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STABILITY ANALYSIS FOR NON–AUTONOMOUS SEMILINEAR EVOLUTION EQUATIONS IN HILBERT SPACES: A PRACTICAL APPROACH

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Abstract. The Luenberger observer design problem for non-autonomous semilinear evolution equations has been the subject of several studies. However, much less interest has been given to the more general infinite-dimensional systems. In this paper, we investigate the global practical uniform stability analysis problem for a certain class of non-autonomous semilinear evolution equations with the associated nominal system is linear and the perturbation term satisfies some conditions. Moreover, we study the compensator design in the practical sense for two classes of non-autonomous semilinear evolution equations having a nominal linear part. We propose some classes of memoryless state linear and nonlinear feedback controllers. We illustrate the theory with an example of a controlled perturbed heat equation.

1. Introduction

Unlike for linear infinite-dimensional systems, the compensator design does not generally hold for nonlinear infinite-dimensional systems. Therefore, the output feedback control problem for non-autonomous semilinear evolution equations is much more challenging than stabilization using full-state feedback. It is well known that the Luenberger observer design problem for semilinear evolution equations by itself is quite challenging. One has to often consider special classes of semilinear evolution equations to solve the observer design problem as well as the output feedback control problem by using Lyapunov techniques. For partial differential equations, a systematic approach to the development of controller and Luenberger observers for infinite-dimensional systems was given in [4, 6, 7, 8, 9, 15, 17]. Alternative direct state space finite-dimensional compensator designs can be found in [4, 5]. In [17], the authors give an observer-based output feedback compensator design for a linear parabolic partial differential equation. In finite-dimensional systems, one simple way of designing a compensator is to first construct a state feedback stabilizer and an observer for the system and then combine the two to design a compensator using the feedback of the observer instead of the state. This is the so-called separation principle (see [2, 3, 10, 14]) and we shall show that this also works for our class of infinite-dimensional systems. The authors in [3] established

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a separation principle for two classes of finite-dimensional systems having nominal linear parts. We will give a compensator design via the separation principle based on analysis results for nonlinear cascaded systems. In 2016, Damak and Hammami [11] studied the practical stabilization for a class of abstract differential equations in Hilbert spaces. However, the asymptotic stability of the solutions of a class of non-autonomous semilinear evolution equations in Banach spaces has been presented in [12].

Motivated by the existing literature [3, 9, 10], we present in this paper a novel procedure for constructing stabilizing compensators for two classes of non-autonomous semilinear evolution equations by using an estimated feedback controller. For the first one, the perturbed term is uniformly bounded by known functions and for the second one the perturbation is bounded by a function that could depend on the time and the output of the system. Thus, based on Lyapunov techniques, we give sufficient conditions to guarantee the global uniform practical stability of the closed-loop systems by using an estimated feedback controller via a global uniform practical stable observer for semilinear evolution equations. A practical approach is obtained.

The rest of this paper is organized as follows: Basic definitions and some preliminary results are presented in section 2. In section 3, we present a compensator design using a state controller for a class of non-autonomous semilinear evolution equations that is uniformly bounded by known functions. In section 4, we solve this problem for non-autonomous semilinear evolution equations in the case when the perturbation is bounded by a known function that could depend on the time and the output of the system. An example is included in section 5 as an application. Section 6 concludes the paper.

2. Mathematical preliminaries

We use the following notation throughout the paper. $\mathbb{R}_+$ denotes the set of all non-negative real numbers. H denotes a Hilbert space with the norm $\|\cdot\|$ and the inner product $\langle\cdot,\cdot\rangle$. For linear normed spaces X,Y let $L(X,Y)$ be the space of bounded linear operators from X to Y and $L(X):=L(X,X)$. A norm in these spaces we denote by $\|\cdot\|$. For an operator A, A^* is the adjoint, $Dom(A)$ is the domain and I is the identity operator. $C(X,Y)$ denotes the space of all continuous functions from X to Y.

Consider the following non-autonomous semilinear evolution equation:

$$\begin{cases} \dot{x}(t) = Ax(t) + B(u(t) + F(t,x(t))), \quad x(t_0) = x_0, \\ y(t) = Cx(t), \end{cases} \tag{2.1}$$

where $t \geqslant t_0 \geqslant 0$ is the time, $x(t) \in H$ is the system state, $u(t) \in U$ is the control input, $y(t) \in Y$ is the measured output, A is the infinitesimal generator of an analytic semigroup $S(t)$, $B \in L(U,H)$, and $C \in L(H,Y)$. The operator $F : \mathbb{R}_+ \times H \longrightarrow H$ is continuous in t and is locally Lipschitz continuous in x, uniformly in t on bounded intervals.

In this paper, we consider mild solutions of (2.1), i.e. solutions of the integral form

$$x(t) = S(t-t_0)x_0 + \int_{t_0}^{t} S(t-s)[Bu(s) + BF(s,x(s))]ds \tag{2.2}$$

belonging to the class $C([t_0,d],H)$ for some $d > t_0$.

Under the condition that F is locally Lipschitz continuous in x, uniformly in t on bounded intervals, it is shown in [16, Theorem 1.4] that equation (2.3) has a unique mild solution on $[t_0,d]$. Moreover, if $d < \infty$, then $\lim_{t\to d} \|x(t)\| = \infty$.

The corresponding system without perturbations, called the nominal system, is described by

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \quad t \geqslant 0. \tag{2.3}$$

Next, we recall the definition of the generator of an exponentially stable semi-group as well as that of the exponential stabilizability and detectability, see Curtain and Zwart [9] for more details.

DEFINITION 2.1. The operator A generates an exponentially stable semigroup $S(t)$ if the initial value problem (2.3) has a unique solution $x(t) = S(t)x_0$, and $\|S(t)\| \leqslant Me^{-\omega t}$, for all $t \geqslant 0$ with some positive numbers M and ω.

The ω is called the decay rate.

If $S(t)$ is exponentially stable, then the solution to the abstract Cauchy problem (2.3) tends to zero exponentially as $t \longrightarrow \infty$.

DEFINITION 2.2. The pair $\{A,B\}$ is said to be exponentially stabilizable if there exists a feedback operator $D \in L(H,U)$, such that the operator $A+BD$ generates an exponentially stable semigroup S_{BD}.

DEFINITION 2.3. The pair $\{A,C\}$ is said to be exponentially detectable if there exists an output injection operator $L \in L(Y,U)$, such that the operator $A+LC$ generates an exponentially stable semigroup S_{LC}.

To study the stability properties of (2.1) with respect to external inputs, we use the notion of practical stabilizability.

