Operators and Matrices
Volume 16, Number 4 (2022), 1097–1123
doi:10.7153/oam-2022-16-72

IMPROVED RATE OF APPROXIMATION BY MODIFICATION OF BASKAKOV OPERATOR

ASHA RAM GAIROLA, AMRITA SINGH, LAXMI RATHOUR* AND VISHNU NARAYAN MISHRA*

(*Communicated by D. Han*)

Abstract. The optimal order of approximation, $|L_n f(x) - f(x)|$ of a linear positive operator $L_n f(x)$ is $1/n$ and can not be improved however smooth the function may be. We remove the positivity of the Baskakov operator $V_n(f;x)$ and introduce its three variants $V_n^{M,i}(f;x)$, $i = 1,2,3$. We prove that the rates of approximation by these operators are improved from the linear order $1/n$ to quadratic order $1/n^2$ and then to cubic order $1/n^3$ for sufficiently smooth functions.

1. Introduction

The Weierstrass theorem states that "A continuous function on a closed interval is uniform limit of the sequence of polynomials". A constructive proof of this theorem was given by Bernstein in 1912 [22]. For a function defined and bounded in the interval $[0,1]$, the nth Bernstein operator $B_n, n \in \mathbb{N}$ is defined by

$$B_n(f,x) := \sum_{k=0}^{n} B_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1.1}$$

where

$$B_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \;\; k = 0,1,\ldots,n.$$

It is known that $B_n(f,x)$ converges to $f(x)$ (see [22]) whenever f is continuous at x. In fact Aramă [6] proved that for a function f in $C[0,1]$ there exist three distinct points u_1, u_2, u_3 such that

$$B_n(f,x) := f(x) + \frac{x(1-x)}{n} [u_1, u_2, u_3; f]. \tag{1.2}$$

Further, convergence is uniform in case f is continuous on $[0,1]$. The Bernstein operators demonstrate interesting analytical and geometric properties e.g. (see [21])

Mathematics subject classification (2020): 41A35, 41A10, 41A25.
Keywords and phrases: Linear operators, approximation by polynomials, rate of convergence.
* Corresponding author.

1. $B_n(f,0)=f(0)$, $B_n(f,1)=f(1)$ i.e. $B_n(f,x)$ have end point interpolation property,
2. $B_n(f,x)\geqslant 0$, for any f with $f(x)\geqslant 0$ on $[0,1]$,
3. $\lim_{n\to\infty} B_n^{(j)}(f,x)=f^{(j)}(x)$ if $f^{(j)}$ is continuous at x,
4. $f\in C[0,1]$ is convex if and only iff for every $n\in\mathbb{N}$, $B_n(f)$ is convex.

In 1960 Korovkin [23] extensively studied approximation by linear positive operators in his work. After that study of approximation by linear positive operators have attracted interest of many researchers. Subsequently, a number of new operators have been introduced as well as modification of classical operators also have been made in order to approximate other classes of function. It is worth mentioning the work of Voronovkaja [53] about the asymptotic error estimation of Bernstein operator and results of [26], [27] and [29].

Since the rate of approximation by linear positive operators is relatively slow, there have been attempts by many researchers for better order of approximation. In this direction certain methods have been introduced. The first one is the method of linear combination of operators introduced by Butzer [11]. These combinations for Bernstein operators B_n are defined by

$$(2^k-1)B_n^{2k}(f;x)=2^kB_{2n}^{2k-1}(f;x)-B_n^{2k-1}(f;x), B_n^0(f;x)=B_n(f;x) \tag{1.3}$$

The following convergence theorem was shown in [11] that states

THEOREM 1. *If $f(x)$ is defined on $[0,1]$ with $|f(x)|\leqslant M$ and if $f^{(2k)}$ exists at the point x, then*

$$\left|B_n^{2k-2}(f,x)-f(x)\right|=O(n^{-k}),$$

and moreover,

$$\left|B_n^{2k}(f,x)-f(x)\right|=o(n^{-k}),\ \text{as}\ n\to\infty, \qquad k=1,2,\ldots$$

Thus the degree of approximation is significantly improved to $O(n^{-k})$, however it requires 2^kn nodes that makes the method of linear combinations of less practical value. Later, Rathore [48] in 1973 and May [35] in 1976 extended the method in [11].

Another technique to improve degree of approximation without requirement of large number of nodes, was introduced by Micchelli [37], (see also [36]) wherein he used combinations of iterates of the Bernstein operator B_n to define the operator $T_{n,M}(f,x)$ as follows

$$T_{n,M}(f,x)=(I-(I-B_n)^M)(f,x)=\sum_{s=1}^{M}(-1)^{s+1}\binom{M}{s}B_n^s(f;x), \tag{1.4}$$

where $B_n^s(f;x)$, $s \in N$ denotes the s-th iterate and $I = B_n^0(f;x)$. In fact Gonska and Zhou in [28] introduced the M-fold Boolean sum of the Bernstein operators B_n by

$$\oplus^M B_n := \underbrace{B_n \oplus B_n \oplus \cdots \oplus B_n}_{M\text{fold}},$$

where the Boolean sum of the operators P and Q on a common linear space X is defined by

$$P \oplus Q = P + Q - PoQ.$$

It is easily observed that

$$\oplus^M B_n = I - (I - B_n)^M = T_{n,M}.$$

In [28] the authors have proved a direct and saturation theorem for the operators $\oplus^M B_n$ defined on the space $C[0,1]$ of continuous functions on $[0,1]$. The authors of [28] proved the following Jackson type quantitative result

$$\|f - \oplus^M B_n f\| \leqslant C\left\{\omega_\phi^{2M}\left(f, \frac{1}{\sqrt{n}}\right) + \|f\| n^{-M}\right\}, \quad M \geqslant 1$$

The authors of [28] predicted the order of the approximation $O(n^{-M})$.

2. Extension of the operator $B_n f(x)$ to $[0,\infty)$

In an attempt to extend Bernstein polynomials to unbounded domain, another sequence of positive linear operators known as Baskakov operator was introduced by Baskakov [7]. These operators are defined by the formulas

$$L_n f(x) = \sum_{k=0}^{\infty} (-x)^k \frac{\phi_n^{(k)}(x)}{k!} f\left(\frac{k}{n}\right), \quad x \in [0,b], \quad b \in R^+ \cup \{\infty\},$$

where the sequence $\phi_n^{(k)}(x)$ of functions satisfy the conditions

1. $\phi_n \in C^\infty[0,b]$,
2. $\phi_n(0) = 1$,
3. $(-1)^k \phi_n^{(k)} \geqslant 0$ and
4. $\phi_n^{(k+1)} = -n\phi_n^{(k)}$ for $b > \max\{0,-c\}$ where c is some integer.

It can be easily verified that the functions $(1+x)^{-n}$ satisfy all the conditions of the sequence $\phi_n(x)$. With $\phi_n(x) = (1+x)^{-n}$ we obtain the following particular case of the Baskakov operator

$$V_n(f;x) = \sum_{k=0}^{\infty} P_{n,k}(x) f\left(\frac{k}{n}\right), \tag{2.1}$$

where the basis function

$$P_{n,k}(x) = \begin{cases} \binom{n+k-1}{k}\left(\frac{x}{1+x}\right)^k (1+x)^{-n} & \text{if } 0 \leqslant k \leqslant n, \\ 0 & \text{if } k < 0 \end{cases}$$

It can be shown that the sequence $V_n(f;x)$ is convergent on $[0,\infty)$ provided the function f is such that $\lim_{x\to\infty} f(x)$ exists. Let $C[0,\infty)$ be the space of continuous functions on $[0,\infty)$. Then we define the following sub class of $C[0,\infty)$.

$$\overline{C}[0,\infty) := \{f \in C[0,\infty) : |f(x)| < \infty\}. \tag{2.2}$$

The Baskakov operators $V_n(f;x)$ share interesting shape preservation properties. For example it interpolates the function $f(x)$ at $x=0$ i.e. $V_n(f;0) = f(0)$. Following are some of the geometric properties of the operators $V_n(f;x)$.

1. For a certain n the function $V_n f, n \in \mathbb{N}$ is increasing for each increasing function f,
2. for a convex function f, the functions $V_n f$, $n \in \mathbb{N}$ are convex,
3. $V_n f(x) \geqslant V_{n+1} f(x) \geqslant f(x)$, $x \in [0,\infty)$.

A number of results about the approximation properties of the operators $V_n(f;x)$ have been discussed in [2], [3], [8]–[15], [46], [50] and [51] and the references therein. It is worth refer to [31] for the rate of convergence of Baskakov type operator by Gupta and [38] for generalization by Mihesan. Further, in [55] the preservation properties of Baskakov-Kantorovich operator are studied.

