IMPROVED RATE OF APPROXIMATION BY MODIFICATION OF BASKAKOV OPERATOR

ASHA RAM GAIROLA, AMRITA SINGH, LAXMI RATHOUR* AND VISHNU NARAYAN MISHRA*

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Abstract. The optimal order of approximation, $|L_n f(x) - f(x)|$ of a linear positive operator
$L_n f(x)$ is $1/n$ and cannot be improved however smooth the function may be. We remove
the positivity of the Baskakov operator $V_n(f;x)$ and introduce its three variants $V_{n,i}(f;x)$, $i = 1, 2, 3$. We prove that the rates of approximation by these operators are improved from the linear order $1/n$ to quadratic order $1/n^2$ and then to cubic order $1/n^3$ for sufficiently smooth functions.

1. Introduction

The Weierstrass theorem states that “A continuous function on a closed interval is uniform limit of the sequence of polynomials”. A constructive proof of this theorem was given by Bernstein in 1912 [22]. For a function defined and bounded in the interval $[0, 1]$, the $n$th Bernstein operator $B_n$, $n \in \mathbb{N}$ is defined by

$$B_n(f, x) := \sum_{k=0}^{n} B_{n,k}(x) f\left(\frac{k}{n}\right),$$

where

$$B_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0, 1, \ldots, n.$$

It is known that $B_n(f, x)$ converges to $f(x)$ (see [22]) whenever $f$ is continuous at $x$. In fact Aramâ [6] proved that for a function $f$ in $C[0, 1]$ there exist three distinct points $u_1, u_2, u_3$ such that

$$B_n(f, x) := f(x) + \frac{x(1-x)}{n} [u_1, u_2, u_3; f].$$

Further, convergence is uniform in case $f$ is continuous on $[0, 1]$. The Bernstein operators demonstrate interesting analytical and geometric properties e.g. (see [21])

* Corresponding author.


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1. \( B_n(f, 0) = f(0), \) \( B_n(f, 1) = f(1) \) i.e. \( B_n(f, x) \) have end point interpolation property,

2. \( B_n(f, x) \geq 0, \) for any \( f \) with \( f(x) \geq 0 \) on \([0, 1],\)

3. \( \lim_{n \to \infty} B_n^{(j)}(f, x) = f^{(j)}(x) \) if \( f^{(j)} \) is continuous at \( x, \)

4. \( f \in C[0, 1] \) is convex if and only iff for every \( n \in \mathbb{N}, \) \( B_n(f) \) is convex.

In 1960 Korovkin [23] extensively studied approximation by linear positive operators in his work. After that study of approximation by linear positive operators have attracted interest of many researchers. Subsequently, a number of new operators have been introduced as well as modification of classical operators also have been made in order to approximate other classes of function. It is worth mentioning the work of Voronovkaja [53] about the asymptotic error estimation of Bernstein operator and results of [26], [27] and [29].

Since the rate of approximation by linear positive operators is relatively slow, there have been attempts by many researchers for better order of approximation. In this direction certain methods have been introduced. The first one is the method of linear combination of operators introduced by Butzer [11]. These combinations for Bernstein operators \( B_n \) are defined by

\[
(2^k - 1)B_n^{2k}(f; x) = 2^k B_{2n}^{2k-1}(f; x) - B_n^{2k-1}(f; x), B_n^0(f; x) = B_n(f; x)
\]

The following convergence theorem was shown in [11] that states

**Theorem 1.** If \( f(x) \) is defined on \([0, 1]\) with \( |f(x)| \leq M \) and if \( f^{(2k)} \) exists at the point \( x, \) then

\[

\left| B_n^{2k-2}(f, x) - f(x) \right| = O(n^{-k}),
\]

and moreover,

\[

\left| B_n^{2k}(f, x) - f(x) \right| = o(n^{-k}), \quad \text{as} \quad n \to \infty, \quad k = 1, 2, \ldots
\]

Thus the degree of approximation is significantly improved to \( O(n^{-k}), \) however it requires \( 2^k n \) nodes that makes the method of linear combinations of less practical value. Later, Rathore [48] in 1973 and May [35] in 1976 extended the method in [11].

Another technique to improve degree of approximation without requirement of large number of nodes, was introduced by Micchelli [37], (see also [36]) wherein he used combinations of iterates of the Bernstein operator \( B_n \) to define the operator \( T_{n,M}(f, x) \) as follows

\[
T_{n,M}(f, x) = (I - (I - B_n)^M)(f, x) = \sum_{s=1}^{M} (-1)^{s+1} \binom{M}{s} B_n^s(f; x),
\]
where \( B_n(f;x) \), \( s \in \mathbb{N} \) denotes the \( s \)-th iterate and \( I = B_n^0(f;x) \). In fact Gonska and Zhou in [28] introduced the \( M \)-fold Boolean sum of the Bernstein operators \( B_n \) by

\[
\bigoplus^M B_n := B_n \bigoplus B_n \bigoplus \cdots \bigoplus B_n,
\]

where the Boolean sum of the operators \( P \) and \( Q \) on a common linear space \( X \) is defined by

\[
P \bigoplus Q = P + Q - P \circ Q.
\]

It is easily observed that

\[
\bigoplus^M B_n = I - (I - B_n)^M = T_{n,M}.
\]

In [28] the authors have proved a direct and saturation theorem for the operators \( \bigoplus^M B_n \) defined on the space \( C[0,1] \) of continuous functions on \([0,1]\). The authors of [28] proved the following Jackson type quantitative result

\[
\| f - \bigoplus^M B_n f \| \leq C \left\{ \omega^2_M \left( f, \frac{1}{\sqrt{n}} \right) + \| f \| n^{-M} \right\}, \quad M \geq 1
\]

The authors of [28] predicted the order of the approximation \( O(n^{-M}) \).