DEFINITION 2.4. ([15, Definition 1]) System (2.1) is practically stabilizable if there exists a continuous feedback control $u : [t_0,\infty) \to U$, such that system (2.1) in closed-loop with $u(t)$ satisfies the following properties:

(i) For any initial condition $x_0 \in H$, there exists a unique mild solution $x(t)$ defined on $[t_0,\infty)$.

(ii) There exist positive scalars ω, c, η, such that the solution of the system (2.1) satisfies

$$\|x(t)\| \leqslant c\|x_0\|e^{-\omega(t-t_0)} + \eta, \quad \forall t \geqslant t_0 \geqslant 0. \tag{2.4}$$

When (i) and (ii) are satisfied for (2.1), we say that (2.1) in closed-loop with $u(t)$ is globally practically uniformly exponentially stable, see [3] for more details.

REMARK 2.5. The inequality (2.4) implies that the trajectory will be ultimately bounded. That is the solution is bounded and approach toward a neighborhood of the origin for sufficiently large t.

DEFINITION 2.6. Let $V : H \to \mathbb{R}_+$ be a Lyapunov function. If $x(t)$ is a solution of (2.1), the time derivative of $V(x(t))$ is defined by

$$\dot{V}(x(t)) = \limsup_{h \to 0^+} \frac{1}{t}(V(x(t+h)) - V(x(t)).$$

Recall that a self-adjoint operator $\mathscr{P} \in L(H)$ is positive if $\langle \mathscr{P}x, x \rangle > 0$ holds for all $x \in H \backslash \{0\}$. A positive operator $\mathscr{P} \in L(H)$ is called coercive if there exists $k > 0$, such that $\langle \mathscr{P}x, x \rangle \geqslant k\|x\|^2$, $\forall x \in Dom(\mathscr{P})$.

PROPOSITION 2.7. ([9, Theorem 5.1.3]) *Suppose that A is the infinitesimal generator of the C_0-semigroup $S(t)$ on the Hilbert space H. Then, $S(t)$ is exponentially stable if and only if there exists a coercive positive self-adjoint operator $\mathscr{P} \in L(H)$, such that*

$$\langle Ax, \mathscr{P}x \rangle + \langle \mathscr{P}x, Ax \rangle = -\langle x, x \rangle, \quad \forall x \in Dom(A). \tag{2.5}$$

The following technical lemma will be needed in our investigations.

LEMMA 2.8. ([18, Lemma 2]) *Let β, $\rho : \mathbb{R}_+ \to \mathbb{R}$ be continuous functions and $y : \mathbb{R}_+ \to \mathbb{R}_+$ is a function, such that*

$$\dot{y}(t) \leqslant \beta(t)y(t) + \rho(t), \quad \forall\, t \geqslant t_0.$$

Then, for any $t \geqslant t_0 \geqslant 0$, we have

$$y(t) \leqslant y(t_0)\exp\left(\int_{t_0}^{t} \beta(v)dv\right) + \int_{t_0}^{t} \exp\left(\int_{s}^{t} \beta(v)dv\right)\rho(s)ds.$$

3. Compensator design by linear state controller

In this section, we establish a compensator design in the practical sense for (2.1) via a linear state estimate controller under some restrictions on the nonlinearities. Indeed, we use the measurements to estimate the full state (the construction of an observer) and apply state feedback on the estimated state.

Let's consider the following assumptions.

$(\mathscr{H}_1)$ The pair $\{A, B\}$ is exponentially stabilizable, that is there exists a constant operator $D \in L(H, U)$ and a coercive positive self-adjoint operator $\mathscr{P}_1$

$$\mu I \leqslant \mathscr{P}_1 \leqslant \|\mathscr{P}_1\| I, \tag{3.1}$$

where $\mu > 0$, which satisfies

$$\langle A_D^* x, \mathscr{P}_1 x \rangle + \langle \mathscr{P}_1 x, A_D x \rangle = -\langle x, x \rangle, \quad \forall x \in Dom(A_D),$$

with $A_D = A + BD$.

$(\mathscr{H}_2)$ The operator $F(t,x)$ satisfies $F(t,0)=0$, $t\geqslant 0$ and

$$\|F(t,x)-F(t,y)\|\leqslant \chi(t)\|x-y\|+\psi(t),\quad \forall t\geqslant 0,\quad \forall x,y\in H, \tag{3.2}$$

where $\chi:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ and $\psi:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ are continuous functions satisfying

$$\int_0^{\infty}\chi(s)\,ds\leqslant M_\chi<+\infty$$

and

$$\int_0^{\infty}\psi^2(s)\,ds\leqslant M_\psi<+\infty.$$

3.1. Practical stabilization

In this part, sufficient conditions are presented to guarantee the practical stabilizability of a class of non-autonomous systems modeled by (2.1) using Lyapunov's techniques.

THEOREM 3.1. *If assumptions* $(\mathscr{H}_1)$ *and* $(\mathscr{H}_2)$ *are fulfilled, then the system* (2.1) *in closed-loop with the linear feedback*

$$u(t)=Dx(t) \tag{3.3}$$

is globally uniformly practically exponentially stable.

Proof. First, note that under the assumptions of the theorem, it is easy to see that equation (2.1) has a unique mild solution $x\in C([t_0,\infty),H)$ for any $x_0\in H$ by applying the results of Pazy [16, Theorem 1.4] and [13, Theorem 3.1] for any $x_0\in H$ and this mild solution is even a classical solution which satisfies (2.2). We consider the following Lyapunov function:

$$V(x(t))=\langle\mathscr{P}_1x(t),x(t)\rangle.$$

Let us compute the time derivative of V with respect to system (2.1) in closed-loop with the controller (3.3). For $x(t)\in Dom(A_D)=Dom(A)$, we have

$$\begin{aligned}\dot V(x(t)) &= \langle\mathscr{P}_1\dot x(t),x(t)\rangle+\langle\mathscr{P}_1x(t),\dot x(t)\rangle\\ &= \langle\mathscr{P}_1[A_Dx(t)+BF(t,x(t))],x\rangle+\langle\mathscr{P}_1x(t),[A_Dx(t)+BF(t,x(t))]\rangle.\end{aligned}$$

Using $(\mathscr{H}_1)$ with the help of Cauchy-Schwartz inequality, we obtain

$$\dot V(x(t))\leqslant -\langle x(t),x(t)\rangle+2\|\mathscr{P}_1\|\|B\|\|F(t,x(t))\|\|x(t)\|.$$