It was proved in [7] that the sequence $V_n(f;x)$ converges uniformly to the function $f(x)$ whenever f is bounded on $[0,\infty)$. Later on, the uniform convergence was also observed for the functions having polynomial growth i.e. functions f satisfying the condition $|f(x)| = O(1+x^m)$, $m \in \{0\} \cup \mathbb{N}$. Moreover, for a twice differentiable function f satisfying the growth condition $\sup|f''(x)|e^{-Ax} < \infty$, Ditzian in [14] have established following direct estimate

THEOREM 2.

$$e^{-Ax}|V_n(f;x) - f(x)| \leqslant \|f''\|_A \frac{x(1+x)}{n}\left(\frac{M+1}{2}\right)$$

for $x \leqslant \eta\sqrt{n}$, $n > 2A$ *where* M *depends only on* A *and* $\eta = \frac{1}{3}\min(A^{-2},1)$ *and* $\|f\|_A = \sup|f(x)|e^{-Ax}$. *It follows from the above theorem that even for a twice differentiable function, the rate of approximation is* $O(n^{-1})$ *and can not be improved. For a twice differentiable function* f, *it can be shown by standard method that if* $f''(x) < \infty$, $x \in [0,\infty)$ *then*

$$\lim_{n\to\infty} n\frac{V_n(f,x) - f(x)}{x+x^2} = \frac{1}{2}f''(x).$$

Hence, the optimal rate of approximation achieved by these operators is $O(n^{-1})$, however smooth the function may be. In fact, we have the values $V_n(t^2,x) = x^2 + \frac{x(1+x)}{n}$ which yields order of approximation $O(n^{-1})$, for the function x^2 while this functions is sufficiently smooth. With the aim for better approximation results, Aral and Gupta in [5] introduced the q-varinat $B_{n,q}$ of the Baskakov operators V_n and proved that the rate of approximation

$$|B_{n,q}(f;x) - f(x)| \leqslant M\omega_2\left(f, \frac{x}{[n]_q}\left(1+\frac{x}{q}\right)\right).$$

Thus, the q-varinat $B_{n,q}$ of the Baskakov operators V_n do not yield better estimates. In fact, the method of q-modifications have been proved useful for $q \geqslant 1$ and some exceptional classes of analytical functions only as proved in [42]. However, the q-varinat $B_{n,q}$ demonstrate some shape preservation properties nicely (see [5]). Finally, it is worth to mention the approach of King to obtain better rates of approximation. King in [34] generalized the classical Bernstein operators B_n and achieved better rate of approximation for his operator $V_n(f)$ as follows.

$$|V_n(f) - f(x)| \leqslant \omega(f,\delta)\left[1 + \frac{1}{\delta}\sqrt{2x(x - r_n^*(x))}\right],$$

where

$$r_n^*(x) = \begin{cases} x^2 & \text{if } n = 1 \\ -\frac{1}{2(n-1)} + \sqrt{\left(\frac{n}{n-1}\right)x^2 + \frac{1}{4(n-1)^2}} & \text{if } n = 2,3,\dots \end{cases}$$

For recent progress in the direction of King's approach it is worth to mention the articles [16]–[52], and [43]–[44]. The Baskakov operators have been modified in several ways in order to look for better results. A certain number of modifications and applications of the Baskakov operators can be found in [4], [10], [13], [19], [45] and [49]. Relevant work in this direction can also be found in [25], and [39]–[41].

Recently, Arab et al. [20] have introduced a new method to get improved order of approximation by the Bernstein operator. By decomposition of the weight function $p_{n,k}(x)$, they defined the Bernstein type operator of first order as

$$B_n^{M,1}(f,x) = \sum_{k=0}^{n} B_{n,k}^{M,1}(x) f\left(\frac{k}{n}\right), \quad x \in [0,1],$$

$$B_{n,k}^{M,1}(x) = a(x,n)B_{n-1,k}(x) + a(1-x,n)B_{n-1,k-1}(x) \tag{2.3}$$

where $a(x,n) = a_1(n)(x) + a_0(n)$, $n = 0,1,\dots,$ $a_0(n)$ and $a_1(n)$ are two unknown sequences which are determined in an appropriate way so that convergence is assured. Similarly, second, third and fourth order Bernstein type operators $B_n^{M,2}(f,x)$, $B_n^{M,3}(f,x)$, $B_n^{M,4}(f,x)$, respectively have been defined. It is worth mentioning that for sufficiently smooth function $f(x)$, $B_n^{M,k}(f,x) - f(x) = O(n^{-k})$, $k = \overline{1,4}$. Thus, there is significant improvement in the order of approximation without including more nodes k/n.

Subsequently, A. M. Acu et al. [1] introduced and studied the Durrmeyer variants of the Bernstein operators and applied the method of Arab et al. [20] to achieve better degree of approximation. Similarly, the Kantorovich variant of the modified Bernstein operators was studied in paper [33]. In a recent paper [24], degree of approximation by certain Durrmeyer type operators is given explicitly in terms of the modulus of smoothness of the function. However, to the best of our knowledge there have not been any such attempts to improve the degree of approximation in the infinite domain $[0,\infty)$ or $\mathbb{R}$. Our aim is to extend the method of Arab et al. [20] for approximation of functions defined on the interval $[0,\infty)$. We modify classical Baskakov operator $V_n(f;x)$ for better degree of approximation.

3. First order operator $V_n^{M,1}(f;x)$

We apply the technique of decomposition of weight function to the classical Baskakov operator (2.1) and introduce our operator as follows:

$$V_n^{M,1}(f;x) = \sum_{k=0}^{\infty} P_{n,k}^{M,1}(x) f\left(\frac{k}{n}\right), \quad x \in [0,\infty) \tag{3.1}$$

where,

$$P_{n,k}^{M,1}(x) = S(\varphi(x),n)P_{n,k-1}(x) + S(1-\varphi(x),n)P_{n-1,k}(x) \tag{3.2}$$

and

$$S(\varphi(x),n) = s_1(n)\varphi(x) + s_0(n), \quad \varphi(x) = \frac{1}{1+x}.$$

Here, $s_1(n)$, $s_0(n)$ are the sequences of n and x. Our operators are generalized in the sense that for the particular values $s_1 = -1$ and $s_0 = 1$ the operator (3.1) becomes the classical operator (2.1). Similarly, the operator (2.1) is a special case of the operators $V_n^{M,2}(f;x)$ and $V_n^{M,3}(f;x)$. By straight forward calculations we obtain the following lemmas.

LEMMA 1. *For $r = 0,1$, we have*

1. $\sum\limits_{k=0}^{\infty} P_{n+r-1,k-r}(x) = 1,$

2. $\sum\limits_{k=0}^{\infty} P_{n+r-1,k-r}(x) \left(\frac{k}{n}\right) = \frac{r}{n} + \frac{(n+r-1)x}{n},$

3. $\sum\limits_{k=0}^{\infty} P_{n+r-1,k-r}(x) \left(\frac{k}{n}\right)^2 = \frac{r^2}{n^2} + (2r+1)\frac{(n+r-1)x}{n^2} + \frac{(n+r-1)(n+r)x^2}{n^2},$

4. $\sum\limits_{k=0}^{\infty} P_{n+r-1,k-r}(x) \left(\frac{k}{n}\right)^3 = \frac{r^3}{n^3} + \frac{(3r^2+3r+1)(n+r-1)x}{n^3} + \frac{3(r+1)(n+r-1)(n+r)x^2}{n^3}$
$+ \frac{(n+r-1)(n+r)(n+r+1)x^3}{n^3}.$

Since the expressions for images of monomials $e_j(x) = x^j$ by a positive linear operator are important to evaluate rate of approximation we make use of the identities of lemma 1, to obtain following

LEMMA 2. *Let* $V_n^{M,1}(f;x)$ *be defined by* (3.1) *and* $e_j(x) = x^j$, $j = 0,1,2$. *Then*

$$V_n^{M,1}(e_0;x) = 2s_0 + s_1,$$
$$V_n^{M,1}(e_1;x) = (2s_0 + s_1)x + \frac{(s_0 + s_1)(1-x)}{n},$$
$$V_n^{M,1}(e_2;x) = (2s_0 + s_1)x^2 + (4s_0 + s_1)\frac{x}{n} + \frac{s_1 x(2-x)}{n} + \frac{(s_0 + s_1)(1-x)}{n}.$$

COROLLARY 1. *If* $2s_0 + s_1 = 1$, *then* $V_n^{M,1}(e_0;x) = 1$ *and for the operator* $V_n^{M,1}(f;x)$ *we have that*

$$V_n^{M,1}((t-x);x) = (s_0 + s_1)\frac{(1-x)}{n},$$
$$V_n^{M,1}((t-x)^2;x) = \frac{1}{n}\left[4s_0 x + 3s_1 x - s_1 x^2\right] + \frac{1}{n}\left[\frac{1}{n} - 2x\right](1-x)(s_0 + s_1).$$

We discuss different cases for the values of s_0 *and* $s_1 = 1$. *We observe the following 7 cases:*

1. $s_0 = 0$ *implies* $s_1 = 1$
2. $s_1 = 0$ *implies* $s_0 = \frac{1}{2}$
3. $0 < s_1 < 1$ *implies* $s_0 > 0$
4. $s_1 = -1$ *implies* $s_0 = 1$
5. $s_1 > 1$ *implies* $s_0 < 0$
6. $-1 < s_1 < 0$ *implies* $s_0 > 0$ *and* $s_0 + s_1 > 0$
7. $s_1 < -1$ *implies* $s_0 > 0$ *and* $s_0 + s_1 < 0$

For the cases (1)–(3), *the operator* (3.1) *is positive and for the cases* (4)–(7) *cases these are negative.*

REMARK 1. It follows by lemma 2 that the monomials e^j are mapped to polynomials of degree at most j. Also on every closed sub interval $[a,b]$ of $[0,\infty)$ the convergence $\lim_{n\to\infty} V_n^{M,1}(e_j;x) = x^j$, $j = 0,1,2$ holds.

The convergence $\lim_{n\to\infty} V_n^{M,1}(e_j;x) = x^j$, $j = 0,1,2$ enable us to present the following convergence result

THEOREM 3. *If* $f \in \overline{C}[0,\infty)$, $2s_0+s_1=1$ *and let* $s_i(n)$, $i=0,1$ *be the sequences for which the operator* (3.1) *is positive and* $s_0(n)$ *is bounded then,*

$$\lim_{n\to\infty} V_n^{M,1}(f;x)=f(x)$$

uniformly on $[0,b]$, $b<\infty$.

Proof. If $s_0(n)$ is bounded then under conditions (1–3) $V_n^{M,1}(f;x)$ is positive. Since,

$$\lim_{n\to\infty} V_n^{M,1}(e_i;x)=e_i(x)$$

uniformly, the proof follows from Korovkin's theorem. □

Under the assumptions of Theorem 3, the operator $V_n^{M,1}$ is positive. For a twice differentiable function f, we have the following Voronovskaja type theorem.