**2. Extension of the operator** \( B_n f(x) \) **to** \([0,\infty)\)

In an attempt to extend Bernstein polynomials to unbounded domain, another sequence of positive linear operators known as Baskakov operator was introduced by Baskakov [7]. These operators are defined by the formulas

\[
L_n f(x) = \sum_{k=0}^{\infty} (-x)^k \phi_n^{(k)}(x) f \left( \frac{k}{n} \right), \quad x \in [0,b], \quad b \in \mathbb{R}^+ \cup \{\infty\},
\]

where the sequence \( \phi_n^{(k)}(x) \) of functions satisfy the conditions

1. \( \phi_n \in C^\infty[0,b] \),
2. \( \phi_n(0) = 1 \),
3. \( (-1)^k \phi_n^{(k)} \geq 0 \) and
4. \( \phi_n^{(k+1)} = -n \phi_n^{(k)} \) for \( b > \max\{0,-c\} \) where \( c \) is some integer.

It can be easily verified that the functions \((1+x)^{-n}\) satisfy all the conditions of the sequence \( \phi_n(x) \). With \( \phi_n(x) = (1+x)^{-n} \) we obtain the following particular case of the Baskakov operator

\[
V_n(f;x) = \sum_{k=0}^{\infty} P_{n,k}(x) f \left( \frac{k}{n} \right), \quad (2.1)
\]
A number of results about the approximation properties of the operators the Baskakov operators Baskakov-Kantorovich operator are studied. and [38] for generalization by Mihesan. Further, in [55] the preservation properties of is worth refer to [31] for the rate of convergence of Baskakov type operator by Gupta where the basis function

\[ P_{n,k}(x) = \begin{cases} \binom{n+k-1}{k} \left( \frac{x}{1+x} \right)^k (1+x)^{-n} & \text{if } 0 \leq k \leq n, \\ 0 & \text{if } k < 0 \end{cases} \]

It can be shown that the sequence \( V_n(f;x) \) is convergent on \([0, \infty)\) provided the function \( f \) is such that \( \lim_{x \to \infty} f(x) \) exists. Let \( C[0,\infty) \) be the space of continuous functions on \([0,\infty)\). Then we define the following sub class of \( C[0,\infty) \).

\[ \overline{C}[0,\infty) := \{ f \in C[0,\infty) : |f(x)| < \infty \} . \] (2.2)

The Baskakov operators \( V_n(f;x) \) share interesting shape preservation properties. For example it interpolates the function \( f(x) \) at \( x = 0 \) i.e. \( V_n(f;0) = f(0) \). Following are some of the geometric properties of the operators \( V_n(f;x) \).

1. For a certain \( n \) the function \( V_nf, n \in \mathbb{N} \) is increasing for each increasing function \( f \),
2. for a convex function \( f \), the functions \( V_nf \), \( n \in \mathbb{N} \) are convex,
3. \( V_nf(x) \geq V_{n+1}f(x) \geq f(x), x \in [0,\infty) \).

A number of results about the approximation properties of the operators \( V_n(f;x) \) have been discussed in [2], [3], [8]–[15], [46], [50] and [51] and the references therein. It is worth refer to [31] for the rate of convergence of Baskakov type operator by Gupta and [38] for generalization by Mihesan. Further, in [55] the preservation properties of Baskakov-Kantorovich operator are studied.

It was proved in [7] that the sequence \( V_n(f;x) \) converges uniformly to the function \( f(x) \) whenever \( f \) is bounded on \([0,\infty)\). Later on, the uniform convergence was also observed for the functions having polynomial growth i.e. functions \( f \) satisfying the condition \( |f(x)| = O(1+x^m), m \in \{0\} \cup \mathbb{N} \). Moreover, for a twice differentiable function \( f \) satisfying the growth condition \( \sup |f''(x)|e^{-Ax} < \infty \), Ditzian in [14] have established following direct estimate

**Theorem 2.**

\[ e^{-Ax}|V_n(f;x) - f(x)| \leq \|f''\|_A \frac{x(1+x)}{n} \left( \frac{M + 1}{2} \right) \]

for \( x \leq \eta \sqrt{n}, n > 2A \) where \( M \) depends only on \( A \) and \( \eta = \frac{1}{3} \min(A^{-2}, 1) \) and \( \|f\|_A = \sup |f(x)|e^{-Ax} \). It follows from the above theorem that even for a twice differentiable function, the rate of approximation is \( O(n^{-1}) \) and can not be improved. For a twice differentiable function \( f \), it can be shown by standard method that if \( f''(x) < \infty, x \in [0,\infty) \) then

\[ \lim_{n \to \infty} \frac{V_n(f,x) - f(x)}{x+x^2} = \frac{1}{2} f''(x). \]
Hence, the optimal rate of approximation achieved by these operators is $O(n^{-1})$, however smooth the function may be. In fact, we have the values $V_n(t^2, x) = x^2 + \frac{x(1+x)}{n}$ which yields order of approximation $O(n^{-1})$, for the function $x^2$ while this function is sufficiently smooth. With the aim for better approximation results, Aral and Gupta in [5] introduced the $q$-variant $B_{n,q}$ of the Baskakov operators $V_n$ and proved that the rate of approximation

$$|B_{n,q}(f; x) - f(x)| \leq M \omega_2 \left( f, \frac{x}{n} \left( 1 + \frac{x}{q} \right) \right),$$

Thus, the $q$-variant $B_{n,q}$ of the Baskakov operators $V_n$ do not yield better estimates. In fact, the method of $q$-modifications have been proved useful for $q \geq 1$ and some exceptional classes of analytical functions only as proved in [42]. However, the $q$-variant $B_{n,q}$ demonstrate some shape preservation properties nicely (see [5]). Finally, it is worth to mention the approach of King to obtain better rates of approximation. King in [34] generalized the classical Bernstein operators $B_n$ and achieved better rate of approximation for his operator $V_n(f)$ as follows.

$$|V_n(f) - f(x)| \leq \omega(f, \delta) \left[ 1 + \frac{1}{\delta} \sqrt{2x(x-r_n(x))} \right],$$

where

$$r_n^+(x) = \begin{cases} x^2 - \frac{1}{2(n-1)} + \sqrt{\frac{n}{(n-1)^2} x^2 + \frac{1}{4(n-1)^2}} & \text{if } n = 1 \\ \frac{1}{2(n-1)} + \sqrt{\frac{n}{(n-1)^2} x^2 + \frac{1}{4(n-1)^2}} & \text{if } n = 2, 3, \ldots \end{cases}$$

For recent progress in the direction of King’s approach it is worth to mention the articles [16]–[52], and [43]–[44]. The Baskakov operators have been modified in several ways in order to look for better results. A certain number of modifications and applications of the Baskakov operators can be found in [4], [10], [13], [19], [45] and [49]. Relevant work in this direction can also be found in [25], and [39]–[41].