It follows by (3.1) and (3.2) that

$$\dot V(x(t))\leqslant -\left(\frac{1}{\|\mathscr{P}_1\|}-\frac{2\|\mathscr{P}_1\|\|B\|\chi(t)}{\mu}\right)V(x(t))+\frac{2\|\mathscr{P}_1\|\|B\|\psi(t)}{\sqrt{\mu}}\sqrt{V(x(t))}. \tag{3.4}$$

Since A generated an analytic semigroup, we can apply a density argument for the operator A to prove that (3.4) hold on the whole H. Let

$$\omega(t) = \sqrt{V(x(t))}.$$

The derivative of ω with respect to time is given by

$$\dot{\omega}(t) \leqslant -\left(\frac{1}{2\|\mathscr{P}_1\|} - \frac{\|\mathscr{P}_1\|\|B\|\chi(t)}{\mu}\right)\omega(t) + \frac{\|\mathscr{P}_1\|\|B\|\psi(t)}{\sqrt{\mu}}. \tag{3.5}$$

By Lemma 2.8, from (3.5) we have

$$\omega(t) \leqslant \omega(t_0)e^{\frac{\|\mathscr{P}_1\|\|B\|M_\chi}{\mu}}e^{-\frac{1}{2\|\mathscr{P}_1\|}(t-t_0)} + \sqrt{\frac{M_\psi}{\mu}}\|\mathscr{P}_1\|^{\frac{3}{2}}\|B\|e^{\frac{\|\mathscr{P}_1\|\|B\|M_\chi}{\mu}}.$$

Then, by (3.1), it follows that

$$\|x(t)\| \leqslant \sqrt{\frac{\|\mathscr{P}_1\|}{\mu}}\|x_0\|e^{\frac{\|\mathscr{P}_1\|\|B\|M_\chi}{\mu}}e^{-\frac{1}{2\|\mathscr{P}_1\|}(t-t_0)} + \frac{\sqrt{M_\psi}}{\mu}\|\mathscr{P}_1\|^{\frac{3}{2}}\|B\|e^{\frac{\|\mathscr{P}_1\|\|B\|M_\chi}{\mu}}.$$

Hence, the system (2.1) in closed-loop with the linear feedback (3.3) is globally uniformly practically exponentially stable. □

3.2. Practical Luenberger observer design

Consider the state system (2.1) with state space H, input space U and output space Y. To design a Luenberger observer, let's consider the following assumption:

($\mathscr{H}_3$) The pair $\{A,C\}$ is exponentially detectable, that is there exists a constant operator $L \in L(Y,H)$ and a coercive positive self-adjoint operator $\mathscr{P}_2$

$$\nu I \leqslant \mathscr{P}_2 \leqslant \|\mathscr{P}_2\|I, \tag{3.6}$$

where $\nu > 0$, which satisfies

$$\langle A_L^* x, \mathscr{P}_2 x\rangle + \langle \mathscr{P}_2 x, A_L x\rangle = -\langle x,x\rangle, \quad \forall x \in Dom(A_L), \tag{3.7}$$

with $A_L = A + LC$.

Consider the following observer, where $\hat{x}$ denotes the estimate of the state vector x:

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + BF(t,\hat{x}(t)) + L(\hat{y}(t) - y(t)), \quad t \geqslant t_0, \\ \hat{y}(t) = C\hat{x}(t), \quad \hat{x}(t_0) = \hat{x}_0. \end{cases} \tag{3.8}$$

Setting

$$e(t) = \hat{x}(t) - x(t),$$

the estimation error $e(t)$ satisfies the following equation

$$\dot{e}(t) = \dot{\hat{x}}(t) - \dot{x}(t) = (A+LC)e(t) + BF(t,\hat{x}(t)) - BF(t,x(t)), \tag{3.9}$$

where $e_0 = \hat{x}_0 - x_0$.

The next result gives sufficient conditions under which the state estimation error is globally uniformly practically exponentially stable, the so-called practical exponential observer.

THEOREM 3.2. *Under assumptions* $(\mathscr{H}_2)$ *and* $(\mathscr{H}_3)$, *the system* (3.8) *is a global uniform practical exponential observer for the system* (2.1).

Proof. First, note that under the assumptions of the theorem, it is easy to see that equation (3.9) has a unique global mild solution $e(t)$ by applying the results of Pazy [16, Theorem 1.4] for any $e_0 \in H$ and this mild solution is even a classical solution. Let's consider the Lyapunov function:

$$W(e(t)) = \langle \mathscr{P}_2 e(t), e(t) \rangle$$

with $e(t) \in Dom(A)$. The time derivative of W along the trajectories of system (3.9) is given by

$$\begin{aligned}
\dot{W}(e(t)) &= \langle \mathscr{P}_2 \dot{e}(t), e(t) \rangle + \langle \mathscr{P}_2 e(t), \dot{e}(t) \rangle \\
&= \langle \mathscr{P}_2 [(A+LC)e(t) + BF(t,\hat{x}(t)) - BF(t,x(t))], e(t) \rangle \\
&\quad + \langle \mathscr{P}_2 e(t), [(A+LC)e(t) + BF(t,\hat{x}(t)) - BF(t,x(t))] \rangle.
\end{aligned}$$

Then, by using the Cauchy-Schwartz inequality, one has

$$\dot{W}(e(t)) \leqslant -\left(\frac{1}{\|\mathscr{P}_2\|} - \frac{2\|\mathscr{P}_2\|\|B\|\chi(t)}{\nu})W(e(t)) \right) + \frac{2\|\mathscr{P}_2\|\|B\|\psi(t)}{\sqrt{\nu}} \sqrt{W(e(t))}. \tag{3.10}$$

By using a density argument for the operator A, to prove that (3.10) hold on the whole H. Let $\xi(t) = \sqrt{W(e(t))}$. The derivative of ξ with respect to time leads to

$$\dot{\xi}(t) \leqslant -\left(\frac{1}{2\|\mathscr{P}_2\|} - \frac{\|\mathscr{P}_2\|\|B\|\chi(t)}{\nu} \right) \xi(t) + \frac{\|\mathscr{P}_2\|\|B\|\psi(t)}{\sqrt{\nu}}.$$

Using Lemma 2.8, we have

$$\xi(t) \leqslant \xi(t_0) e^{\frac{\|\mathscr{P}_2\|\|B\|M_\chi}{\nu}} e^{-\frac{1}{2\|\mathscr{P}_2\|}(t-t_0)} + \sqrt{\frac{M_\psi}{\nu}} \|\mathscr{P}_2\|^{\frac{3}{2}} \|B\| e^{\frac{\|\mathscr{P}_2\|\|B\|M_\chi}{\nu}}.$$