THEOREM 4. *Let* f *be a twice differentiable function such that* $f'' \in \overline{C}[0,\infty)$ *and suppose that the operator* (3.1) *is positive. Then*

(a) *If* $s_0(n)+s_1(n)=0$ *and a second derivative* $f''(x)$ *exists at the certain point* $x\in[0,\infty)$ *then we have*

$$\lim_{n\to\infty} 2n(V_n^{M,1}(f;x)-(2s_0(n)+s_1(n))f(x))=\lim_{n\to\infty}(4s_0x+(3x-x^2)s_1)f''(x).$$

(b) *If* $s_0(n)+s_1(n)\neq 0$ *and the first derivative* $f'(x)$ *exists at the certain point* $x\in[0,\infty)$ *then we have*

$$\lim_{n\to\infty} n(V_n^{M,1}(f;x)-f(x))=\lim_{n\to\infty}(1-x)(s_0+s_1)f'(x).$$

Proof. (a) When $s_0(n)+s_1(n)=0$, we have

$$f\left(\frac{k}{n}\right)=f(x)+f'(x)\left(\frac{k}{n}-x\right)+\frac{f''(x)}{2}\left(\frac{k}{n}-x\right)^2+\mu\left(\frac{k}{n}-x\right)\left(\frac{k}{n}-x\right)^2 \tag{3.3}$$

where the function $\mu(t)$ is continuous and $\lim_{t\to 0}\mu(t)=0$. Multiplying equation (3.3) by $P_{n,k}^{M,1}(x)$ and then summation over k leads to

$$V_n^{M,1}(f;x)=f(x)V_n^{M,1}(e_0;x))+f'(x)V_n^{M,1}(e_1-x;x)+\frac{f''(x)}{2}V_n^{M,1}((e_1-x)^2);x)$$
$$+\sum_{k=0}^{\infty}P_{n,k}^{M,1}(x)\mu\left(\frac{k}{n}-x\right)\left(\frac{k}{n}-x\right)^2.$$

For $s_0+s_1=0$, $V_n^{M,1}(e_1-x;x)=\sum\limits_{k=0}^{\infty}P_{n,k}^{M,1}(x)\left(\frac{k}{n}-x\right)=0$. So we have,

$$V_n^{M,1}(f;x)=f(x)+\frac{1}{n}\left[4xs_0+(3x-x^2)s_1\right]\frac{f''(x)}{2}+\sum_{k=0}^{\infty}P_{n,k}^{M,1}(x)\mu\left(\frac{k}{n}-x\right)\left(\frac{k}{n}-x\right)^2. \tag{3.4}$$

By continuity of the function μ, for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|\frac{k}{n} - x| < \delta$ then $\mu\left(\frac{k}{n} - x\right) < \varepsilon$. Denote,

$$P := \left\{ k \in N_0 : \left| \frac{k}{n} - x \right| < \delta \right\}$$

and

$$Q := \left\{ k \in N_0 : \left| \frac{k}{n} - x \right| \geqslant \delta \right\}.$$

Then,

$$\begin{aligned}
&\sum_{k=0}^{\infty} P_{n,k}^{M,1}(x)\mu\left(\frac{k}{n} - x\right)\left(\frac{k}{n} - x\right)^2 \\
&\leqslant \sum_{k\in P} P_{n,k}^{M,1}(x)\left|\mu\left(\frac{k}{n} - x\right)\right|\left(\frac{k}{n} - x\right)^2 + \sum_{k\in Q} P_{n,k}^{M,1}(x)\left|\mu\left(\frac{k}{n} - x\right)\right|\left(\frac{k}{n} - x\right)^2 \\
&\leqslant \varepsilon \sum_{k\in P} P_{n,k}^{M,1}(x) + \frac{M}{\delta^2}\sum_{k\in Q} P_{n,k}^{M,1}(x)\left(\frac{k}{n} - x\right)^2 \\
&= \varepsilon + \frac{M}{\delta^2}(V_n^{M,1}(e_2;x) - 2xV_n^{M,1}(e_1;x) + x^2V_n^{M,1}(e_0;x)) \\
&= \varepsilon + \frac{M}{\delta^2}\frac{4s_0x + (3x - x^2)s_1}{n} < \varepsilon.
\end{aligned}$$

Here, $M = \sup_{0\leqslant t<\infty} |\mu(t)|$, $f'' \in \overline{C}[0,\infty)$, $0 \leqslant x \leqslant b < \infty$. The proof for $s_0 + s_1 = 0$ is complete. The case for $s_0 + s_1 \neq 0$ is similar. This completes the proof. □

REMARK 2. The rate of convergence $|V_n^{M,1}(f;x) - f(x)|$ for function having second order continuous derivatives is therefore $O(n^{-1})$ provided $s_0(n)$ and $s_1(n)$ are bounded sequences.

We prove an estimate for rate of approximation by $V_n^{M,1}(f;x)$ in following theorem.

THEOREM 5. *If $f \in \overline{C}[0,\infty)$, $0 \leqslant x \leqslant b < \infty$ and the operator (3.1) is positive then,*

$$|V_n^{M,1}(f;x) - f(x)| \leqslant (3|s_1| + 1)(b + 2)\omega\left(\frac{1}{\sqrt{n}}\right).$$

Proof. We have

$$V_n^{M,1}(f;x) = S(\varphi(x),n)\sum_{k=0}^{\infty} P_{n,k-1}(x)f\left(\frac{k}{n}\right) + S(1-\varphi(x),n)\sum_{k=0}^{\infty} P_{n-1,k}(x)f\left(\frac{k}{n}\right).$$

Using $2s_0+s_1=1$,

$$\begin{aligned}
|V_n^{M,1}(f;x)-f(x)| &\leqslant |S(\varphi(x),n)|\sum_{k=0}^{\infty}P_{n,k-1}(x)\left|f\left(\frac{k}{n}\right)-f(x)\right| \\
&\quad +|S(1-\varphi(x),n)|\sum_{k=0}^{\infty}P_{n-1,k}(x)\left|f\left(\frac{k}{n}\right)-f(x)\right| \\
&\leqslant |S(\varphi(x),n)|\sum_{k=0}^{\infty}P_{n,k-1}(x)\omega\left(\left|\frac{k}{n}-x\right|\right) \\
&\quad +|S(1-\varphi(x),n)|\sum_{k=0}^{\infty}P_{n-1,k}(x)\omega\left(\left|\frac{k}{n}-x\right|\right).
\end{aligned}$$

By the inequality,

$$\omega(\lambda\delta)\leqslant(1+\lambda)\omega(\delta),\quad \lambda\geqslant 0$$

we get

$$\begin{aligned}
|V_n^{M,1}(f;x)-f(x)| &\leqslant |S(\varphi(x),n)|\left(1+\sqrt{n}\sum_{k=0}^{\infty}P_{n,k-1}(x)\left|\frac{k}{n}-x\right|\right)\omega\left(\frac{1}{\sqrt{n}}\right) \\
&\quad +|S(1-\varphi(x),n)|\left(1+\sqrt{n}\sum_{k=0}^{\infty}P_{n-1,k}(x)\left|\frac{k}{n}-x\right|\right)\omega\left(\frac{1}{\sqrt{n}}\right).
\end{aligned} \tag{3.5}$$

Now using the Schwarz's inequality, we obtain

$$\begin{aligned}
\sum_{k=0}^{\infty}P_{n,k-1}(x)\left|\frac{k}{n}-x\right| &\leqslant \left(\sum_{k=0}^{\infty}P_{n,k-1}(x)\left(\frac{k}{n}-x\right)^2\right)^{\frac{1}{2}}\left(\sum_{k=0}^{\infty}P_{n,k-1}(x)\right)^{\frac{1}{2}} \\
&\leqslant \left(\sum_{k=0}^{\infty}P_{n,k-1}(x)\left(\frac{k}{n}\right)^2-2x\sum_{k=0}^{\infty}P_{n,k-1}(x)\left(\frac{k}{n}\right)+x^2\sum_{k=0}^{\infty}P_{n,k-1}(x)\right)^{\frac{1}{2}}.
\end{aligned}$$

Using lemma 1, we get for $x\in[0,b)$, $b\leqslant\infty$

$$\begin{aligned}
\sum_{k=0}^{\infty}P_{n,k-1}x\left|\frac{k}{n}-x\right| &\leqslant \left(\frac{x^2}{n}+\frac{x}{n}+\frac{1}{n^2}\right)^{\frac{1}{2}} \\
&\leqslant \frac{(1+x)}{\sqrt{n}}\sqrt{\frac{x}{1+x}+\frac{1}{n}\frac{1}{(1+x)^2}} \\
&\leqslant \frac{1+x}{\sqrt{n}}\sqrt{\left(1-\frac{1}{1+x}\right)+\frac{1}{n}\left(\frac{1}{1+x}\right)^2} \\
&\leqslant \frac{1+x}{\sqrt{n}}.
\end{aligned} \tag{3.6}$$

Similarly,

$$\sum_{k=0}^{\infty} P_{n,k-1}(x)\left|\frac{k}{n}-x\right| \leqslant \left(\sum_{k=0}^{\infty} P_{n,k-1}(x)\left(\frac{k}{n}-x\right)^2\right)^{\frac{1}{2}}\left(\sum_{k=0}^{\infty} P_{n,k-1}(x)\right)^{\frac{1}{2}}$$
$$\leqslant \frac{1+x}{\sqrt{n}} \tag{3.7}$$

Making use of (3.6) and (3.7) in (3.5) we obtain

$$|V_n^{M,1}(f;x)-f(x)| \leqslant (|S(\varphi(x),n)|+|S(1-\varphi(x),n)|)\left(1+\sqrt{n}\frac{(1+x)}{\sqrt{n}}\right)\omega\left(\frac{1}{\sqrt{n}}\right)$$
$$\leqslant (|S(\varphi(x),n)|+|S(1-\varphi(x),n)|)(x+2)\omega\left(\frac{1}{\sqrt{n}}\right). \tag{3.8}$$

In view of the equality $2s_0+s_1=1$, and $0<\varphi(x)\leqslant 1$, it follows that

$$|S(\varphi(x),n)|+|S(1-\varphi(x),n)| \leqslant |s_1\varphi|+\left|\frac{1-s_1}{2}\right|+|s_1-s_1\varphi+\frac{1}{2}-\frac{s_1}{2}| \leqslant 3|s_1|+1. \tag{3.9}$$

Combining these estimates we finally get

$$|V_n^{M,1}(f;x)-f(x)| \leqslant (3|s_1|+1)(2+x)\omega\left(\frac{1}{\sqrt{n}}\right).$$

Hence the proof is completed. □

The error estimate by the sequence $V_n^{M,1}(f;x)$ is at least as good as those by the usual Baskakov operators $V_n(f;x)$. Our next theorem discusses the convergence when the sequences $s_1(n), s_0(n)$ are convergent.