Recently, Arab et al. [20] have introduced a new method to get improved order of approximation by the Bernstein operator. By decomposition of the weight function $p_{n,k}(x)$, they defined the Bernstein type operator of first order as

$$B_{n,1}^{M,1}(f; x) = \sum_{k=0}^{n} B_{n,k}^{M,1}(x)f\left(\frac{k}{n}\right), \quad x \in [0, 1],$$

$$B_{n,k}^{M,1}(x) = a(x, n)B_{n-1,k}(x) + a(1-x, n)B_{n-1,k-1}(x) \tag{2.3}$$

where $a(x, n) = a_1(n)(x) + a_0(n), \quad n = 0, 1, \ldots, a_0(n)$ and $a_1(n)$ are two unknown sequences which are determined in an appropriate way so that convergence is assured. Similarly, second, third and fourth order Bernstein type operators $B_{n}^{M,2}(f; x)$, $B_{n}^{M,3}(f; x)$, $B_{n}^{M,4}(f; x)$, respectively have been defined. It is worth mentioning that for sufficiently smooth function $f(x)$, $B_{n}^{M,k}(f; x) - f(x) = O(n^{-k})$, $k = 1,4$. Thus, there is significant improvement in the order of approximation without including more nodes $k/n$. 
Subsequently, A. M. Acu et al. [1] introduced and studied the Durrmeyer variants of the Bernstein operators and applied the method of Arab et al. [20] to achieve better degree of approximation. Similarly, the Kantorovich variant of the modified Bernstein operators was studied in paper [33]. In a recent paper [24], degree of approximation by certain Durrmeyer type operators is given explicitly in terms of the modulus of smoothness of the function. However, to the best of our knowledge there have not been any such attempts to improve the degree of approximation in the infinite domain \([0, \infty)\) or \(\mathbb{R}\). Our aim is to extend the method of Arab et al. [20] for approximation of functions defined on the interval \([0, \infty)\). We modify classical Baskakov operator \(V_n(f;x)\) for better degree of approximation.

### 3. First order operator \(V_n^{M,1}(f;x)\)

We apply the technique of decomposition of weight function to the classical Baskakov operator (2.1) and introduce our operator as follows:

\[
V_n^{M,1}(f;x) = \sum_{k=0}^{\infty} P_{n,k}^{M,1}(x)f\left(\frac{k}{n}\right), \quad x \in [0, \infty)
\]

where,

\[
P_{n,k}^{M,1}(x) = S(\varphi(x), n)P_{n,k-1}(x) + S(1 - \varphi(x), n)P_{n-1,k}(x)
\]

and

\[
S(\varphi(x), n) = s_1(n)\varphi(x) + s_0(n), \quad \varphi(x) = \frac{1}{1+x}.
\]

Here, \(s_1(n), s_0(n)\) are the sequences of \(n\) and \(x\). Our operators are generalized in the sense that for the particular values \(s_1 = -1\) and \(s_0 = 1\) the operator (3.1) becomes the classical operator (2.1). Similarly, the operator (2.1) is a special case of the operators \(V_n^{M,2}(f;x)\) and \(V_n^{M,3}(f;x)\). By straightforward calculations we obtain the following lemmas.

**Lemma 1.** For \(r = 0, 1\), we have

1. \(\sum_{k=0}^{\infty} P_{n+r-1,k-r}(x) = 1\),

2. \(\sum_{k=0}^{\infty} P_{n+r-1,k-r}(x) \left(\frac{k}{n}\right) = \frac{r}{n} + \frac{(n+r-1)x}{n}\),

3. \(\sum_{k=0}^{\infty} P_{n+r-1,k-r}(x) \left(\frac{k}{n}\right)^2 = \frac{r^2}{n^2} + (2r + 1) \frac{(n+r-1)x}{n^2} + \frac{(n+r-1)(n+r)x^2}{n^2}\),

4. \(\sum_{k=0}^{\infty} P_{n+r-1,k-r}(x) \left(\frac{k}{n}\right)^3 = \frac{r^3}{n^3} + \frac{(3r^2+3r+1)(n+r-1)x}{n^3} + \frac{3(r+1)(n+r-1)(n+r)x^2}{n^3} + \frac{(n+r-1)(n+r)(n+r+1)x^3}{n^3}\).
Since the expressions for images of monomials $e_j(x) = x^j$ by a positive linear operator are important to evaluate rate of approximation we make use of the identities of lemma 1, to obtain following

**Lemma 2.** Let $V_n^{M,1}(f; x)$ be defined by (3.1) and $e_j(x) = x^j$, $j = 0, 1, 2$. Then

$$V_n^{M,1}(e_0; x) = 2s_0 + s_1,$$
$$V_n^{M,1}(e_1; x) = (2s_0 + s_1)x + \frac{(s_0 + s_1)(1-x)}{n},$$
$$V_n^{M,1}(e_2; x) = (2s_0 + s_1)x^2 + \frac{s_1x(2-x)}{n} + \frac{(s_0 + s_1)(1-x)}{n}.$$

**Corollary 1.** If $2s_0 + s_1 = 1$, then $V_n^{M,1}(e_0; x) = 1$ and for the operator $V_n^{M,1}(f; x)$ we have that

$$V_n^{M,1}((t-x); x) = (s_0 + s_1)\frac{(1-x)}{n},$$
$$V_n^{M,1}((t-x)^2; x) = \frac{1}{n} \left[ 4s_0x + 3s_1x - s_1x^2 \right] + \frac{1}{n} \left[ \frac{1}{n} - 2x \right] (1-x)(s_0 + s_1).$$

