3) $H \cong B^3_{p+q+k,p,q}$, $(k \equiv 1 \pmod 2,\ k \geqslant 1)$ or
(4) $H \cong B^4_{p+q+k,p,q}$, $(k \equiv 1 \pmod 2,\ k \geqslant 1)$ or
(5) $H \cong B^5_{p+q+l,p,q}$, $(l \equiv 0 \pmod 2,\ l \geqslant 2)$ or
(6) $H \cong B^6_{p+q+l,p,q}$, $(l \equiv 0 \pmod 2,\ l \geqslant 2)$.

By Lemma 3.1, $\lambda_\alpha(G) \geqslant \lambda_\alpha(H)$, the equality holds if and only if $H \cong G$.

*Case (i).* $H \cong B^1_{p+q+k,p,q}$, $(k \equiv 1 \pmod 2,\ k \geqslant 1)$.

Insert $n-p-q-k-1$ vertices into the directed path $P_{k+2}$ such that the resulting bipartite digraph is $B^1_{n-1,p,q}$, then $\lambda_\alpha(H) \geqslant \lambda_\alpha(B^1_{n-1,p,q})$ by using Lemma 3.3 repeatedly $n-p-q-k-1$ times, and thus $\lambda_\alpha(G) > \lambda_\alpha(H) \geqslant \lambda_\alpha(B^1_{n-1,p,q}) \geqslant \lambda_\alpha(B^5_{n,p,q})$ by Theorem 4.8.

*Case (ii).* $H \cong B^2_{p+q+k,p,q}$, $(k \equiv 1 \pmod 2,\ k \geqslant 1)$.

Insert $n-p-q-k-1$ vertices into the directed path $P_{k+2}$ such that the resulting bipartite digraph is $B^2_{n-1,p,q}$, then $\lambda_\alpha(H) \geqslant \lambda_\alpha(B^2_{n-1,p,q})$ by using Lemma 3.3 repeatedly $n-p-q-k-1$ times, and thus $\lambda_\alpha(G) > \lambda_\alpha(H) \geqslant \lambda_\alpha(B^2_{n-1,p,q}) \geqslant \lambda_\alpha(B^1_{n-1,p,q}) \geqslant \lambda_\alpha(B^5_{n,p,q})$ by Theorems 4.3 and 4.8.

*Case (iii).* $H \cong B^3_{p+q+k,p,q}$, $(k \equiv 1 \pmod 2,\ k \geqslant 1)$.

Insert $n-p-q-k-1$ vertices into the directed cycle $C_{k+1}$ such that the resulting bipartite digraph is $B^3_{n-1,p,q}$, then $\lambda_\alpha(H) \geqslant \lambda_\alpha(B^3_{n-1,p,q})$ by using Lemma 3.3 repeatedly $n-p-q-k-1$ times, and thus $\lambda_\alpha(G) > \lambda_\alpha(H) \geqslant \lambda_\alpha(B^3_{n-1,p,q}) > \lambda_\alpha(B^1_{n-1,p,q}) \geqslant \lambda_\alpha(B^5_{n,p,q})$ by Theorems 4.5 and 4.8.

*Case (iv).* $H \cong B^4_{p+q+k,p,q}$, $(k \equiv 1 \pmod 2,\ k \geqslant 1)$.

Insert $n-p-q-k-1$ vertices into the directed cycle $C_{k+1}$ such that the resulting bipartite digraph is $B^4_{n-1,p,q}$, then $\lambda_\alpha(H) \geqslant \lambda_\alpha(B^4_{n-1,p,q})$ by using Lemma 3.3 repeatedly $n-p-q-k-1$ times, and thus $\lambda_\alpha(G) > \lambda_\alpha(H) \geqslant \lambda_\alpha(B^4_{n-1,p,q}) > \lambda_\alpha(B^2_{n-1,p,q}) \geqslant \lambda_\alpha(B^1_{n-1,p,q}) \geqslant \lambda_\alpha(B^5_{n,p,q})$ by Theorems 4.6, 4.3 and 4.8.

*Case* $(v)$. $H \cong B^5_{p+q+l,p,q}$, $\quad (l \equiv 0 \pmod 2,\ l \geqslant 2)$.

Insert $n-p-q-l$ vertices into the directed path $P_{l+2}$ such that the resulting bipartite digraph is $B^5_{n,p,q}$, then $\lambda_\alpha(H) \geqslant \lambda_\alpha(B^5_{n,p,q})$ by using Lemma 3.3 repeatedly $n-p-q-l$ times. Hence, by Theorem 4.4, we have

(1) For $\alpha = 0$, $\lambda_\alpha(G) \geqslant \lambda_\alpha(H) \geqslant \lambda_\alpha(B^5_{n,p,q}) = \lambda_\alpha(B^6_{n,p,q})$.