Therefore, by (3.6), it follows that

$$\|e(t)\| \leqslant \sqrt{\frac{\|\mathscr{P}_2\|}{\nu}} \|e_0\| e^{\frac{\|\mathscr{P}_2\|\|B\|M_\chi}{\nu}} e^{-\frac{1}{2\|\mathscr{P}_2\|}(t-t_0)} + \frac{\sqrt{M_\psi}}{\nu} \|\mathscr{P}_2\|^{\frac{3}{2}} \|B\| e^{\frac{\|\mathscr{P}_2\|\|B\|M_\chi}{\nu}}.$$

Thus, the error equation (3.9) is globally uniformly practically exponentially stable. Consequently, the system (3.8) is a global uniform practical exponential Luenberger observer for the system (2.1). $\square$

3.3. The compensator design

Now, in order to obtain a compensator design for (2.1). We consider the system (2.1) controlled by the linear feedback control $u(t) = D\hat{x}(t)$ and estimated with the Luenberger observer (3.8).

THEOREM 3.3. *Consider the controlled system* (2.1) *and suppose that assumptions* $(\mathscr{H}_1)$, $(\mathscr{H}_2)$ *and* $(\mathscr{H}_3)$ *hold. If* $D \in L(H,U)$ *and* $L \in L(Y,U)$ *are such that* $A+BD$ *and* $A+LC$ *generate exponentially stable semigroups, then the controller* $u(t) = D\hat{x}(t)$, *where* $\hat{x}$ *is the Luenberger observer with output injection* L, *uniformly practically exponentially stabilizes closed-loop system. The stabilizing compensator is given by*

$$\begin{cases} \dot{\hat{x}}(t) = (A+LC)\hat{x}(t) + Bu(t) + BF(t,\hat{x}(t)) - Ly(t), \\ u(t) = D\hat{x}(t). \end{cases} \tag{3.11}$$

Proof. Under assumptions $(\mathscr{H}_1)$ and $(\mathscr{H}_3)$, there exist operators D and L, such that $S_{BD}(t)$ and $S_{LC}(t)$ are exponentially stable. Combining the abstract differential equations, we see that the closed-loop system is given by the dynamics of the extended state $x^e = \begin{pmatrix} \hat{x} & e \end{pmatrix}^T$,

$$\dot{x^e}(t) = \mathscr{A}x^e(t) + \mathscr{F}(t,x), \tag{3.12}$$

where

$$\mathscr{A} = \begin{pmatrix} A+BD & LC \\ 0 & A+LC \end{pmatrix}$$

and

$$\mathscr{F}(t,x) = \begin{pmatrix} BF(t,\hat{x}(t)) \\ BF(t,\hat{x}(t)) - BF(t,x(t)) \end{pmatrix}.$$

As LC and BD are bounded linear operators, $A+BD$ and $A+LC$ generate C_0-semigroups on H. So, $\mathscr{A}$ is the infinitesimal generator of a C_0-semigroups on $H \times H$, see [9]. Observe that the operator $\mathscr{F}$ is also locally Lipschitz continuous in x, uniformly in t on bounded intervals. Then, it easy to see that equation (3.12) has a unique classical solution $x^e(t)$ which is defined on $[t_0, \infty)$.

Let us define the following Lyapunov function:

$$U(x^e(t)) = \varsigma V(\hat{x}(t)) + W(e(t)),$$

where $V(\hat{x}(t)) = \langle \mathscr{P}_1\hat{x}(t), \hat{x}(t)\rangle$, $W(e(t)) = \langle \mathscr{P}_2 e(t), e(t)\rangle$ and $\varsigma > 0$ is a Lyapunov parameter to be determined. Let $\hat{x}(t) \in Dom(A)$ and $e(t) \in Dom(A)$. Then, the time derivative of U along the trajectories of system (3.12) is given by

$$\begin{aligned} \dot{U}(x^e(t)) &= \alpha\dot{V}(\hat{x}(t)) + \dot{W}(e(t)) \\ &= \varsigma(\langle \mathscr{P}_1\dot{\hat{x}}(t), \hat{x}(t)\rangle + \langle \mathscr{P}_1\hat{x}(t), \dot{\hat{x}}(t)\rangle) + \langle \mathscr{P}_2\dot{e}(t), e\rangle + \langle \mathscr{P}_2 e(t), \dot{e}(t)\rangle \\ &= \varsigma(\langle \mathscr{P}_1[A\hat{x}(t) + BD\hat{x}(t) + BF(t,\hat{x}(t)) + LCe(t)], \hat{x}(t)\rangle \\ &\quad + \langle \mathscr{P}_1\hat{x}(t), A\hat{x}(t) + BD\hat{x}(t) + BF(t,\hat{x}(t)) + LCe(t)\rangle \\ &\quad + \langle \mathscr{P}_2[(A+LC)e(t) + BF(t,\hat{x}(t)) - BF(t,x(t))], e(t)\rangle \\ &\quad + \langle \mathscr{P}_2 e(t), (A+LC)e(t) + BF(t,\hat{x}(t)) - BF(t,x(t))\rangle. \end{aligned}$$

Thus, by using the Cauchy-Schwartz inequality, one has

$$\begin{aligned}\dot{U}(x^e(t)) \leqslant \varsigma\Big(&-\frac{1}{\|\mathscr{P}_1\|}V(\hat{x}(t))+2\|\mathscr{P}_1\|\|B\|\chi(t)\|\hat{x}(t)\|^2+2\|\mathscr{P}_1\|\|B\|\psi(t)\|\hat{x}(t)\|\\&+2\|\mathscr{P}_1\|\|LCe(t)\|\|\hat{x}(t)\|\Big)-\frac{1}{\|\mathscr{P}_2\|}W(e(t))+2\|\mathscr{P}_2\|\|B\|\chi(t)\|e(t)\|^2\\&+2\|\mathscr{P}_2\|\|B\|\psi(t)\|e(t)\|.\end{aligned}$$

Let $\varepsilon > 0.$ By applying Young's inequality

$$2\|\hat{x}\|\|e\| \leqslant \frac{1}{\varepsilon}\|\hat{x}\|^2+\varepsilon\|e\|^2,$$

we have

$$\begin{aligned}\dot{U}(x^e(t)) \leqslant \varsigma\Big(&-\frac{1}{\|\mathscr{P}_1\|}+\frac{2\|\mathscr{P}_1\|\|B\|\chi(t)}{\mu}+\frac{\|\mathscr{P}_1\|\|LC\|}{\varepsilon\mu}\Big)V(\hat{x}(t))\\&+\Big(-\frac{1}{\|\mathscr{P}_2\|}+\frac{2\|\mathscr{P}_2\|\|B\|\chi(t)}{\nu}+\frac{\varsigma\varepsilon\|\mathscr{P}_1\|\|LC\|}{\nu}\Big)W(e(t))\\&+\frac{2\varsigma\|\mathscr{P}_1\|\|B\|\psi(t)}{\sqrt{\mu}}\sqrt{V(\hat{x}(t))}+\frac{2\|\mathscr{P}_2\|\|B\|\psi(t)}{\sqrt{\nu}}\sqrt{W(e(t))}.\end{aligned}$$