We discuss different cases for the values of $s_0$ and $s_1 = 1$. We observe the following 7 cases:

1. $s_0 = 0$ implies $s_1 = 1$
2. $s_1 = 0$ implies $s_0 = \frac{1}{2}$
3. $0 < s_1 < 1$ implies $s_0 > 0$
4. $s_1 = -1$ implies $s_0 = 1$
5. $s_1 > 1$ implies $s_0 < 0$
6. $-1 < s_1 < 0$ implies $s_0 > 0$ and $s_0 + s_1 > 0$
7. $s_1 < -1$ implies $s_0 > 0$ and $s_0 + s_1 < 0$

For the cases (1)–(3), the operator (3.1) is positive and for the cases (4)–(7) cases these are negative.

**Remark 1.** It follows by lemma 2 that the monomials $e_j$ are mapped to polynomials of degree at most $j$. Also on every closed sub interval $[a, b]$ of $[0, \infty)$ the convergence $\lim_{n \to \infty} V_n^{M,1}(e_j; x) = x^j$, $j = 0, 1, 2$ holds.

The convergence $\lim_{n \to \infty} V_n^{M,1}(e_j; x) = x^j$, $j = 0, 1, 2$ enable us to present the following convergence result
THEOREM 3. If \( f \in \mathcal{C}[0, \infty) \), \( 2s_0 + s_1 = 1 \) and let \( s_i(n), i = 0, 1 \) be the sequences for which the operator (3.1) is positive and \( s_0(n) \) is bounded then,
\[
\lim_{n \to \infty} V_{n}^{M,1}(f; x) = f(x)
\]
uniformly on \([0, b], b < \infty\).

Proof. If \( s_0(n) \) is bounded then under conditions (1–3) \( V_{n}^{M,1}(f;x) \) is positive. Since,
\[
\lim_{n \to \infty} V_{n}^{M,1}(e_i; x) = e_i(x)
\]
uniformly, the proof follows from Korovkin’s theorem. \( \square \)

Under the assumptions of Theorem 3, the operator \( V_{n}^{M,1} \) is positive. For a twice differentiable function \( f \), we have the following Voronovskaja type theorem.

THEOREM 4. Let \( f \) be a twice differentiable function such that \( f'' \in \mathcal{C}[0, \infty) \) and suppose that the operator (3.1) is positive. Then

(a) If \( s_0(n) + s_1(n) = 0 \) and a second derivative \( f''(x) \) exists at the certain point \( x \in [0, \infty) \) then we have
\[
\lim_{n \to \infty} 2n(V_{n}^{M,1}(f; x) - (2s_0(n) + s_1(n))f(x)) = \lim_{n \to \infty} (4s_0x + (3x - x^2)s_1)f''(x).
\]

(b) If \( s_0(n) + s_1(n) \neq 0 \) and the first derivative \( f'(x) \) exists at the certain point \( x \in [0, \infty) \) then we have
\[
\lim_{n \to \infty} n(V_{n}^{M,1}(f; x) - f(x)) = \lim_{n \to \infty} (1-x)(s_0 + s_1)f'(x).
\]

Proof. (a) When \( s_0(n) + s_1(n) = 0 \), we have
\[
f\left(\frac{k}{n}\right) = f(x) + f'(x)\left(\frac{k}{n} - x\right) + \frac{f''(x)}{2}\left(\frac{k}{n} - x\right)^2 + \mu\left(\frac{k}{n} - x\right)\left(\frac{k}{n} - x\right)^2 \quad (3.3)
\]
where the function \( \mu(t) \) is continuous and \( \lim_{t \to 0} \mu(t) = 0 \). Multiplying equation (3.3) by \( P_{n,k}^{M,1}(x) \) and then summation over \( k \) leads to
\[
V_{n}^{M,1}(f; x) = f(x)V_{n}^{M,1}(e_0; x) + f'(x)V_{n}^{M,1}(e_1 - x; x) + \frac{f''(x)}{2}V_{n}^{M,1}((e_1 - x)^2; x) + \sum_{k=0}^{\infty} P_{n,k}^{M,1}(x)\mu\left(\frac{k}{n} - x\right)\left(\frac{k}{n} - x\right)^2.
\]
For \( s_0 + s_1 = 0 \), \( V_{n}^{M,1}(e_1 - x; x) = \sum_{k=0}^{\infty} P_{n,k}^{M,1}(x)\left(\frac{k}{n} - x\right) = 0 \). So we have,
\[
V_{n}^{M,1}(f; x) = f(x) + \frac{1}{n}\left[4xs_0 + (3x - x^2)s_1\right]\frac{f''(x)}{2} + \sum_{k=0}^{\infty} P_{n,k}^{M,1}(x)\mu\left(\frac{k}{n} - x\right)\left(\frac{k}{n} - x\right)^2. \quad (3.4)
\]
By continuity of the function $\mu$, for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|\frac{k}{n} - x| < \delta$ then $\mu\left(\frac{k}{n} - x\right) < \varepsilon$. Denote,

$$P := \left\{ k \in N_0 : \left|\frac{k}{n} - x\right| < \delta \right\}$$

and

$$Q := \left\{ k \in N_0 : \left|\frac{k}{n} - x\right| \geq \delta \right\}.$$

Then,

$$\sum_{k=0}^{\infty} P_{n,k}^{M,1}(x) \mu\left(\frac{k}{n} - x\right) \left(\frac{k}{n} - x\right)^2 \leq \sum_{k \in P} P_{n,k}^{M,1}(x) \mu\left(\frac{k}{n} - x\right) \left(\frac{k}{n} - x\right)^2 + \sum_{k \in Q} P_{n,k}^{M,1}(x) \mu\left(\frac{k}{n} - x\right) \left(\frac{k}{n} - x\right)^2$$

$$\leq \varepsilon \sum_{k \in P} P_{n,k}^{M,1}(x) + \frac{M}{\delta^2} \sum_{k \in Q} P_{n,k}^{M,1}(x) \left(\frac{k}{n} - x\right)^2$$

$$= \varepsilon + \frac{M}{\delta^2} \left(V_n^{M,1}(e_2;x) - 2xV_n^{M,1}(e_1;x) + x^2 V_n^{M,1}(e_0;x)\right)$$

$$= \varepsilon + \frac{M}{\delta^2} \frac{4s_0 + (3x - x^2)s_1}{n} < \varepsilon.$$