THEOREM 6. *Let $f \in \overline{C}[0,\infty)$ and $2s_0+s_1=1$. Then, for all convergent sequences $s_0(n)$ and $s_1(n)$ satisfying the cases when the operator is non positive,*

$$\lim_{n\to\infty} V_n^{M,1}(f;x) = f(x)$$

uniformly on $[0,b]$, $b<\infty$.

Proof. We prove the convergence theorem when the operator (3.1) is not positive. For this we divide our proof in two cases:

Case 1. When the unknown sequences $s_i(n)$, $i=0,1$ are convergent. We have

$$V_n^{M,1}(f;x) = S(\varphi(x),n)\sum_{k=0}^{\infty} P_{n,k-1}(x)f\left(\frac{k}{n}\right)+S((1-\varphi),n)\sum_{k=0}^{\infty} P_{n-1,k}(x)f\left(\frac{k}{n}\right).$$

We write,

$$V_n^{M,1}(f;x) = W_{n,1}(f;x)-W_{n,2}(f;x),$$

where

$$W_{n,1}(f;x)=s_1\varphi(x)\sum_{k=0}^{\infty}P_{n,k-1}(x)f\left(\frac{k}{n}\right)+s_1\sum_{k=0}^{\infty}P_{n-1,k}(x)f\left(\frac{k}{n}\right)$$

and

$$W_{n,2}(f;x)=s_1\varphi(x)\sum_{k=0}^{\infty}P_{n-1,k}(x)f\left(\frac{k}{n}\right)-s_0\sum_{k=0}^{\infty}P_{n-1,k}(x)f\left(\frac{k}{n}\right)-s_0\sum_{k=0}^{\infty}P_{n,k-1}(x)f\left(\frac{k}{n}\right).$$

By direct calculations we have

$$W_{n,1}(e_0,x)=s_1\left(1+\frac{1}{1+x}\right),$$

$$W_{n,1}(e_1,x)=s_1\left(1+\frac{1}{1+x}\right)x+\frac{s_1}{n}\left(\frac{1}{1+x}-x\right),$$

and

$$W_{n,1}(e_2,x)=s_1\left(1+\frac{1}{1+x}\right)x^2+\frac{s_1}{1+x}\left(x^2+3x+\frac{1}{n}\right)-\frac{s_1}{n^2}\left(nx^2-nx+x\right).$$

Similarly for $W_{n,2}(f;x)$ we have that

$$W_{n,2}(e_0,x)=\left(s_1\left(1+\frac{1}{1+x}\right)-1\right),$$

$$W_{n,2}(e_1,x)=\left(s_1\left(1+\frac{1}{1+x}\right)-1\right)x-\frac{1}{n}\left(\frac{s_1x}{1+x}+\frac{1-s_1}{2}(1+x)\right),$$

and

$$\begin{aligned}W_{n,2}(e_2,x)=&\left(s_1\left(1+\frac{1}{1+x}\right)-1\right)x^2-\frac{s_1}{n(1+x)}\left(x^2-x+\frac{x}{n}\right)\\&+\frac{s_1-1}{2n}\left(4x-\frac{x}{n}+\frac{1}{n}\right).\end{aligned}$$

Let $\lim_{n\to\infty}s_1(n)=l_1$. Since, the sequence (2.1) converges uniformly on $[0,b]$, it follows that

$$\lim_{n\to\infty}W_{n,1}(f;x)=l_1\left(1+\frac{1}{1+x}\right)f(x)$$

uniformly and similarly

$$\lim_{n\to\infty}W_{n,2}(f;x)=\left(l_1\left(1+\frac{1}{1+x}\right)-1\right)f(x)$$

uniformly. Therefore, the limit

$$\lim_{n\to\infty}V_n^{M,1}(f;x)=f(x)$$

holds uniformly on $[0,b]$.

Case 2. When the unknown sequences are bounded. The proof in this case is similar to Case 1. Hence, the proof is completed. □

4. Second order operator $V_n^{M,2}(f;x)$

In order to improve the order of approximation of classical Baskakov operator we further decompose the Baskakov operator $V_n(f;x)$ and get a new modified operator. Thus, we introduced new modified Baskakov operator $V_n^{M,2}(f;x)$ of order two by

$$V_n^{M,2}(f;x) = \sum_{k=0}^{\infty} P_{n,k}^{M,2}(x) f\left(\frac{k}{n}\right), \tag{4.1}$$

where

$$P_{n,k}^{M,2}(x) = S(\varphi(x),n)P_{n,k-2}(x) + R(\varphi,n)P_{n-1,k-1}(x) + S(1-\varphi(x),n)P_{n-2,k}(x) \tag{4.2}$$

and

$$\begin{aligned} S(\varphi(x),n) &= s_2\varphi^2 + s_1\varphi + s_0 \\ S(1-\varphi(x),n) &= s_2(1-\varphi)^2 + s_1(1-\varphi) + s_0 \\ R(\varphi,n) &= r_0\varphi(1-\varphi). \end{aligned}$$

REMARK 3. Our method is natural generalization of the operator $V_n(f;x)$ in the sense that if we put $s_2 = 1$, $s_1 = -2$, $s_0 = 1$, $r_0 = 2$ we recover the Baskakov operator (2.1).

LEMMA 3. *For $r = 0,1,2$ there hold the identities*

1. $\sum_{k=0}^{\infty} P_{n+r-2,k-r}(x) = 1$

2. $\sum_{k=0}^{\infty} P_{n+r-2,k-r}(x)\left(\frac{k}{n}\right) = \frac{r}{n} + \left(\frac{n+r-2}{n}\right)x$

3. $\sum_{k=0}^{\infty} P_{n+r-2,k-r}(x)\left(\frac{k}{n}\right)^2 = \frac{r^2}{n^2} + (2r+1)\left(\frac{n+r-2}{n^2}\right)x + \left(\frac{(n+r-1)(n+r-2)}{n^2}\right)x^2$

4. $\sum_{k=0}^{\infty} P_{n+r-2,k-r}(x)\left(\frac{k}{n}\right)^3 = \frac{r^3}{n^3} + (3r^2+3r+1)\left(\frac{n+r-2}{n^3}\right)x$
$+3(r+1)\left(\frac{(n+r-1)(n+r-2)}{n^3}\right)x^2 + \left(\frac{(n+r)(n+r-1)(n+r-2)}{n^3}\right)x^3.$

Using lemma 3 we find the values for sequences s_i, and r_i for the operator $V_n^{M,2}(f;x)$. By straight forward calculations

$$V_n^{M,2}(e_0;x) = 2s_0 + s_1 + s_2 - \varphi(2s_2 - r_0) + \varphi^2(2s_2 - r_0).$$

We assume the conditions

$$2s_0 + s_1 + s_2 = 1 \tag{4.3}$$

and

$$2s_2 - r_0 = 0. \tag{4.4}$$

Using the condition (4.4) we obtain

$$V_n^{M,2}(e_1;x) = (2s_0+s_1+s_2)x + \left(\frac{2(s_0+s_1+s_2)(1-x)}{n}\right).$$

Next, using the condition (4.3), we get

$$V_n^{M,2}(e_1;x) = x + 2(1-s_0)\frac{(1-x)}{n}.$$

Now we set $s_0 = 1,$ in the evaluation for $V_n^{M,2}(e_1;x).$ This yields

$$V_n^{M,2}(e_1;x) = x.$$

Next, we have

$$V_n^{M,2}(e_2;x) = (2s_0+s_1+s_2)x^2 + (-2s_0-3s_1-3s_2)\frac{x^2}{n} + (6s_0+5s_1+5s_2)\frac{x}{n}$$
$$+ (2s_0+2s_1+2s_2)\frac{x^2}{n^2} + (-2s_0-4s_1-6s_2)\frac{x}{n^2} + (s_0+s_1+s_2)\frac{4}{n^2}$$

Since, we have $2s_0+s_1+s_2 = 1,$ $s_0 = 1$ so that

$$s_1 = -1 - s_2. \tag{4.5}$$

Using these values together with the condition $2s_0+s_1+s_2 = 1,$ we get

$$V_n^{M,2}(e_2;x) = x^2 + \frac{x(2+n+nx-2s_2)}{n^2}.$$

Next, we set

$$s_2 = \frac{n(1+x)}{2}. \tag{4.6}$$

Finally, we evaluate the sequences's $s_i,$ r_i as

$$s_0 = 1, \quad s_2 = \frac{n(1+x)}{2}, \quad r_0 = n(1+x), \quad s_1 = -1 - \frac{n(1+x)}{2}.$$

With these values of s_0, s_1, s_2, r_0 the operator $V_n^{M,2}(f;x)$ is finally defined as

$$V_n^{M,2}(f;x) = \sum_{k=0}^{\infty} P_{n,k}^{M,2}(x) f\left(\frac{k}{n}\right), \tag{4.7}$$

where

$$P_{n,k}^{M,2}(x) = \left(\frac{n}{2}\left(\frac{1}{1+x}\right) - \left(1+\frac{n(1+x)}{2}\right)\left(\frac{1}{1+x}\right)+1\right)P_{n,k-2}(x)$$
$$+ n\left(\frac{x}{1+x}\right)P_{n-1,k-1}(x)$$
$$+ \left(\frac{n}{2}\left(\frac{x^2}{1+x}\right) - \left(1+\frac{n(1+x)}{2}\right)\left(\frac{x}{1+x}\right)+1\right)P_{n-2,k}(x).$$

NOTE 1. We will show that the (4.7) has order of approximation $O\left(\frac{1}{n^2}\right)$. First, we find the estimates of monomials and then extend the results for any arbitrary function with preassigned smoothness in Theorem 7.