Let

$$\varepsilon = \frac{2\|\mathscr{P}_1\|^2\|LC\|}{\mu}.$$

Choose $\varsigma,$ such that $\frac{1}{\|\mathscr{P}_2\|}-\frac{\varsigma\varepsilon\|\mathscr{P}_1\|\|LC\|}{\nu} > 0.$ Then, let

$$\varsigma = \frac{\mu\nu}{4\|\mathscr{P}_1\|^3\|\mathscr{P}_2\|\|LC\|^2}.$$

It yields,

$$\begin{aligned}\dot{U}(x^e(t)) \leqslant \varsigma\Big(&-\frac{1}{2\|\mathscr{P}_1\|}+\frac{2\|\mathscr{P}_1\|\|B\|\chi(t)}{\mu}\Big)V(\hat{x}(t))\\&+\Big(-\frac{1}{2\|\mathscr{P}_2\|}+\frac{2\|\mathscr{P}_2\|\|B\|\chi(t)}{\nu}\Big)W(e(t))\\&+\lambda_1\psi(t)(\sqrt{\varsigma V(\hat{x}(t))}+\sqrt{W(e(t))}),\end{aligned}$$

where

$$\lambda_1 = \max\left(\frac{2\sqrt{\varsigma}\|\mathscr{P}_1\|\|B\|}{\sqrt{\mu}},\frac{2\|\mathscr{P}_2\|\|B\|}{\sqrt{\nu}}\right).$$

It follows that

$$\dot{U}(x^e(t)) \leqslant (-\lambda_2+\lambda_3\chi(t))U(x^e)+2\lambda_1\psi(t)\sqrt{U(x^e(t))},$$

where

$$\lambda_2 = \min\left(\frac{\varsigma}{2\|\mathscr{P}_1\|}, \frac{1}{2\|\mathscr{P}_2\|}\right),$$

and

$$\lambda_3 = \max\left(\frac{2\varsigma\|\mathscr{P}_1\|\|B\|}{\mu}, \frac{2\|\mathscr{P}_2\|\|B\|}{\nu})\right).$$

Let,

$$\varpi(t) = \sqrt{U(x^e(t))},$$

which implies that

$$\dot{\varpi}(t) \leqslant \frac{1}{2}(-\lambda_2 + \lambda_3\chi(t))\varpi(t) + \lambda_1\psi(t). \tag{3.13}$$

Applying Lemma 2.8, inequality (3.13) gives

$$\varpi(t) \leqslant \varpi(t_0)e^{\frac{\lambda_3}{2}M_\chi}e^{\frac{-\lambda_2}{2}(t-t_0)} + \lambda_1\sqrt{\frac{M_\psi}{\lambda_2}}e^{\frac{\lambda_3}{2}M_\chi}.$$

Therefore,

$$\|\hat{x}(t)\| \leqslant \frac{e^{\frac{\lambda_3}{2}M_\chi}}{\sqrt{\mu\varsigma}}\left[\max(\sqrt{\varsigma\|\mathscr{P}_1\|}, \sqrt{\|\mathscr{P}_2\|})(\|\hat{x}_0\| + \|e_0\|)e^{\frac{-\lambda_2}{2}(t-t_0)} + \lambda_1\sqrt{\frac{M_\psi}{\lambda_2}}\right]. \tag{3.14}$$

From (3.14), we can see that the system (3.12) is globally uniformly practically exponentially stable. □

4. Compensator design by nonlinear state controller

In this section, we shall suppose some assumptions more than considered in Section 3 and examine a compensator design in the practical sense for the class of non-autonomous semilinear evolution equations modeled by (2.1), that is, the designed state nonlinear feedback law remains valid when the control law implemented with the estimated states. We propose the following assumption.

$(\mathscr{H}_4)$ There exists a nonnegative scalar function $\kappa(.,.)$, such that

$$\|F(t,x(t))\| \leqslant \kappa(t,y(t)),$$

for all $t \in \mathbb{R}_+$, $x(t) \in H$ and $y(t) \in Y$ is the known output function given by (2.1).

4.1. Practical stabilization

Now, we shall construct a nonlinear feedback law which makes the system (2.1) is globally practically uniformly exponentially stable.

We have the following result.

THEOREM 4.1. *Under assumptions $(\mathscr{H}_1)$ and $(\mathscr{H}_4)$, the nonlinear control system (2.1) is practically stabilizable by the nonlinear feedback*

$$u(t) = u_1(t) + u_2(t), \tag{4.1}$$

where

$$u_1(t) = Dx(t),$$

$$u_2(t) = -\frac{B^* \mathscr{P}_1 x(t) \kappa^2(t,y)}{\|B^* \mathscr{P}_1 x(t)\| \kappa(t,y(t)) + \tau_1},$$

where τ_1 is a positive constant and $D \in L(H,U)$ is the operator given in $(\mathscr{H}_1)$.

Proof. By the state feedback controller (4.1), the nonlinear closed-loop system

$$\dot{x}(t) = (A+BD)x(t) - \frac{BB^* \mathscr{P}_1 x(t) \kappa^2(t,y)}{\|B^* \mathscr{P}_1 x(t)\| \kappa(t,y(t)) + \tau_1} + BF(t,x(t))$$

has a unique mild solution on $[t_0,\infty)$ for any $x_0 \in H$ by using the result of Pazy [16, Theorem 1.4] and this mild solution is even a classical solution. We consider the following Lyapunov function:

$$W_1(x(t)) = \langle \mathscr{P}_1 x(t), x(t) \rangle.$$

Let us compute the time derivative of W_1 with respect to system (2.1) in closed-loop with the controller (4.1). For $x(t) \in Dom(A_D) = Dom(A)$, we obtain

$$\begin{aligned} \dot{W}_1(x(t)) &= \langle \mathscr{P}_1 \dot{x}(t), x(t) \rangle + \langle \mathscr{P}_1 x(t), \dot{x}(t) \rangle \\ &\leqslant -\langle x(t), x(t) \rangle + 2\|B^* \mathscr{P}_1 x(t)\| \|F(t,x(t))\| - \frac{2\|B^* \mathscr{P}_1 x(t)\|^2 \kappa^2(t,y)}{\|B^* \mathscr{P}_1 x(t)\| \kappa(t,y(t)) + \tau_1} \\ &\leqslant -\frac{1}{\|\mathscr{P}_1\|} W_1(x(t)) + \frac{2\|B^* \mathscr{P}_1 x(t)\| \kappa(t,y(t)) \tau_1}{\|B^* \mathscr{P}_1 x(t)\| \kappa(t,y(t)) + \tau_1}. \end{aligned}$$