Here, $M = \sup_{0 \leq t < \infty} |\mu(t)|$, $f'' \in \overline{C}[0, \infty)$, $0 \leq x \leq b < \infty$. The proof for $s_0 + s_1 = 0$ is complete. The case for $s_0 + s_1 \neq 0$ is similar. This completes the proof. □

**Remark 2.** The rate of convergence $|V_n^{M,1}(f;x) - f(x)|$ for function having second order continuous derivatives is therefore $O(n^{-1})$ provided $s_0(n)$ and $s_1(n)$ are bounded sequences.

We prove an estimate for rate of approximation by $V_n^{M,1}(f;x)$ in following theorem.

**Theorem 5.** If $f \in \overline{C}[0, \infty)$, $0 \leq x \leq b < \infty$ and the operator (3.1) is positive then,

$$|V_n^{M,1}(f;x) - f(x)| \leq (3|s_1| + 1)(b + 2)\omega\left(\frac{1}{\sqrt{n}}\right).$$

**Proof.** We have

$$V_n^{M,1}(f;x) = S(\phi(x),n) \sum_{k=0}^{\infty} P_{n,k-1}(x)f\left(\frac{k}{n}\right) + S(1 - \phi(x),n) \sum_{k=0}^{\infty} P_{n-1,k}(x)f\left(\frac{k}{n}\right).$$
Using $2s_0 + s_1 = 1$,

$$|V_n^{M,1}(f; x) - f(x)| \leq |S(\phi(x), n)| \sum_{k=0}^{\infty} P_{n,k-1}(x) \left| f\left(\frac{k}{n}\right) - f(x) \right|$$

$$+ |S(1 - \phi(x), n)| \sum_{k=0}^{\infty} P_{n-1,k}(x) \left| f\left(\frac{k}{n}\right) - f(x) \right|$$

$$\leq |S(\phi(x), n)| \sum_{k=0}^{\infty} P_{n,k-1}(x) \omega \left(\left| \frac{k}{n} - x \right| \right)$$

$$+ |S(1 - \phi(x), n)| \sum_{k=0}^{\infty} P_{n-1,k}(x) \omega \left(\left| \frac{k}{n} - x \right| \right).$$

By the inequality,

$$\omega(\lambda, \delta) \leq (1 + \lambda) \omega(\delta), \quad \lambda \geq 0$$

we get

$$|V_n^{M,1}(f; x) - f(x)| \leq |S(\phi(x), n)| \left(1 + \sqrt{n} \sum_{k=0}^{\infty} P_{n,k-1}(x) \frac{k}{n} - x \right) \omega \left(\frac{1}{\sqrt{n}}\right)$$

$$+ |S(1 - \phi(x), n)| \left(1 + \sqrt{n} \sum_{k=0}^{\infty} P_{n-1,k}(x) \frac{k}{n} - x \right) \omega \left(\frac{1}{\sqrt{n}}\right).$$

Now using the Schwarz’s inequality, we obtain

$$\sum_{k=0}^{\infty} P_{n,k-1}(x) \frac{k}{n} - x \leq \left(\sum_{k=0}^{\infty} P_{n,k-1}(x) \left(\frac{k}{n} - x\right)^2 \right)^{\frac{1}{2}} \left(\sum_{k=0}^{\infty} P_{n,k-1}(x) \right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{k=0}^{\infty} P_{n,k-1}(x) \left(\frac{k}{n}\right)^2 - 2x \sum_{k=0}^{\infty} P_{n,k-1}(x) \left(\frac{k}{n}\right) + x^2 \sum_{k=0}^{\infty} P_{n,k-1}(x) \right)^{\frac{1}{2}}.$$

Using lemma 1, we get for $x \in [0, b), b \leq \infty$

$$\sum_{k=0}^{\infty} P_{n,k-1}x \frac{k}{n} - x \leq \left(\frac{x^2}{n} + \frac{x}{n} + \frac{1}{n^2}\right)^{\frac{1}{2}}$$

$$\leq \frac{1 + x}{\sqrt{n}} \sqrt{\frac{x}{1 + x} + \frac{1}{n(1 + x)^2}}$$

$$\leq \frac{1 + x}{\sqrt{n}} \sqrt{\left(1 - \frac{1}{1 + x}\right) + \frac{1}{n}\left(\frac{1}{1 + x}\right)^2}$$

$$\leq \frac{1 + x}{\sqrt{n}}.$$

(3.6)
Similarly,
\[
\sum_{k=0}^{\infty} P_{n,k-1}(x) \left| \frac{k}{n} - x \right| \leq \left( \sum_{k=0}^{\infty} P_{n,k-1}(x) \left( \frac{k}{n} - x \right)^2 \right)^{\frac{1}{2}} \left( \sum_{k=0}^{\infty} P_{n,k-1}(x) \right)^{\frac{1}{2}} \leq \frac{1+x}{\sqrt{n}}
\]
(3.7)

Making use of (3.6) and (3.7) in (3.5) we obtain
\[
|V_{n}^{M,1}(f;x) - f(x)| \leq (|S(\phi(x),n)| + |S(1 - \phi(x),n)|) \left( 1 + \sqrt{n} \frac{(1+x)}{\sqrt{n}} \right) \omega \left( \frac{1}{\sqrt{n}} \right)
\]
\[
\leq (|S(\phi(x),n)| + |S(1 - \phi(x),n)|)(x+2)\omega \left( \frac{1}{\sqrt{n}} \right).
\]
(3.8)