(2) For $0 < \alpha < 1$, $\lambda_\alpha(G) \geqslant \lambda_\alpha(H) \geqslant \lambda_\alpha(B^5_{n,p,q})$.

*Case* $(vi)$. $H \cong B^6_{p+q+l,p,q}$, $\quad (l \equiv 0 \pmod 2,\ l \geqslant 2)$.

Insert $n-p-q-l$ vertices into the directed path $P_{l+2}$ such that the resulting bipartite digraph is $B^6_{n,p,q}$, then $\lambda_\alpha(H) \geqslant \lambda_\alpha(B^6_{n,p,q})$ by using Lemma 3.3 repeatedly $n-p-q-l$ times. Hence, by Theorem 4.4, we have

(1) For $\alpha = 0$, $\lambda_\alpha(G) \geqslant \lambda_\alpha(H) \geqslant \lambda_\alpha(B^5_{n,p,q}) = \lambda_\alpha(B^6_{n,p,q})$.

(2) For $0 < \alpha < 1$, $\lambda_\alpha(G) \geqslant \lambda_\alpha(H) \geqslant \lambda_\alpha(B^6_{n,p,q}) \geqslant \lambda_\alpha(B^5_{n,p,q})$.

Combining the above six cases, we have

(1) For $\alpha = 0$, $\lambda_\alpha(G) \geqslant \lambda_\alpha(B^5_{n,p,q}) = \lambda_\alpha(B^6_{n,p,q})$ and the equality holds if and only if $G \cong B^5_{n,p,q}$ or $G \cong B^6_{n,p,q}$.

(2) For $0 < \alpha < 1$, $\lambda_\alpha(G) \geqslant \lambda_\alpha(B^5_{n,p,q})$ and the equality holds if and only if $G \cong B^5_{n,p,q}$. $\square$

THEOREM 4.10. *Let* $p \geqslant q \geqslant 2$, $p+q \leqslant n-1$, $n \not\equiv p+q \pmod 2$ *and* $G \in \mathscr{G}_{n,p,q}$. *Then* $\lambda_\alpha(G) \geqslant \lambda_\alpha(B^1_{n,p,q})$ *and the equality holds if and only if* $G \cong B^1_{n,p,q}$.

*Proof.* Since $G \in \mathscr{G}_{n,p,q}$, $\overleftrightarrow{K_{p,q}}$ is a proper subdigraph of $G$. Since $G$ is strongly connected, it is possible to obtain a digraph $H$ from $G$ by deleting vertices and arcs in a way such that one has a subdigraph $\overleftrightarrow{K_{p,q}}$. Therefore

(1) $H \cong B^1_{p+q+k,p,q}$, $\quad (k \equiv 1 \pmod 2,\ k \geqslant 1)$ or

(2) $H \cong B^2_{p+q+k,p,q}$, $\quad (k \equiv 1 \pmod 2,\ k \geqslant 1)$ or

(3) $H \cong B^3_{p+q+k,p,q}$, $\quad (k \equiv 1 \pmod 2,\ k \geqslant 1)$ or

(4) $H \cong B^4_{p+q+k,p,q}$, $\quad (k \equiv 1 \pmod 2,\ k \geqslant 1)$ or

(5) $H \cong B^5_{p+q+l,p,q}$, $\quad (l \equiv 0 \pmod 2,\ l \geqslant 2)$ or

(6) $H \cong B^6_{p+q+l,p,q}$, $\quad (l \equiv 0 \pmod 2,\ l \geqslant 2)$.

By Lemma 3.1, $\lambda_\alpha(G) \geqslant \lambda_\alpha(H)$, the equality holds if and only if $H \cong G$.

*Case* $(i)$. $H \cong B^1_{p+q+k,p,q}$, $\quad (k \equiv 1 \pmod 2,\ k \geqslant 1)$.

Insert $n-p-q-k$ vertices into the directed path $P_{k+2}$ such that the resulting bipartite digraph is $B^1_{n,p,q}$, then $\lambda_\alpha(H) \geqslant \lambda_\alpha(B^1_{n,p,q})$ by using Lemma 3.3 repeatedly $n-p-q-k$ times, and thus $\lambda_\alpha(G) \geqslant \lambda_\alpha(H) \geqslant \lambda_\alpha(B^1_{n,p,q})$.