Then, one gets

$$\dot{W}_1(x(t)) \leqslant -\frac{1}{\|\mathscr{P}_1\|} W_1(x(t)) + 2\tau_1. \tag{4.2}$$

By using a density argument for the operator A, to prove that (4.2) hold on the whole H. A simple computation shows that

$$W_1(x(t)) \leqslant W_1(x_0) e^{-\frac{1}{\|\mathscr{P}_1\|}(t-t_0)} + 2\tau_1 \|\mathscr{P}_1\| \cdot$$

Thus,

$$\|x(t)\| \leqslant \sqrt{\frac{\|\mathscr{P}_1\|}{\mu}}\|x_0\|e^{-\frac{1}{2\|\mathscr{P}_1\|}(t-t_0)} + \sqrt{\frac{2\tau_1\|\mathscr{P}_1\|}{\mu}}.$$

Therefore, the system (2.1) in closed-loop with the nonlinear feedback (4.1) is globally practically uniformly exponentially stable. □

4.2. Practical Luenberger observer design

The goal is to design a Luenberger observer for the system (2.1), such that the practical global exponential stability of the resulting error system can be guaranteed.

We propose the following assumption.

($\mathscr{H}_5$) There exists an operator $N \in L(U,Y)$ that satisfies $B^*\mathscr{P}_2 = NC$, where $\mathscr{P}_2$ is the solution of the Lyapunov equation (3.7).

First, we construct a Luenberger observer for (2.1) of the following form:

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + \phi(t,e(t),y(t)) + L(\hat{y}(t) - y(t)), \quad t \geqslant t_0 \geqslant 0 \\ \hat{y}(t) = C\hat{x}(t), \quad \hat{x}(t_0) = \hat{x}_0, \end{cases} \tag{4.3}$$

where

$$\phi(t,e(t),y(t)) = -\frac{\mathscr{P}_2^* C^* N^* NCe(t)\kappa^2(t,y(t))}{\|NCe(t)\|\kappa(t,y(t)) + \tau_2}, \tag{4.4}$$

with τ_2 is a positive constant and $e(t) = \hat{x}(t) - x(t)$.

The error equation is given by

$$\dot{e}(t) = \dot{\hat{x}}(t) - \dot{x}(t) = (A + LC)e(t) + \phi(t,e(t),y(t)) - BF(t,x(t)), \tag{4.5}$$

where $e_0 = \hat{x}_0 - x_0$.

We have the following theorem.

THEOREM 4.2. *Under assumptions* ($\mathscr{H}_3$), ($\mathscr{H}_4$) *and* ($\mathscr{H}_5$), *the system* (4.3) *is a practical exponential Luenberger observer for the system* (2.1).

Proof. It is easy to we see that system (4.5) has a unique classical solution $e(t)$, which is defined on $[t_0,\infty)$. Let's consider the Lyapunov function:

$$Z_1(e(t)) = \langle \mathscr{P}_2 e(t), e(t)\rangle,$$

with $e(t) \in Dom(A)$. Then, the time derivative of Z_1 along the trajectories of system

(4.5) is given by

$$\begin{aligned}
\dot{Z}_1(e(t)) &= \langle \mathscr{P}_2\dot{e}(t), e(t)\rangle + \langle \mathscr{P}_2 e(t), \dot{e}(t)\rangle \\
&\leqslant -\langle e(t), e(t)\rangle + 2\|B^*\mathscr{P}_2 e(t)\|\kappa(t,y) - \frac{2\|NCe(t)\|^2\kappa^2(t,y(t))}{\|NCe(t)\|\kappa(t,y(t))+\tau_2} \\
&\leqslant -\|e(t)\|^2 + \frac{2\|B^*\mathscr{P}_2 e(t)\|\kappa(t,y(t))\tau_2}{\|NCe(t)\|\kappa(t,y(t))+\tau_2} \\
&\leqslant -\frac{1}{\|\mathscr{P}_2\|} Z_1(e(t)) + 2\tau_2.
\end{aligned}$$

It follows that,

$$Z_1(e(t)) \leqslant Z_1(e_0)e^{-\frac{1}{\|\mathscr{P}_2\|}(t-t_0)} + 2\tau_2\|\mathscr{P}_2\|. \tag{4.6}$$

By applying a density argument for the operator A, to prove that (4.6) hold on the whole H. Therefore,

$$\|e(t)\| \leqslant \sqrt{\frac{\|\mathscr{P}_2\|}{\nu}}\|e_0\|e^{-\frac{1}{2\|\mathscr{P}_2\|}(t-t_0)} + \sqrt{\frac{2\|\mathscr{P}_2\|\tau_2}{\nu}}. \tag{4.7}$$

From (4.7), we can see that the system (4.5) is globally uniformly practically exponentially stable. Consequently, the system (4.3) is a global uniform practical exponential Luenberger observer for the system (2.1). □

4.3. The compensator design

We consider the system (2.1) controlled by the nonlinear feedback law

$$u(t) = D\hat{x}(t) + u_2(t), \tag{4.8}$$

where

$$u_2(t) = -\frac{B^*\mathscr{P}_1\hat{x}(t)\kappa^2(t,\hat{y}(t))}{\|B^*\mathscr{P}_1\hat{x}(t)\|\kappa(t,\hat{y}(t))+\tau_1}, \quad \tau_1 > 0$$

and estimated with the Luenberger observer (4.3).

Then, we have the following theorem.