In view of the equality \(2s_0 + s_1 = 1\), and \(0 < \phi(x) \leq 1\), it follows that
\[
|S(\phi(x),n)| + |S(1 - \phi(x),n)| \leq |s_1\phi| + \left| \frac{1-s_1}{2} \right| + |s_1 - s_1\phi + \frac{1}{2} - \frac{s_1}{2}| \leq 3|s_1| + 1.
\]
(3.9)

Combining these estimates we finally get
\[
|V_{n}^{M,1}(f;x) - f(x)| \leq (3|s_1| + 1)(2+x)\omega \left( \frac{1}{\sqrt{n}} \right).
\]

Hence the proof is completed. \(\square\)

The error estimate by the sequence \(V_{n}^{M,1}(f;x)\) is at least as good as those by the usual Baskakov operators \(V_n(f;x)\). Our next theorem discusses the convergence when the sequences \(s_1(n), s_0(n)\) are convergent.

**THEOREM 6.** Let \(f \in \mathcal{C}[0, \infty)\) and \(2s_0 + s_1 = 1\). Then, for all convergent sequences \(s_0(n)\) and \(s_1(n)\) satisfying the cases when the operator is non positive,
\[
\lim_{n \to \infty} V_{n}^{M,1}(f;x) = f(x)
\]
uniformly on \([0,b], b < \infty\).

**Proof.** We prove the convergence theorem when the operator (3.1) is not positive. For this we divide our proof in two cases:

Case 1. When the unknown sequences \(s_i(n), i = 0, 1\) are convergent. We have
\[
V_{n}^{M,1}(f;x) = S(\phi(x),n) \sum_{k=0}^{\infty} P_{n,k-1}(x)f \left( \frac{k}{n} \right) + S((1 - \phi),n) \sum_{k=0}^{\infty} P_{n-1,k}(x)f \left( \frac{k}{n} \right).
\]

We write,
\[
V_{n}^{M,1}(f;x) = W_{n,1}(f;x) - W_{n,2}(f;x),
\]
where
\[ W_{n,1}(f; x) = s_1 \varphi(x) \sum_{k=0}^{\infty} P_{n,k-1}(x)f \left( \frac{k}{n} \right) + s_1 \sum_{k=0}^{\infty} P_{n-1,k}(x)f \left( \frac{k}{n} \right) \]
and
\[ W_{n,2}(f; x) = s_1 \varphi(x) \sum_{k=0}^{\infty} P_{n-1,k}(x)f \left( \frac{k}{n} \right) - s_0 \sum_{k=0}^{\infty} P_{n-1,k}(x)f \left( \frac{k}{n} \right) - s_0 \sum_{k=0}^{\infty} P_{n,k-1}(x)f \left( \frac{k}{n} \right). \]

By direct calculations we have
\[ W_{n,1}(e_0, x) = s_1 \left( 1 + \frac{1}{1+x} \right), \]
\[ W_{n,1}(e_1, x) = s_1 \left( 1 + \frac{1}{1+x} \right) x + \frac{s_1}{n} \left( \frac{1}{1+x} - x \right), \]
and
\[ W_{n,1}(e_2, x) = s_1 \left( 1 + \frac{1}{1+x} \right) x^2 + \frac{s_1}{1+x} \left( x^2 + 3x + \frac{1}{n} \right) - \frac{s_1}{n^2} (nx^2 - nx + x). \]

Similarly for \( W_{n,2}(f; x) \) we have that
\[ W_{n,2}(e_0, x) = s_1 \left( 1 + \frac{1}{1+x} \right) - 1, \]
\[ W_{n,2}(e_1, x) = s_1 \left( 1 + \frac{1}{1+x} \right) - 1 \left( x - \frac{1}{n} \left( \frac{s_1x}{1+x} + \frac{1-s_1}{2}(1+x) \right) \right), \]
and
\[ W_{n,2}(e_2, x) = s_1 \left( 1 + \frac{1}{1+x} \right) - 1 \left( x^2 - \frac{s_1}{n(1+x)} \left( x^2 - x + \frac{x}{n} \right) \right) + \frac{s_1-1}{2n} \left( 4x - \frac{x}{n} + \frac{1}{n} \right). \]

Let \( \lim_{n \to \infty} s_1(n) = l_1 \). Since, the sequence (2.1) converges uniformly on \([0, b]\), it follows that
\[ \lim_{n \to \infty} W_{n,1}(f; x) = l_1 \left( 1 + \frac{1}{1+x} \right) f(x) \]
uniformly and similarly
\[ \lim_{n \to \infty} W_{n,2}(f; x) = \left( l_1 \left( 1 + \frac{1}{1+x} \right) - 1 \right) f(x) \]
uniformly. Therefore, the limit
\[ \lim_{n \to \infty} V_n^{M,1}(f; x) = f(x) \]
holds uniformly on \([0, b]\).

Case 2. When the unknown sequences are bounded. The proof in this case is similar to Case 1. Hence, the proof is completed. \( \square \)
4. Second order operator $V_n^{M.2}(f;x)$

In order to improve the order of approximation of classical Baskakov operator we further decompose the Baskakov operator $V_n(f;x)$ and get a new modified operator. Thus, we introduced new modified Baskakov operator $V_n^{M.2}(f;x)$ of order two by

$$V_n^{M.2}(f;x) = \sum_{k=0}^{\infty} P_{n,k}^{M.2}(x) f\left(\frac{k}{n}\right),$$

(4.1)

where

$$P_{n,k}^{M.2}(x) = S(\varphi(x),n)P_{n,k-2}(x) + R(\varphi,n)P_{n-1,k-1}(x) + S(1-\varphi(x),n)P_{n-2,k}(x)$$

(4.2)

and

$$S(\varphi(x),n) = s_2\varphi^2 + s_1\varphi + s_0$$
$$S(1-\varphi(x),n) = s_2(1-\varphi)^2 + s_1(1-\varphi) + s_0$$
$$R(\varphi,n) = r_0\varphi(1-\varphi).$$

Remark 3. Our method is natural generalization of the operator $V_n(f;x)$ in the sense that if we put $s_2 = 1, s_1 = -2, s_0 = 1, r_0 = 2$ we recover the Baskakov operator (2.1).