LEMMA 4. *For the operator* (4.7), *we have*

1. $V_n^{M,2}(1,x) = 1$,
2. $V_n^{M,2}(t,x) = x$,
3. $V_n^{M,2}(t^2,x) = x^2 + \frac{2x}{n^2}$,
4. $V_n^{M,2}(t^3,x) = x^3 + \frac{2x^3+6x^2-2x}{n^2} + \frac{6x}{n^3}$.

LEMMA 5. *We have*

1. $V_n^{M,2}((t-x);x) = 0$,
2. $V_n^{M,2}((t-x)^2;x) = \frac{2x}{n^2}$,
3. $V_n^{M,2}((t-x)^3;x) = \frac{2x(x^2-1)}{n^2} + \frac{6x}{n^3}$.

Finally, we have following theorem on the operator $V_n^{M,2}(f;x)$.

THEOREM 7. *Let* $f \in \overline{C}[0,\infty)$ *and* $f'''(x)$ *exists at* $x \in [0,b]$, $b < \infty$. *Then,*

$$\lim_{n\to\infty} n^2(V_n^{M,2}(f;x) - f(x)) = x(1+x)\left(\frac{f''(x)}{1+x} + \frac{x-1}{3}f'''(x)\right).$$

Moreover, the results holds uniformly if $f'''(x)$ *is continuous on* $[0,b]$.

Proof. By linearity of $V_n^{M,2}(f;x)$ and smoothness of f

$$V_n^{M,2}(f,x) = f(x) + f'(x)V_n^{M,2}((t-x);x) + \frac{1}{2}f''(x)V_n^{M,2}((t-x)^2;x)$$
$$+ \frac{1}{6}f'''(x)V_n^{M,2}((t-x)^3;x) + V_n^{M}(\varepsilon(t,x)(t-x)^3;x),$$

where $\varepsilon(t,x)$ is bounded on $[0,b]$ and $\lim_{t\to x}\varepsilon(t,x) = 0$. By the lemma 5 we have

$$V_n^{M,2}(f;x) = f(x) + f''(x)\left(\frac{x}{n^2}\right) + \frac{1}{3}f'''(x)\left(\frac{x(x^2-1)}{n^2} + \frac{3x}{n^3}\right) + V_n^{M,2}(\varepsilon(t)(t-x)^3;x).$$

The proof now follows along the lines similar to Theorem 4. $\square$

THEOREM 8. *We have*

$$|V_n^{M,2}(f;x) - f(x)| \leqslant \phi(\varphi,n)(x+2)\omega\left(\frac{1}{\sqrt{n}}\right)$$

uniformly on $[0,b], b < \infty$, *where* $\phi(\varphi,n) = |S(\varphi(x),n)| + |R(\varphi,n)| + |S(1-\varphi(x),n)|$.

Proof. By using the values $2s_0+s_1+s_2=1$, $2s_2-r_0=0$

$$
\begin{aligned}
&\left|V_n^{M,2}(f;x)-f(x)\right|\\
&=\left|V_n^{M,2}(f;x)-(2s_0+s_1+s_2)f(x)\right|\\
&=\left|\sum_{k=0}^{\infty}P_{n,k}^{M,2}(x)f\left(\frac{k}{n}\right)-(2s_0+s_1+s_2)f(x)\right|\\
&\leqslant\Bigg|\sum_{k=0}^{\infty}(S(\varphi(x),n)P_{n,k-2}(x)+R(\varphi,n)P_{n-1,k-1}(x)\\
&\quad+S(1-\varphi(x),n)P_{n-2,k}(x))f\left(\frac{k}{n}\right)\\
&\quad-(S(\varphi(x),n)+R(\varphi,n)+S(1-\varphi(x),n))f(x)\Bigg|\\
&\leqslant\left|\sum_{k=0}^{\infty}S(\varphi(x),n)P_{n,k-2}(x)\left(f\left(\frac{k}{n}\right)-f(x)\right)\right|\\
&\quad+\left|\sum_{k=0}^{\infty}R(\varphi,n)P_{n-1,k-1}(x)\left(f\left(\frac{k}{n}\right)-f(x)\right)\right|\\
&\quad+\left|\sum_{k=0}^{\infty}S(1-\varphi(x),n)P_{n-2,k}(x)\left(f\left(\frac{k}{n}\right)-f(x)\right)\right|.
\end{aligned}
$$

Now, by the inequality

$$\left|f\left(\frac{k}{n}\right)-f(x)\right|\leqslant\omega\left(\left|\frac{k}{n}-x\right|\right)$$

we have that

$$\omega\left|\frac{k}{n}-x\right|=\omega\left(\sqrt{n}\left|\frac{k}{n}-x\right|\frac{1}{\sqrt{n}}\right)\leqslant\left(1+\sqrt{n}\left|\frac{k}{n}-x\right|\right)\omega\left(\frac{1}{\sqrt{n}}\right).$$

Hence,

$$
\begin{aligned}
\left|V_n^{M,2}(f;x)-f(x)\right|\leqslant&\sum_{k=0}^{\infty}|S(\varphi(x),n)P_{n,k-2}(x)|\left|\left(1+\sqrt{n}\left|\frac{k}{n}-x\right|\right)\right|\omega\left(\frac{1}{\sqrt{n}}\right)\\
&+\sum_{k=0}^{\infty}|R(\varphi,n)P_{n-1,k-1}(x)|\left|\left(1+\sqrt{n}\left|\frac{k}{n}-x\right|\right)\right|\omega\left(\frac{1}{\sqrt{n}}\right)\\
&+\sum_{k=0}^{\infty}|S(1-\varphi(x),n)P_{n-2,k}(x)|\left|\left(1+\sqrt{n}\left|\frac{k}{n}-x\right|\right)\right|\omega\left(\frac{1}{\sqrt{n}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \omega\left(\frac{1}{\sqrt{n}}\right)\left[|S(\varphi(x),n)|\left(1+\sqrt{n}\sum_{k=0}^{\infty}P_{n,k-2}(x)\left|\frac{k}{n}-x\right|\right)\right]\\
&\quad+\left[|R(\varphi,n)|\left(1+\sqrt{n}\sum_{k=0}^{\infty}P_{n-1,k-1}(x)\left|\frac{k}{n}-x\right|\right)\right]\\
&\quad+\left[|S(1-\varphi(x),n)|\left(1+\sqrt{n}\sum_{k=0}^{\infty}P_{n-2,k}(x)\left|\frac{k}{n}-x\right|\right)\right].
\end{aligned}
$$

An application of Schwarz's inequality yields

$$
\begin{aligned}
\sum_{k=0}^{\infty}P_{n,k-2}(x)\left|\frac{k}{n}-x\right| &\leqslant \left(\sum_{k=0}^{\infty}P_{n,k}(x)\left(\frac{k}{n}-x\right)^2\right)^{\frac{1}{2}}\left(\sum_{k=0}^{\infty}P_{n,k}(x)\right)^{\frac{1}{2}}\\
&=\left(x^2-2x\left(x+\frac{2}{n}\right)+x^2+\frac{x^2}{n}+\frac{5x}{n}+\frac{4}{n^2}\right)^{\frac{1}{2}}\\
&=\left(\frac{x^2+x}{n}+\frac{4}{n^2}\right)^{\frac{1}{2}}\\
&=\sqrt{\frac{x(1+x)}{n}+\frac{4}{n^2}}\\
&=\frac{(1+x)}{\sqrt{n}}\sqrt{\left(1-\frac{1}{1+x}\right)+\frac{4}{n}\left(\frac{1}{1+x}\right)^2}\\
&\leqslant \frac{1+x}{\sqrt{n}}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\sum_{k=0}^{\infty}P_{n-1,k-1}(x)\left|\frac{k}{n}-x\right| &\leqslant \left(\sum_{k=0}^{\infty}P_{n-1,k-1}(x)\left(\frac{k}{n}-x\right)^2\right)^{\frac{1}{2}}\left(\sum_{k=0}^{\infty}P_{n-1,k-1}(x)\right)^{\frac{1}{2}}\\
&=\left(x^2-2x\left(x-\frac{x}{n}+\frac{1}{n}\right)+x^2-\frac{x^2}{n}+\frac{3x}{n}-\frac{3x}{n^2}+\frac{1}{n^2}\right)^{\frac{1}{2}}\\
&=\sqrt{\frac{x(1+x)}{n}+\frac{2x}{n}-\frac{5x}{n^2}+\frac{1}{n^2}}\\
&\leqslant \frac{1+x}{\sqrt{n}}
\end{aligned}
$$

and

$$
\sum P_{n-2,k}(x)\left|\frac{k}{n}-x\right| \leqslant \frac{(1+x)}{\sqrt{n}}.
$$

Finally, combining these results we obtain,

$$\begin{aligned}|V_n^{M,2}(f;x)-f(x)| &\leqslant (|S(\varphi(x),n)|+|R(\varphi,n)|+|S(1-\varphi(x),n)|)(1+1+x)\omega\left(\frac{1}{\sqrt{n}}\right)\\ &\leqslant \phi(\varphi,n)(x+2)\omega\left(\frac{1}{\sqrt{n}}\right).\end{aligned}$$

Hence the proof is completed. □

REMARK 4. Although the degree of approximation in Theorem 8 is of order $O\left(\frac{1}{\sqrt{n}}\right)$, it is significantly improved to $O\left(\frac{1}{n^2}\right)$ for smoother functions as observed in Theorem 7.