THEOREM 4.3. *Consider the system* (2.1) *and assume that assumptions* $(\mathscr{H}_1)$, $(\mathscr{H}_3)$, $(\mathscr{H}_4)$ *and* $(\mathscr{H}_5)$ *hold. If* $D \in L(H,U)$ *and* $L \in L(Y,U)$ *are such that* $A+BD$ *and* $A+LC$ *generate exponentially stable semigroups, then the controller* (4.8)*, where* $\hat{x}$ *is the Luenberger observer with output injection* L*, uniformly practically exponentially stabilizes closed-loop system. The stabilizing compensator is given by*

$$\begin{cases} \dot{\hat{x}}(t) = (A+LC)\hat{x}(t) + Bu(t) + BF(t,\hat{x}(t)) + \phi(t,e(t),y(t)) - Ly(t), \\ u(t) = D\hat{x}(t) + u_2(t), \end{cases} \tag{4.9}$$

where

$$u_2(t) = -\frac{B^*\mathscr{P}_1\hat{x}(t)\kappa^2(t,\hat{y}(t))}{\|B^*\mathscr{P}_1\hat{x}(t)\|\kappa(t,\hat{y}(t))+\tau_1},$$

with τ_1 is a positive constant, ϕ is a known function given by (4.4) and $e(t)=\hat{x}(t)-x(t)$.

Proof. Under assumptions $(\mathscr{H}_2)$ and $(\mathscr{H}_3)$, there exist operators D and L, such that $S_{BD}(t)$ and $S_{LC}(t)$ are exponentially stable. Combining the abstract differential equations, we see that the closed-loop system is given by the dynamics of the extended state $x^e=\begin{pmatrix}\hat{x} & e\end{pmatrix}^T$,

$$\dot{x^e}(t)=\mathscr{A}x^e(t)+\mathscr{F}(t,x), \tag{4.10}$$

where

$$\mathscr{A}=\begin{pmatrix} A+BD & LC \\ 0 & A+LC \end{pmatrix}$$

and

$$\mathscr{F}(t,x)=\begin{pmatrix} Bu_2(t)+\phi(t,e(t),y(t)) \\ \phi(t,e(t),y(t))-BF(t,x(t)) \end{pmatrix}.$$

Using the same argument as the proof of Theorem 3.3, equation (4.10) has a unique classical solution $x^e(t)$ which is defined on $[t_0,\infty)$.

Let us define the following Lyapunov function:

$$Y(x^e(t))=\theta W_1(\hat{x}(t))+Z_1(e(t)),$$

where $W_1(\hat{x}(t))=\langle \mathscr{P}_1\hat{x}(t),\hat{x}(t)\rangle$, $Z_1(e)=\langle \mathscr{P}_2 e(t),e(t)\rangle$ and $\theta>0$ is a Lyapunov parameter to be determined. The time derivative of Y along the trajectories of system (4.10) is given as follows:

$$\begin{aligned}\dot{Y}(x^e(t)) \leqslant & -\frac{\theta}{\|\mathscr{P}_1\|}W_1(\hat{x}(t))-\frac{1}{\|\mathscr{P}_2\|}Z_1(e(t))+2\theta\|\mathscr{P}_1\|\|LCe(t)\|\|\hat{x}(t)\| \\ & +2\langle e(t)\mathscr{P}_2,\phi(t,e(t),y(t))-BF(t,x(t))\rangle+2\theta\langle \hat{x}(t)\mathscr{P}_1,Bu_2(t)+\phi(t,e(t),y)\rangle.\end{aligned}$$

Since,

$$\begin{aligned}2\langle e(t)\mathscr{P}_2,\phi(t,e(t),y)-BF(t,x)\rangle &\leqslant -2\frac{\|NCe(t)\|^2\kappa^2(t,y(t))}{\|NCe(t)\|\kappa(t,y(t))+\tau_2}+2\|NCe(t)\|\kappa(t,y(t)) \\ &\leqslant \frac{2\|NCe(t)\|\kappa(t,y(t))\tau_2}{\|NCe(t)\|\kappa(t,y(t))+\tau_2}\leqslant 2\tau_2,\end{aligned}$$

and

$$\begin{aligned}2\theta\langle \hat{x}(t)\mathscr{P}_1,Bu_2(t)+\phi(t,e(t),y(t))\rangle &\leqslant -2\theta\frac{\|B^*\mathscr{P}_1\hat{x}(t)\|^2\kappa^2(t,y(t))}{\|B^*\mathscr{P}_1\hat{x}(t)\|\kappa(t,y(t))+\tau_1} \\ &\quad +2\theta\|B^*\mathscr{P}_1\hat{x}(t)\|\kappa(t,y(t)) \\ &\leqslant \frac{2\theta\|B^*\mathscr{P}_1\hat{x}(t)\|\kappa(t,y(t))\tau_1}{\|B^*\mathscr{P}_1\hat{x}(t)\|\kappa(t,y(t))+\tau_1}\leqslant 2\theta\tau_1,\end{aligned}$$

we have

$$\dot{Y}(x^e(t))\leqslant -\frac{\theta}{\|\mathscr{P}_1\|}W_1(\hat{x}(t))-\frac{1}{\|\mathscr{P}_2\|}Z_1(e(t))+2\theta\|\mathscr{P}_1\|\|LCe(t)\|\|\hat{x}(t)\|+2\tau_2+2\theta\tau_1.$$

Let $\varepsilon > 0$. Using Young's inequality

$$2\|\hat{x}(t)\|\|e(t)\| \leqslant \frac{1}{\varepsilon}\|\hat{x}(t)\|^2 + \varepsilon\|e(t)\|^2,$$

we can continue the above estimates as

$$\begin{aligned}\dot{Y}(x^e(t)) \leqslant& \Big(-\frac{1}{\|\mathscr{P}_1\|} + \frac{\|\mathscr{P}_1\|\|LC\|}{\mu\varepsilon}\Big)\theta W_1(\hat{x}(t)) \\ &+ \left(-\frac{1}{\|\mathscr{P}_2\|} + \frac{\theta\varepsilon\|\mathscr{P}_1\|\|LC\|}{\nu}\right) Z_1(e(t)) + 2\tau_2 + 2\theta\tau_1.\end{aligned}$$

Let,

$$\varepsilon = \frac{2\|\mathscr{P}_1\|^2\|LC\|}{\mu}.$$

Also, choose for this value of ε the scalar θ, such that $\frac{1}{\|\mathscr{P}_2\|} - \frac{\theta\varepsilon\|\mathscr{P}_1\|\|LC\|}{\nu} > 0$. Then, let

$$\theta = \frac{\mu\nu}{4\|\mathscr{P}_1\|^3\|\mathscr{P}_2\|\|LC\|^2}.$$

It yields,

$$\dot{Y}(x^e(t)) \leqslant -\sigma Y(x^e(t)) + 2\tau_2 + 2\theta\tau_1,$$

with

$$\sigma = \min\left(\frac{\theta}{2\|\mathscr{P}_1\|}, \frac{1}{2\|\mathscr{P}_2\|}\right).$$

It follows that,

$$Y(x^e(t)) \leqslant Y(x_0^e)e^{-\sigma(t-t_0)} + \frac{2\tau_2 + 2\theta\tau_1}{\sigma},$$

where $x_0^e = (\hat{x}_0, e_0)$.