Lemma 3. For $r = 0, 1, 2$ there hold the identities

1. $\sum_{k=0}^{\infty} P_{n+r-2,k-r}(x) = 1$

2. $\sum_{k=0}^{\infty} P_{n+r-2,k-r}(x) \left(\frac{k}{n}\right) = \frac{x}{n} + \left(\frac{n+r-2}{n}\right) x$

3. $\sum_{k=0}^{\infty} P_{n+r-2,k-r}(x) \left(\frac{k}{n}\right)^2 = \frac{x^2}{n^2} + (2r+1)\left(\frac{n+r-2}{n^2}\right) x + \left(\frac{(n+r-1)(n+r-2)}{n^2}\right) x^2$

4. $\sum_{k=0}^{\infty} P_{n+r-2,k-r}(x) \left(\frac{k}{n}\right)^3 = \frac{x^3}{n^3} + (3r^2 + 3r + 1)\left(\frac{n+r-2}{n^3}\right) x$
$$+ 3(r+1)\left(\frac{(n+r-1)(n+r-2)}{n^3}\right) x^2 + \left(\frac{(n+r)(n+r-1)(n+r-2)}{n^3}\right) x^3.$$

Using lemma 3 we find the values for sequences $s_i$, and $r_i$ for the operator $V_n^{M.2}(f;x)$. By straightforward calculations

$$V_n^{M.2}(e_0;x) = 2s_0 + s_1 + s_2 - \varphi(2s_2 - r_0) + \varphi^2(2s_2 - r_0).$$

We assume the conditions

$$2s_0 + s_1 + s_2 = 1$$

(4.3)

and

$$2s_2 - r_0 = 0.$$
Using the condition (4.4) we obtain
\[ V_{n}^{M,2}(e_1; x) = (2s_0 + s_1 + s_2)x + \left( \frac{2(s_0 + s_1 + s_2)(1-x)}{n} \right). \]

Next, using the condition (4.3), we get
\[ V_{n}^{M,2}(e_1; x) = x + 2(1-s_0)\frac{(1-x)}{n}. \]

Now we set \( s_0 = 1 \), in the evaluation for \( V_{n}^{M,2}(e_1; x) \). This yields
\[ V_{n}^{M,2}(e_1; x) = x. \]

Next, we have
\[ V_{n}^{M,2}(e_2; x) = (2s_0 + s_1 + s_2)x^2 + (-2s_0 - 3s_1 - 3s_2)\frac{x^2}{n} + (6s_0 + 5s_1 + 5s_2)\frac{x}{n} \]
\[ + (2s_0 + 2s_1 + 2s_2)\frac{x^2}{n^2} + (-2s_0 - 4s_1 - 6s_2)\frac{x}{n^2} + (s_0 + s_1 + s_2)\frac{4}{n^2} \]

Since, we have \( 2s_0 + s_1 + s_2 = 1 \), \( s_0 = 1 \) so that
\[ s_1 = -1 - s_2. \] \hspace{1cm} (4.5)

Using these values together with the condition \( 2s_0 + s_1 + s_2 = 1 \), we get
\[ V_{n}^{M,2}(e_2; x) = x^2 + \frac{x(2 + n + nx - 2s_2)}{n^2}. \]

Next, we set
\[ s_2 = \frac{n(1+x)}{2}. \] \hspace{1cm} (4.6)

Finally, we evaluate the sequences’s \( s_i, r_i \) as
\[ s_0 = 1, \quad s_2 = \frac{n(1+x)}{2}, \quad r_0 = n(1+x), \quad s_1 = -1 - \frac{n(1+x)}{2}. \]

With these values of \( s_0, s_1, s_2, r_0 \) the operator \( V_{n}^{M,2}(f; x) \) is finally defined as
\[ V_{n}^{M,2}(f; x) = \sum_{k=0}^{\infty} P_{n,k}^{M,2}(x) f \left( \frac{k}{n} \right), \] \hspace{1cm} (4.7)

where
\[ P_{n,k}^{M,2}(x) = \left( \frac{n}{2} \left( \frac{1}{1+x} \right) - \left( 1 + n\frac{(1+x)}{2} \right) \left( \frac{1}{1+x} \right) + 1 \right) P_{n,k-2}(x) \]
\[ + n \left( \frac{x}{1+x} \right) P_{n-1,k-1}(x) \]
\[ + \left( \frac{n}{2} \left( \frac{x^2}{1+x} \right) - \left( 1 + n\frac{(1+x)}{2} \right) \left( \frac{x}{1+x} \right) + 1 \right) P_{n-2,k}(x). \]
NOTE 1. We will show that the (4.7) has order of approximation \( O \left( \frac{1}{n^2} \right) \). First, we find the estimates of monomials and then extend the results for any arbitrary function with preassigned smoothness in Theorem 7.

**LEMMA 4.** For the operator (4.7), we have
1. \( V_n^{M,2}(1, x) = 1 \),
2. \( V_n^{M,2}(t, x) = x \),
3. \( V_n^{M,2}(t^2, x) = x^2 + \frac{2x}{n^2} \),
4. \( V_n^{M,2}(t^3, x) = x^3 + \frac{2x^2 + 6x^2 - 2x + 6x}{n^2} \).

**LEMMA 5.** We have
1. \( V_n^{M,2}((t - x); x) = 0 \),
2. \( V_n^{M,2}((t - x)^2; x) = \frac{2x}{n^2} \),
3. \( V_n^{M,2}((t - x)^3; x) = \frac{2x(x^2 - 1)}{n^2} + \frac{6x}{n^2} \).

Finally, we have following theorem on the operator \( V_n^{M,2}(f; x) \).