5. Third order operator $V_n^{M,3}(f;x)$

For further improvement in the degree of approximation we define the third order modified Baskakov operator by

$$V_n^{M,3}(f;x)=\sum_{k=0}^{\infty}P_{n,k}^{M,3}(x)f\left(\frac{k}{n}\right) \tag{5.1}$$

where,

$$\begin{aligned}P_{n,k}^{M,3}(x)=&\,S(\varphi(x),n)P_{n,k-4}(x)+R(\varphi,n)P_{n-1,k-3}(x)\\ &+V(\varphi,n)P_{n-2,k-2}(x)+R(1-\varphi,n)P_{n-3,k-1}(x)\\ &+S(1-\varphi(x),n)P_{n-4,k}(x)\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}S(\varphi(x),n)&=s_4\varphi^4+s_3\varphi^3+s_2\varphi^2+s_1\varphi+s_0,\\ R(\varphi,n)&=r_4\varphi^4+r_3\varphi^3+r_2\varphi^2+r_1\varphi+r_0,\\ V(\varphi,n)&=v_0(\varphi(1-\varphi)^2).\end{aligned}$$

REMARK 5. For the operator (5.1) we note that if we put $s_4=1$, $s_3=-4$, $s_2=6$, $s_1=-4$, $s_0=1$, $r_4=-4$, $r_3=12$, $r_2=-12$, $r_1=4$, $r_0=0$ and $v_0=6$ then it reduces in the classical Baskakov operator (2.1).

LEMMA 6. *For* $r=0,1,2,3,4$ *we have*

1. $\sum\limits_{k=0}^{\infty}P_{n+r-4,k-r}(x)=1$
2. $\sum\limits_{k=0}^{\infty}P_{n+r-4,k-r}(x)\left(\frac{k}{n}\right)=\frac{r}{n}+\frac{(n+r-4)x}{n}$
3. $\sum\limits_{k=0}^{\infty}P_{n+r-4,k-r}(x)\left(\frac{k}{n}\right)^2=\frac{r^2}{n^2}+\frac{(2r+1)(n+r-4)x}{n^2}+\frac{(n+r-4)(n+r-3)x^2}{n^2}$

4. $\sum_{k=0}^{\infty} P_{n+r-4,k-r}(x)\left(\frac{k}{n}\right)^3 = \frac{r^3}{n^3} + (3r^2+3r+1)\frac{(n+r-4)x}{n^3}$
$+3(r+1)\frac{(n+r-4)(n+r-3)x^2}{n^3} + \frac{(n+r-4)(n+r-3)(n+r-2)x^3}{n^3}.$

Using the technique as for the operator $V_n^{M,2}(f;x)$, we find following sequences

$$\begin{aligned}
s_4 &= 1+\frac{23n(1+x)}{12}+\frac{n^2(1+x)^2}{8},\\
s_3 &= -4-\frac{14n(1+x)}{3}-\frac{n^2(1+x)^2}{4},\\
s_2 &= 6+\frac{10n(1+x)}{3}+\frac{n^2(1+x)^2}{8},\\
s_1 &= -4-\frac{7n(1+x)}{12},\\
s_0 &= 1,\\
r_4 &= -4-\frac{23n(1+x)}{3}-\frac{n^2(1+x)^2}{2},\\
r_3 &= 12+17n(1+x)+n^2(1+x)^2,\\
r_2 &= -12-\frac{31n(1+x)}{3}-\frac{n^2(1+x)^2}{2},\\
r_1 &= 4+n(1+x),\\
r_0 &= 0,\\
v_0 &= 6+\frac{23n(1+x)}{2}+\frac{3n^2(1+x)^2}{4}.
\end{aligned}$$

Using the values of sequences s_i, r_i where $i = 0,1,2,3,4$ and v_0 we find the moments of operator (5.1) in the next lemma.

LEMMA 7. *The operator $V_n^{M,3}(f;x)$ verifies*

1. $V_n^{M,3}(e_0;x) = 1,$
2. $V_n^{M,3}(e_1;x) = x,$
3. $V_n^{M,3}(e_2;x) = x^2,$
4. $V_n^{M,3}(e_3;x) = x^3,$
5. $V_n^{M,3}(e_4;x) = x^4 + \frac{12x(1+x)^3}{n^3}.$

Finally we have that

COROLLARY 2.

$$V_n^{M,3}((e_1-x)^j;x) = 0, \quad j = 0,1,2,3$$

and

$$V_n^{M,3}((t-x)^4;x) = \frac{12x(1+x)^3}{n^3} = O\left(\frac{1}{n^3}\right), \quad x \in [0,\infty).$$

THEOREM 9. *Let* $f \in \overline{C}[0,\infty)$. *Then the operator* (5.1) *together with the values of sequences* $v_0, s_i, r_i, i = 0,\dots,4$ *converges uniformly to* $f(x)$.

THEOREM 10. *Let* f *be a function such that* $f^{(iv)}(x) \in \overline{C}[0,\infty)$. *Then,*

$$\lim_{n\to\infty} n^3\left(V_n^{M,3}(f;x) - f(x)\right) = \frac{1}{2}x(x+1)^3 f^{(iv)}(x).$$

REMARK 6. We observe that $s_i(n)$, $r_i(n)$ and $v_0(n)$ are the sequences of functions i.e. depend on x while in the case of Bernstein operators (see [20]), the unknown sequences are numerical sequences only. The degree of approximation by the operators $V_n^{M,3}(f;x)$ is $O(n^{-3})$ for sufficiently smooth functions.

6. Numerical verification

Having been establishing the orders $O(n^{-2})$ and $O(n^{-3})$ we apply the operators $V_n^{M,i}(f;x)$, $k = 2,3$ to the various smooth and non-smooth functions. First, we choose $\exp(-x/10)$, $0 \leqslant x \leqslant 30$ for approximation by $V_n^{M,2}(f;x)$ for the degrees $n = 5$ and $n = 10$.

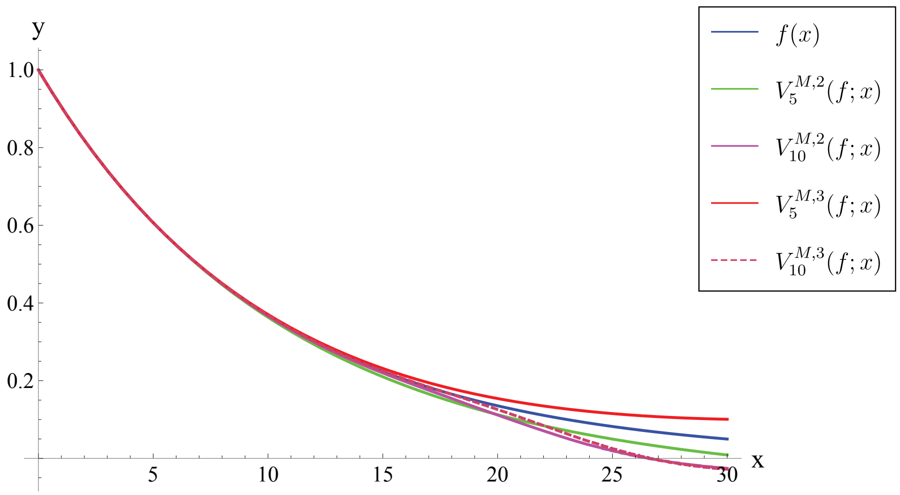

Figure 1: *Comparison of* $V_n^{M,2}(f;x)$ *and* $V_n^{M,3}(f;x)$ *for* $\exp(-x/10)$

The large error are obtained near the end $x = 30$ which is due to the small values of n and truncation of the series for $V_n^{M,2}(f;x)$ at $k = 400$. For approximation in large interval we need to enhance both the degree of polynomial as well as degree of partial sums in $V_n^{M,2}(f;x)$. The tables 1, 2 provide the absolute errors, values of $f(x)$ and the polynomials $V_n^{M,2}(f;x)$, $n=5,10$.

Table 1: *Comparison of* $V_5^{M,2}(f;x)$ *and* $V_{10}^{M,2}(f;x)$ *with respect to the function* $f(x)=\exp(-x/10)$.

		$V_n^{M,2}(f;x)$		$\lvert f(x)-V_n^{M,2}(f;x)\rvert$	
x	$f(x)$	$n=5$	$n=10$	$n=5$	$n=10$
3	0.740818	0.741429	0.74097	0.00061227	0.000151318
6	0.548812	0.548313	0.548662	0.000498981	0.00014917
9	0.40657	0.403135	0.405634	0.00343497	0.000935989
12	0.301194	0.293361	0.299085	0.00783306	0.0021096
15	0.22313	0.209947	0.219608	0.0131828	0.00352233
18	0.165299	0.14631	0.160272	0.0189886	0.00502694
21	0.122456	0.0976175	0.115956	0.0248389	0.00650006
24	0.090718	0.0602921	0.0828675	0.0304258	0.00785048
27	0.0672055	0.0316653	0.0581868	0.0355403	0.00901875
30	0.0497871	0.00973072	0.0398142	0.0400564	0.00997283

The maximum absolute errors by $V_5^{M,2}(f;x)$ and $V_{10}^{M,2}(f;x)$ are 0.0400564 and 0.00997283 at $x=30$, while the errors by $V_5^{M,3}(f;x)$ and $V_{10}^{M,3}(f;x)$ are 0.051334 and 0.00584135 obtained at $x=30$ again. It should be noted that there is slight difference between the values of error shown in table and the figure near end point because in software calculations some values are too small to represent as a normalized machine number so there is loss of precision near end point.

Table 2: *Comparison of* $V_5^{M,3}(f;x)$ *and* $V_{10}^{M,3}(f;x)$ *with respect to the function* $f(x)=\exp(-x/10)$.