*Case* $(ii)$. $H \cong B^2_{p+q+k,p,q}$, $\quad (k \equiv 1 \pmod 2,\ k \geqslant 1)$.

Insert $n-p-q-k$ vertices into the directed path $P_{k+2}$ such that the resulting bipartite digraph is $B^2_{n,p,q}$, then $\lambda_\alpha(H) \geqslant \lambda_\alpha(B^2_{n,p,q})$ by using Lemma 3.3 repeatedly $n-p-q-k$ times, and thus $\lambda_\alpha(G) \geqslant \lambda_\alpha(H) \geqslant \lambda_\alpha(B^2_{n,p,q}) \geqslant \lambda_\alpha(B^1_{n,p,q})$ by Theorem 4.3.

*Case* (*iii*). $H \cong B^3_{p+q+k,p,q}$, $\quad (k \equiv 1 \pmod 2,\ k \geqslant 1)$.

Insert $n-p-q-k$ vertices into the directed cycle $C_{k+1}$ such that the resulting bipartite digraph is $B^3_{n,p,q}$, then $\lambda_\alpha(H) \geqslant \lambda_\alpha(B^3_{n,p,q})$ by using Lemma 3.3 repeatedly $n-p-q-k$ times, and thus $\lambda_\alpha(G) \geqslant \lambda_\alpha(H) \geqslant \lambda_\alpha(B^3_{n,p,q}) > \lambda_\alpha(B^1_{n,p,q})$ by Theorem 4.5.

*Case* (*iv*). $H \cong B^4_{p+q+k,p,q}$, $\quad (k \equiv 1 \pmod 2,\ k \geqslant 1)$.

Insert $n-p-q-k$ vertices into the directed cycle $C_{k+1}$ such that the resulting bipartite digraph is $B^4_{n,p,q}$, then $\lambda_\alpha(H) \geqslant \lambda_\alpha(B^4_{n,p,q})$ by using Lemma 3.3 repeatedly $n-p-q-k$ times, and thus $\lambda_\alpha(G) \geqslant \lambda_\alpha(H) \geqslant \lambda_\alpha(B^4_{n,p,q}) > \lambda_\alpha(B^2_{n,p,q}) \geqslant \lambda_\alpha(B^1_{n,p,q})$ by Theorems 4.6 and 4.3.

*Case* (*v*). $H \cong B^5_{p+q+l,p,q}$, $\quad (l \equiv 0 \pmod 2,\ l \geqslant 2)$.

Insert $n-p-q-l-1$ vertices into the directed path $P_{l+2}$ such that the resulting bipartite digraph is $B^5_{n-1,p,q}$, then $\lambda_\alpha(H) \geqslant \lambda_\alpha(B^5_{n-1,p,q})$ by using Lemma 3.3 repeatedly $n-p-q-l-1$ times. Hence, by Theorem 4.7, we have $\lambda_\alpha(G) > \lambda_\alpha(H) \geqslant \lambda_\alpha(B^5_{n-1,p,q}) > \lambda_\alpha(B^1_{n,p,q})$.

*Case* (*vi*). $H \cong B^6_{p+q+l,p,q}$, $\quad (l \equiv 0 \pmod 2,\ l \geqslant 2)$.

Insert $n-p-q-l-1$ vertices into the directed path $P_{l+2}$ such that the resulting bipartite digraph is $B^6_{n-1,p,q}$, then $\lambda_\alpha(H) \geqslant \lambda_\alpha(B^6_{n-1,p,q})$ by using Lemma 3.3 repeatedly $n-p-q-l-1$ times. Hence, by Theorems 4.4 and 4.7, we have $\lambda_\alpha(G) > \lambda_\alpha(H) \geqslant \lambda_\alpha(B^6_{n-1,p,q}) \geqslant \lambda_\alpha(B^5_{n-1,p,q}) > \lambda_\alpha(B^1_{n,p,q})$.

Combining the above six cases, we have $\lambda_\alpha(G) \geqslant \lambda_\alpha(B^1_{n,p,q})$ and the equality holds if and only if $G \cong B^1_{n,p,q}$. $\square$

*Acknowledgements.* The authors would like to thank the anonymous referee for his or her helpful comments which help to improve the presentation of the manuscript.

## REFERENCES

[1] A. BERMAN, R. J. PLEMMONS, *Nonnegative Matrices in the Mathematical Sciences*, New York: Academic Press, 1979.