Hence,

$$\|\hat{x}(t)\| \leqslant \frac{1}{\sqrt{\theta}}\left[\max(\sqrt{\theta\|\mathscr{P}_1\|}, \sqrt{\|\mathscr{P}_2\|})(\|\hat{x}_0\| + \|e_0\|)e^{-\frac{\sigma}{2}(t-t_0)} + \sqrt{\frac{2\tau_2 + 2\theta\tau_1}{\sigma}}\right].$$

Therefore, the cascade system (4.10) is globally uniformly practically exponentially stable. $\square$

5. Application

We consider the controlled perturbed heat equation

$$\begin{cases} \dfrac{\partial x(\zeta,t)}{\partial t} = \dfrac{\partial^2 x(\zeta,t)}{\partial^2\zeta} + b(\zeta)u(t) + \dfrac{b(\zeta)}{\sqrt{1+t^2}}x(\zeta,t) + b(\zeta)e^{-t}\sin(x(\zeta,t)), \\ \dfrac{\partial x}{\partial \zeta}(0,t) = 0 = \dfrac{\partial x}{\partial\zeta}(1,t), \quad x(\zeta,0) = x_0(\zeta), \\ y(t) = \displaystyle\int_0^1 c(\zeta)x(\zeta,t)d\zeta. \end{cases} \tag{5.1}$$

where $x(\zeta,t)$ represents the temperature at position ζ at time $t \geqslant 0$ and x_0 represents the initial temperature profile, $u(t)$ the addition of heat along the bar and b,c represents the shaping functions around the control ω_0 and the sensing point ω_1, respectively

$$b(\zeta) = \frac{1}{2\delta}\mathbf{1}_{[\omega_0-\delta,\omega_0+\delta]}(\zeta)$$

and

$$c(\zeta) = \frac{1}{2\kappa}\mathbf{1}_{[\omega_1-\kappa,\omega_1+\kappa]}(\zeta),$$

with $[\omega_0-\delta,\omega_0+\delta]\cap[\omega_1-\kappa,\omega_1+\kappa]=\emptyset$, and

$$\mathbf{1}_{[\vartheta,\upsilon]}(x) = \begin{cases} 1, \text{ if } \vartheta \leqslant x \leqslant \upsilon \\ 0, \text{ elsewhere.} \end{cases}$$

Notice that b and c in this example are both elements in $L^2(0,1)$ for a fixed positive constants δ and κ. To treat this system in the abstract form (2.1), we choose $H = L^2(0,1)$, $U=\mathbb{C}$ and $Y=\mathbb{C}$. Define operator $A : Dom(A)\subset H \to H$ by $Ah = \dfrac{\partial^2 h}{\partial^2\zeta}$, with domain

$$Dom(A) = \Big\{h\in L^2(0,1), h, \frac{\partial h}{\partial \zeta} \text{ are absolutely continuous,}$$
$$\frac{\partial^2 h}{\partial^2\zeta}\in L^2(0,1) \text{ and } \frac{dh}{d\zeta}(0) = \frac{dh}{d\zeta}(1) = 0\Big\}.$$

The input operator B is defined by

$$Bu = b(\zeta)u$$

and the measured output operator C by

$$Cx = \int_0^1 c(\zeta)x(\zeta,t)d\zeta.$$

The nonlinear operator is defined by

$$F(t,x(\zeta,t)) = \frac{1}{\sqrt{1+t^2}}x(\zeta,t) + e^{-t}\sin(x(\zeta,t))\cdot$$

On the other hand, A has the eigenvalues $0, -n^2\pi^2$, $n\geqslant 1$ and the corresponding eigenvectors $\{1, \sqrt{2}\cos(n\pi\zeta),\ n\geqslant 1\}$. From [9], we know that A generates an analytic semigroup $S(t)$. We choose a stabilizing feedback

$$u(t) = Dx(t), \tag{5.2}$$

with $Dx = -3\langle x,1\rangle$. It is easy to verify that $A+BD$ has the eigenvalues -3, $-(n\pi)^2$, $n\geqslant 1$. Then, the pair $\{A,B\}$ is exponentially stabilizable. One can see that Assumption

$(\mathscr{H}_2)$, is fulfilled with $\chi(t)=\dfrac{1}{\sqrt{1+t^2}}$ and $\psi(t)=e^{-t}$. Thus, using Theorem 3.1, the system (5.1) is globally practically uniformly exponentially stable with the controller (5.2). In this case, all states trajectories are bounded and approach toward a neighborhood of the origin.

Moreover, we choose a stabilizing output injection such that $Ly=-3y\phi_0=-3y.1$. The system $A+LC$ has the eigenvalues $-3,\ -(n\pi)^2$, $n\geqslant 1$. Therefore, the pair $\{A,C\}$ is exponentially detectable. Consequently, all hypotheses of Theorem 3.3 are satisfied. We conclude that a stabilizing compensator is given by

$$\begin{cases} \dfrac{\partial \hat{x}(\zeta,t)}{\partial t}=\dfrac{\partial^2 \hat{x}(\zeta,t)}{\partial^2 \zeta}-\dfrac{3}{2\kappa}\displaystyle\int_{\omega_1-\kappa}^{\omega_1+\kappa}\hat{x}(\zeta,t)d\zeta \\ +\frac{1}{2\delta}\mathbf{1}_{[\omega_0-\delta,\omega_0+\delta]}(x)[u(t)+\frac{1}{\sqrt{1+t^2}}\hat{x}(\zeta,t)+e^{-t}\sin(\hat{x}(\zeta,t))]+3y(t), \\ \dfrac{\partial \hat{x}}{\partial \zeta}(0,t)=0=\dfrac{\partial \hat{x}}{\partial \zeta}(1,t), \quad \hat{x}(\zeta,0)=\hat{x}_0(\zeta), \quad t\geqslant 0, \\ u(t)=-3\int_0^1\hat{x}(\zeta,t)d\zeta. \end{cases} \tag{5.3}$$

6. Conclusion

In this paper, we have provided a compensator design for two classes of non-autonomous semilinear evolution equations. It is shown that the system can be practically stabilized by means of an estimated state feedback given by a designated Luenberger observer. We presented, how under the assumptions of stabilizability and detectability of the pairs $\{A,B\}$ and $\{A,C\}$, we can construct a stabilizing feedback law and a Luenberger observer. An application has been introduced to validate the developed methods.

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