		$V_n^{M,3}(f;x)$		$\lvert f(x)-V_n^{M,3}(f;x)\rvert$	
x	$f(x)$	$n=5$	$n=10$	$n=5$	$n=10$
3	0.740818	0.740826	0.740826	0.0000575333	.00000734
6	0.548812	0.549313	0.548876	0.000501069	0.0000639825
9	0.40657	0.408365	0.406798	0.00179559	0.000228654
12	0.301194	0.305531	0.301743	0.00433724	0.000548682
15	0.22313	0.231511	0.224179	0.00838108	0.00104858
18	0.165299	0.179335	0.167033	0.0140365	0.00173079
21	0.122456	0.143747	0.125053	0.0212904	0.00259683
24	0.090718	0.120757	0.0943438	0.0300393	0.00362582
27	0.0672055	0.107325	0.0718925	0.0401196	0.00468703
30	0.0497871	0.101121	0.0553377	0.051334	0.0055506

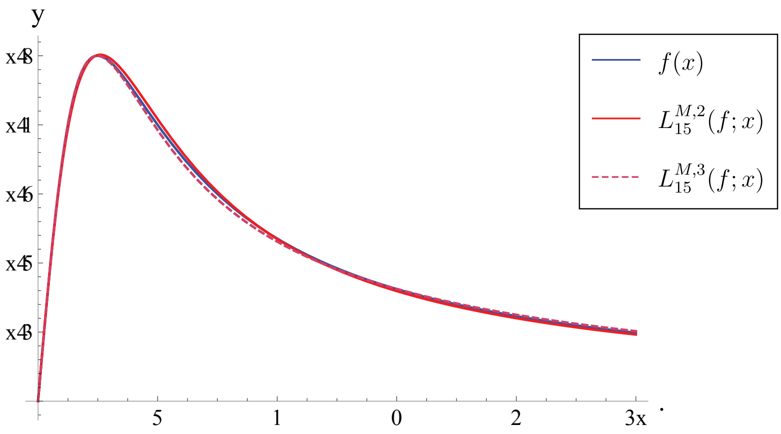

Figure 2: *Approximation of a smooth functions by* $V_n^{M,i}(f;x)$

Our next example discusses convergence of $V_n^{M,2}(f;x)$ and $V_n^{M,3}(f;x)$ for $n = 15$ to the function

$$f(x) = \frac{x}{x^2+1}. \tag{6.1}$$

It is clear that $f \in \overline{C}[0,\infty)$. Table 3 provides a comparison of the error bounds by $V_{15}^{M,2}(f;x)$ and $V_{15}^{M,3}(f;x)$ at different points.

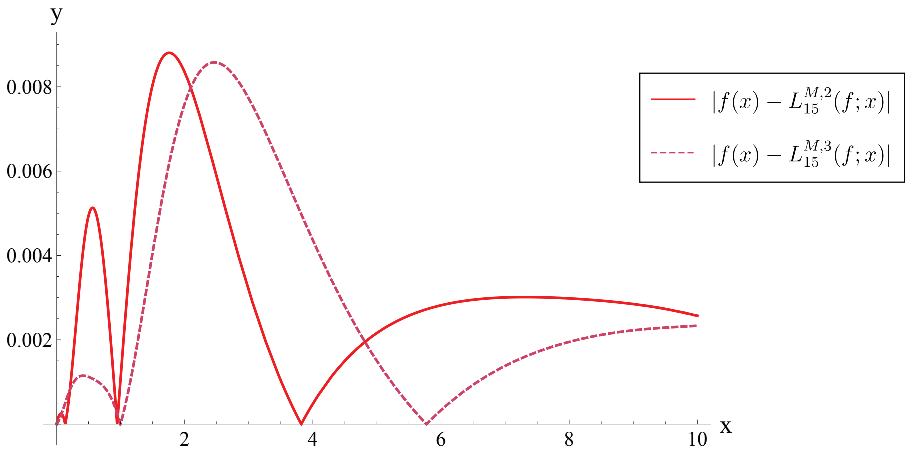

Figure 3: *Comparison of absolute errors by* $|f(x) - V_n^{M,i}(f;x)|$ *for* $i = 2,\ i = 3.$

Fig. 3 verifies that the absolute error $|f(x) - V_n^{M,3}(f;x)|$ by third order operator is less than the corresponding error $|f(x) - V_n^{M,2}(f;x)|$ by the second order operator $V_n^{M,2}(f;x)$.

Table 3: *Comparison of* $V_{15}^{M,2}(f;x)$ *and* $V_{15}^{M,3}(f;x)$ *with respect to the function* 6.1.

		$V_n^{M,i}(f;x)$		$\lvert f(x)-V_n^{M,i}(f;x)\rvert$	
x	$f(x)$	$i=2$	$i=3$	$i=2$	$i=3$
1	0.5	0.501087	0.499991	0.00108733	0.000008679
2	0.4	0.408345	0.392403	0.00834536	0.00759716
3	0.3	0.303171	0.292277	0.00317085	0.00772254
4	0.235294	0.234811	0.230935	0.000483567	0.00435922
5	0.192308	0.190146	0.190814	0.00216195	0.00149363
6	0.162162	0.159344	0.162493	0.00281795	0.000330981
7	0.14	0.136995	0.141383	0.00300469	0.00138254
8	0.123077	0.120094	0.125029	0.0029827	0.00195226
9	0.109756	0.106885	0.111993	0.0028715	0.00223698
10	0.0990099	0.0962838	0.101366	0.00272612	0.00235607

Next, we discuss convergence of $V_n^{M,3}(f;x)$ for a non-differentiable function $f(x)$ defined by

$$f(x)=\begin{cases} x^2, & \text{if } 0\leqslant x<2\\ 4, & \text{if } 2\leqslant x<6,\\ (x-8)^2, & \text{if } 6\leqslant x\leqslant 8,\\ 0, & \text{if } x>8. \end{cases} \tag{6.2}$$

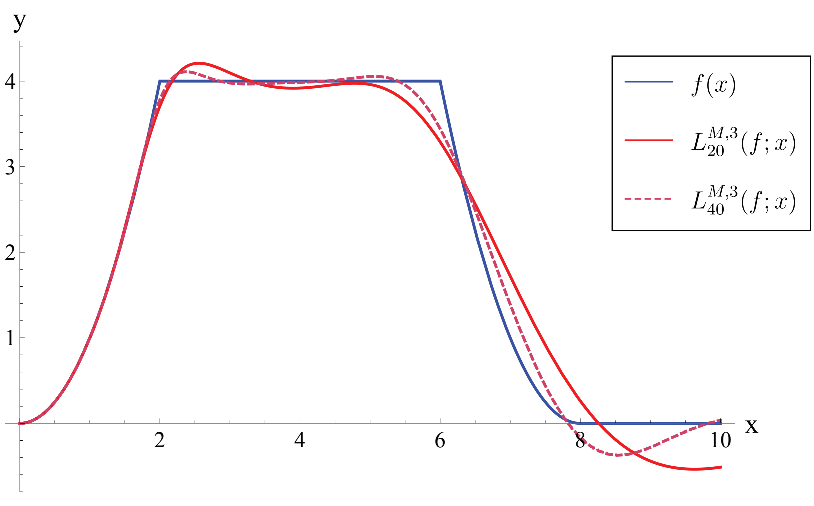

Figure 4: *Approximation of Discontinuous functions by* $V_n^{M,3}(f;x)$

Table 4, discusses the comparison of values of the function with respect to the value by $V_n^{M,3}(f;x)$ for $n=20$ and $n=40$.

Table 4: *Comparison of* $V_{20}^{M,3}(f;x)$ *and* $V_{40}^{M,3}(f;x)$ *with respect to the function* 6.2.

		$V_n^{M,3}(f;x)$		$\lvert f(x)-V_n^{M,3}(f;x)\rvert$	
x	$f(x)$	$n=20$	$n=40$	$n=20$	$n=40$
1	1	0.994772	0.999724	0.00522774	0.00027647
2	4	3.70739	3.78039	0.292613	0.219613
3	4	4.09573	3.97765	0.0957321	0.0223536
4	4	3.91882	3.98593	0.0811822	0.0140728
5	4	3.96041	4.05347	0.0395879	0.0534708
6	4	3.29052	3.4432	0.709483	0.556804
7	1	1.72108	1.39419	0.721076	0.39419
8	0	0.263065	-0.0637303	0.263065	0.0637303

Conclusions

It is shown that the absolute error, $|f(x)-V_n^{M,i}(f;x)|$ for a sufficiently smooth function f is of order n^{-i}. It is worth pointing out that in order to achieve higher rate of approximation, it is better to use a suitable modification $V_n^{M,2}(f;x)$ rather than increasing the degree of ordinary operator $V_n(f;x)$. It is observed that our method is applicable to approximate non-smooth functions too. However, the choice of the degree of polynomials $V_n^{M,i}(f;x)$ depends on the length of the interval taken for the approximation. The approximation of unbounded functions on $[0,\infty)$ with certain growth conditions i.e. functions f satisfying the inequalities $|f(x)| \leqslant C\phi(x)$ for given well behaved functions $\phi(x)$ can be interesting and significant topic for future investigations. For weighted approximation and numerical quadrature such functions are quite suitable so that the interested readers can consider this as open area of study.

Competing interests. The authors declare that they have no competing interests.

Acknowledgements. The authors are thankful to the anonymous reviewers for their careful reading and valuable suggestions.

REFERENCES

[1] A. M. ACU, V. GUPTA AND G. TACHEV, *Better numerical approximation by Durrmeyer type operators*, Results Math. **74** (3) (2019).

[2] J. A. ADELL, F.G. BADIA AND J. DE LA CAL, *On the iterates of some Bernstein type operators*, J. Math. Anal. Appl. **209**, 529–541 (1997).

[3] F. ALTOMARE AND E. MANGINO, *On a generalization of Baskakov operators*, Rev. Roumaine Math. Pures Appl. **44**, 683–706 (1999).

[4] A. ARAL AND H. ERBAY, *Parametric generalization of Baskakov operators*, Math. Commun. **24** (1), 119–131 (2019).