[2] S. T. CHEN, S. L. CHEN, W. Q. LIU, *The minimum spectral radius of strongly connected bipartite digraphs with complete bipartite subdigraph*, Quantitative Logic and Soft Computing 2016, Springer International Publishing 2017, 659–669.

[3] H. A. GANIE, M. BAGHIPUR, *On the generalized adjacency spectral radius of digraphs*, Linear Multilinear Algebra, 70 (2022), 3497–3510.

[4] G. Q. GUO, J. LIU, *Some results on the spectral radius of generalized $\infty$ and $\theta$-digraphs*, Linear Algebra Appl., 437 (2012), 2200–2208.

[5] R. A. HORN, C. R. JOHNSON, *Matrix Analysis*, Cambridge University Press, New York, 1985.

[6] W. X. HONG, L. H. YOU, *Spectral radius and signless Laplacian spectral radius of strongly connected digraphs*, Linear Algebra Appl., 457 (2014), 93–113.

[7] J. LI, B. ZHOU, *On the spectral radius of strongly connected digraphs*, Bull Iranian Math. Soc., 41 (2015), 381–387.

[8] H. Q. LIN, S. W. DRURY, *The maximum perron roots of digraphs with some given parameters*, Discrete Math., 313 (2013), 2607–2613.

[9] X. H. LI, L. G. WANG S. Y ZHANG, *The signless Laplacian spectral radius of some strongly connected digraphs*, Indian J. Pure Appl. Math., 49 (2018), 113–127.

[10] H. Q. LIN, J. L. SHU, *Spectral radius of digraphs with given dichromatic number*, Linear Algebra Appl., 434 (2011), 2462–2467.

[11] H. Q. LIN, J. L. SHU, *A note on the spectral characterization of strongly connected bicyclic digraphs*, Linear Algebra Appl., 436 (2012), 2524–2530.
[12] H. Q. LIN, J. L. SHU, Y. R. WU, G. L. YU, *Spectral radius of strongly connected digraphs*, Discrete Math., 312 (2012), 3663–3669.
[13] H. Q. LIN, X. G. LIU, J. XUE, *Graphs determined by their $A_\alpha$-spectra*, Discrete Math., 342 (2019), 441–450.
[14] J. P. LIU, X. Z. W, J. S. CHEN, B. L. LIU, *The $A_\alpha$ spectral radius characterization of some digraphs*, Linear Algebra Appl., 563 (2019), 63–74.
[15] X. G. LIU, S. Y. LIU, *On the $A_\alpha$-characteristic polynomial of a graph*, Linear Algebra Appl., 546 (2018), 274–288.
[16] V. NIKIFOROV, *Merging the $A$- and $Q$-spectral theories*, Applicable Analysis and Discrete Math., 11 (2017), 81–107.
[17] V. NIKIFOROV, O. ROJO, *On the $\alpha$-index of graphs with pendent paths*, Linear Algebra Appl., 550 (2018), 87–104.
[18] V. NIKIFOROV, O. ROJO, *A note on the positive semidefiniteness of $A_\alpha(G)$*, Linear Algebra Appl., 519 (2017), 156–163.
[19] W. G. XI, L. G. WANG, *Sharp upper bounds on the signless Laplacian spectral radius of strongly connected digraphs*, Discuss. Math. Graph Theory, 36 (2016), 977–988.
[20] W. G. XI, L. G. WANG, *The signless Laplacian and distance signless Laplacian spectral radius of digraphs with some given parameters*, Discrete Appl. Math., 227 (2017), 136–141.
[21] W. G. XI, W. SO, L. G. WANG, *On the $A_\alpha$ spectral radius of digraphs with given parameters*, Linear Multilinear Algebra, 70 (2022), 2248–2263.
[22] J. XUE, H. Q. LIN, S. T. LIU, J. L. SHU, *On the $A_\alpha$-spectral radius of a graph*, Linear Algebra Appl., 550 (2018), 105–120.

(Received September 15, 2021)

*Weige Xi*
*College of Science*
*Northwest A&F University*
*Yangling, Shaanxi 712100, China*
*and*
*School of Mathematics and Statistics*
*Northwestern Polytechnical University*
*Xi'an, Shaanxi 710129, P.R. China*

*Ligong Wang*
*School of Mathematics and Statistics*
*Northwestern Polytechnical University*
*Xi'an, Shaanxi 710129, P.R. China*
*and*
*Xi'an-Budapest Joint Research Center for Combinatorics*
*Northwestern Polytechnical University*
*Xi'an, Shaanxi 710129, P.R. China*
*e-mail:* xiyanxwg@163.com
lgwangmath@163.com