[5] A. ARAL AND V. GUPTA, *Generalized q-Baskakov operators*, Math. Slovaca **61** (4), 619–634 (2011).

[6] O. ARAMĂ, *Some properties concerning the sequence of polynomials of S.N. Bernstein*, Studii si Cerc. (Cluj) **8**, 195–210 (1957) (in Romanian).

[7] V. BASKAKOV, *An instance of a sequence of linear positive operators in the space of continuous functions*, Doklady Akademii Nauk SSSR **113**, 249–251 (1957).

[8] M. BECKER, *Global approximation theorems for Szász-Mirakjan and Baskakov operators in polynomial weight spaces*, Indiana Univ. Math. J. **27**, 127–142 (1978).

[9] S. BERNSTEIN, *Complément à l'article de E. Voronovskaja Détermination de la forme, etc.*, Compt. Rend. l'acad. sci. URSS **4**, 86–92 (1932).

[10] J. BUSTAMANTE, AND M. GARCÍA, J. JESÚS AND J. M. QUESADA, *Baskakov operators and Jacobi weights: pointwise estimates*, J. Inequal. Appl. **1**, 1–17 (2021).

[11] P. L. BUTZER, *Linear combinations of Bernstein polynomials*, Canad. J. Math **5** (2), 559–567 (1953).

[12] F. CAO, C. DING, Z. XU, *On multivariate Baskakov operator*, J. Math. Anal. Appl. **307**, 274–291 (2005).

[13] G. DATTOLI, S. LORENZUTTA AND C. CESARANO, *Bernstein Polynomials and Operational Methods*, J. Comput. Anal. Appl. **8** (4) (2006).

[14] Z. DITZIAN, *On global inverse theorems of Szász and Baskakov operators*, Can. J. Math. XXXI, **2**, 255–263 (1979).

[15] Z. DITZIAN, V. TOTIK, *Moduli of Smoothness*, Springer, New York, (1987).

[16] Z. FINTA, *A Quantitative Variant of Voronovskaja's Theorem for King-Type Operators*, Constr. Math. Anal. **2**, 3, 124–129 (2019).

[17] M. BIROU, *A note about some general King-type operators*, Ann. Tiberiu Popoviciu Semin. Funct. Equ. Approx. Convexity **12**, 3–16 (2014).

[18] G. BAŞCANBAZ-TUNCA, H. G. İNCE-İLARSLAN, AND A. ERENÇIN, *Bivariate Bernstein type operators*, Appl. Math. Comput. **273**, 543–552 (2016).

[19] A. R. GAIROLA, S. MAINDOLA, L. RATHOUR, L. N. MISHRA, V. N. MISHRA, *Better uniform approximation by new Bivariate Bernstein Operators*, Int. J. Anal. Appl., vol. **20**, ID: 60, (2022), pp. 1–19, https://doi.org/10.28924/2291-8639-20-2022-60.

[20] H. KHOSRAVIAN-ARAB, M. DEHGHAN, M. R. ESLAHCHI, *A new approach to improve the order of approximation of the Bernstein operators: Theory and applications*, Numer. Algorithms **77** (1), 111–150 (2018).

[21] G. G. LORENTZ, *Zur theorie der polynome von S. Bernstein*, Mat. Sb (N.S.) **2**, 543–556 (1937) (in German).

[22] G. G. LORENTZ, *Bernstein Polynomials*, Amer. Math. Soc. (2012).

[23] P. P. KOROVKIN, *Linear Operators and Approximation Theory*, First Revised Edition, ed. A. R. Gairola, Hindustan Publications Corporations, India (2017).

[24] A. R. GAIROLA, K. KUMAR AND L. N. MISHRA, *Degree of approximation by certain Durrmeyer type operator*, Discontinuity Nonlinearity Complex. **11** (2), 253–273 (2022).

[25] R. B. GANDHI, DEEPMALA, V. N. MISHRA, *Local and global results for modified Szász-Mirakjan operators*, Math. Method. Appl. Sci., vol. **40**, issue 7, (2017), pp. 2491–2504, doi:10.1002/mma.4171.

[26] I. GAVREA AND M. IVAN, *An answer to a conjecture on Bernstein operators*, J. Math. Anal. Appl. **390** (1), 86–92 (2012).

[27] H. GONSKA, *On the degree of approximation in Voronovskaja's theorem*, Stud. Univ. Babeş Bolyai Math. **52** (3), 103–115 (2007).

[28] H. H. GONSKA AND X. L. ZHOU, *Approximation theorems for the iterated Boolean sums of Bernstein operators*, J. Comput. Appl. Math. **53**, 21–31, (1994).

[29] H. GONSKA AND I. RASA, *Asymptotic behavior of differentiated Bernstein polynomials*, Mat. Vesnik **61** (1), 53–60 (2009).

[30] H. GONSKA, P. PIŢUL AND I. RAŞA, *General King-type operators*, Results Math. **53** (3), 279–286 (2009).

[31] V. GUPTA, *A note on modified Baskakov type operators*, Approx. Theory Appl. **10**, 74–78 (1994).

[32] V. GUPTA AND G. TACHEV, *Approximation with Positive Linear Operators and Linear Combinations*, Springer, Cham (2017).

[33] V. GUPTA, G. TACHEV AND A. M. ACU, *Modified Kantorovich operators with better approximation properties*, Numer. Algorithms (2018).

[34] J. P. KING, *Positive linear operators which preserve x^2*, Acta. Math. Hungar **3** (99), 203–208, (2003).

[35] C. P. MAY, *Saturation and inverse theorems for combinations of a class of exponential type opertors*, Can. J. Math. XXVIII, 1224–1250 (1976).

[36] C. A. MICCHELLI, *Saturation classes and iterates of operators*, Ph. D. Thesis, Stanford University (1969).

[37] C. A. MICCHELLI, *The saturation class and iterates of the Bernstein polynomials*, J. Approx. Theory **8** (1), 1–18 (1973).

[38] V. MIHESAN, *Uniform approximation with positive linear operators generated by generalized Baskakov method*, Automat. Comput. Appl. Math. **7**, 34–37 (1998).

[39] L. N. MISHRA, M. RAIZ, L. RATHOUR, V. N. MISHRA, *Tauberian theorems for weighted means of double sequences in intuitionistic fuzzy normed spaces*, Yugosl. J. Oper. Res. **32** (3), (2022), 377–388, https://doi.org/10.2298/YJOR210915005M.

[40] V. N. MISHRA, K. KHATRI, L. N. MISHRA, DEEPMALA, *Inverse result in simultaneous approximation by Baskakov-Durrmeyer-Stancu operators*, J. Inequal. Appl. 2013, 2013:586, doi:10.1186/1029-242X-2013-586.

[41] V. N. MISHRA, K. KHATRI, L. N. MISHRA, *On Simultaneous Approximation for Baskakov-Durrmeyer-Stancu type operators*, J. Ultra Sci. Phys. Sci., vol. **24**, no. (3) A, 2012, pp. 567–577.

[42] S. OSTROVSKA, *q-Bernstein polynomials and their iterates*, J. Approx. Theory **123** (2), 232–255 (2003).

[43] M. A. ÖZARSLAN, O. DUMAN AND C. KAANOĞLU, *Rates of convergence of certain King-type operators for functions with derivative of bounded variation*, Math. Comput. Model. **52**, (1–2), 334–345 (2010).

[44] M. A. ÖZARSLAN AND H. AKTUĞLU, *Local approximation properties for certain King type operators*, Filomat **27** (1), 173–181 (2013).

[45] P. K. PATEL, V. N. MISHRA AND M. ÖRKCÜ, *Some approximation properties of the generalized Baskakov operators*, J. Interdiscip. Math. **21** (3), 611–622 (2018).

[46] S. PETHE, *On the Baskakov operator*, Indian J. Math. **26**, 43–48 (1984).

[47] D. POPA, *Voronovskaja type theorems for King type operators*, Results Math. **75** (3), 1–28 (2020).

[48] R. K. S. RATHORE, *Linear combinations of linear positive operators and generating relations in special functions*, Ph.D. Thesis, IIT Delhi, (1973).

[49] X.-J. TANG, X.-C. WANG AND H. YI, *Weighted simultaneous approximation of the linear combinations of Baskakov operators*, Complex. **2020** (2020).

[50] V. TOTIK, *Uniform approximation by Baskakov and Meyer-König and Zeller operators*, Period. Math. Hung. **14**, 209–228 (1984).

[51] V. TOTIK, *Uniform approximation by exponential-type operators*, J. Math. Anal. Appl. **132**, 238–246 (1988).

[52] F. USTA, *On approximation properties of a new construction of Baskakov operators*, Adv. Difference Equ. **1**, 1–13 (2021).

[53] E. VORONOWSKAJA, *Détermination de la forme asymptotique d'approximation des fonctions par des polynômes de Bernstein*, C. R. Acad. Sci. URSS 79–95 (1932).

[54] A. WAFI AND S. KHATOON, *On the order of approximation of functions by generalized Baskakov operators*, Indian J. Pure Appl. Math. **35** (3), 347–358 (2004).

[55] C. ZHANG AND Z. ZHU, *Preservation properties of the Baskakov-Kantorovich operators*, Comput. Math. Appl. **57**, 1450–1455 (2009).

(Received June 13, 2022)

Asha Ram Gairola
Department of Mathematics
Doon University
Dehradun-248001 (Uttarakhand), India
e-mail: ashagairola@gmail.com

Amrita Singh
Department of Mathematics
Doon University
Dehradun-248001 (Uttarakhand), India
e-mail: ami449628@gmail.com

Laxmi Rathour
Ward Number – 16, Bhagatbandh, Anuppur 484 224
Madhya Pradesh, India
e-mail: laxmirathour817@gmail.com

Vishnu Narayan Mishra
Department of Mathematics
Indira Gandhi National Tribal University
Lalpur, Amarkantak-484 887, M.P., India
e-mail: vishnunarayanmishra@gmail.com
vnm@igntu.ac.in