**THEOREM 7.** Let \( f \in \mathcal{C}[0, \infty) \) and \( f'''(x) \) exists at \( x \in [0, b] \), \( b < \infty \). Then,
\[
\lim_{n \to \infty} n^2 (V_n^{M,2}(f; x) - f(x)) = x(1 + x) \left( \frac{f''(x)}{1 + x} + \frac{x - 1}{3} f'''(x) \right).
\]
Moreover, the results holds uniformly if \( f'''(x) \) is continuous on \([0, b]\).

**Proof.** By linearity of \( V_n^{M,2}(f; x) \) and smoothness of \( f \)
\[
V_n^{M,2}(f; x) = f(x) + f'(x)V_n^{M,2}((t - x); x) + \frac{1}{2} f''(x)V_n^{M,2}((t - x)^2; x) + \frac{1}{6} f'''(x)V_n^{M,2}((t - x)^3; x) + V_n^{M}(\varepsilon(t, x)(t - x)^3; x),
\]
where \( \varepsilon(t, x) \) is bounded on \([0, b]\) and \( \lim_{x \to x} \varepsilon(t, x) = 0 \). By the lemma 5 we have
\[
V_n^{M,2}(f; x) = f(x) + f''(x) \left( \frac{x}{n^2} \right) + \frac{1}{3} f'''(x) \left( \frac{x(x^2 - 1)}{n^2} + \frac{3x}{n^2} \right) + V_n^{M,2}(\varepsilon(t)(t - x)^3; x).
\]
The proof now follows along the lines similar to Theorem 4. \( \square \)

**THEOREM 8.** We have
\[
|V_n^{M,2}(f; x) - f(x)| \leq \phi(\varphi, n)(x + 2) \omega \left( \frac{1}{\sqrt{n}} \right)
\]
uniformly on \([0, b], b < \infty\), where \( \phi(\varphi, n) = |S(\varphi(x), n)| + |R(\varphi, n)| + |S(1 - \varphi(x), n)| \).
Proof. By using the values $2s_0 + s_1 + s_2 = 1$, $2s_2 - r_0 = 0$

$$|V_n^{M,2}(f; x) - f(x)|$$

$$= |V_n^{M,2}(f; x) - (2s_0 + s_1 + s_2)f(x)|$$

$$= \left| \sum_{k=0}^{\infty} P_{n,k}^{M,2}(x)f\left(\frac{k}{n}\right) - (2s_0 + s_1 + s_2)f(x) \right|$$

$$\leq \left| \sum_{k=0}^{\infty} (S(\phi(x), n)P_{n,k-2}(x) + R(\phi, n)P_{n-1,k-1}(x) + S(1 - \phi(x), n)P_{n-2,k}(x))f\left(\frac{k}{n}\right) \right. - (S(\phi(x), n) + R(\phi, n) + S(1 - \phi(x), n))f(x)$$

$$\leq \left| \sum_{k=0}^{\infty} S(\phi(x), n)P_{n,k-2}(x)\left(f\left(\frac{k}{n}\right) - f(x)\right) \right|$$

$$+ \left| \sum_{k=0}^{\infty} R(\phi, n)P_{n-1,k-1}(x)\left(f\left(\frac{k}{n}\right) - f(x)\right) \right|$$

$$+ \left| \sum_{k=0}^{\infty} S(1 - \phi(x), n)P_{n-2,k}(x)\left(f\left(\frac{k}{n}\right) - f(x)\right) \right| .$$

Now, by the inequality

$$f\left(\frac{k}{n}\right) - f(x) \leq \omega\left(\frac{k}{n} - x\right)$$

we have that

$$\omega\left|\frac{k}{n} - x\right| = \omega\left(\sqrt{n}\left|\frac{k}{n} - x\right| \frac{1}{\sqrt{n}}\right) \leq \left(1 + \sqrt{n}\left|\frac{k}{n} - x\right|\right) \omega\left(\frac{1}{\sqrt{n}}\right) .$$

Hence,

$$|V_n^{M,2}(f; x) - f(x)| \leq \sum_{k=0}^{\infty} |S(\phi(x), n)P_{n,k-2}(x)| \left(1 + \sqrt{n}\left|\frac{k}{n} - x\right|\right) \omega\left(\frac{1}{\sqrt{n}}\right)$$

$$+ \sum_{k=0}^{\infty} |R(\phi, n)P_{n-1,k-1}(x)| \left(1 + \sqrt{n}\left|\frac{k}{n} - x\right|\right) \omega\left(\frac{1}{\sqrt{n}}\right)$$

$$+ \sum_{k=0}^{\infty} |S(1 - \phi(x), n)P_{n-2,k}(x)| \left(1 + \sqrt{n}\left|\frac{k}{n} - x\right|\right) \omega\left(\frac{1}{\sqrt{n}}\right) .$$
\[
\leq \omega \left( \frac{1}{\sqrt{n}} \right) \left[ |S(\phi(x),n)| \left( 1 + \sqrt{n} \sum_{k=0}^{\infty} P_{n,k-2}(x) \left| \frac{k}{n} - x \right| \right) \right] \\
+ \left[ |R(\phi,n)| \left( 1 + \sqrt{n} \sum_{k=0}^{\infty} P_{n-1,k-1}(x) \left| \frac{k}{n} - x \right| \right) \right] \\
+ \left[ |S(1-\phi(x),n)| \left( 1 + \sqrt{n} \sum_{k=0}^{\infty} P_{n-2,k}(x) \left| \frac{k}{n} - x \right| \right) \right].
\]

An application of Schwarz’s inequality yields

\[
\sum_{k=0}^{\infty} P_{n,k-2}(x) \left| \frac{k}{n} - x \right| \leq \left( \sum_{k=0}^{\infty} P_{n,k}(x) \left( \frac{k}{n} - x \right)^2 \right)^{\frac{1}{2}} \left( \sum_{k=0}^{\infty} P_{n,k}(x) \right)^{\frac{1}{2}} \\
= \left( x^2 - 2x \left( x + \frac{2}{n} \right) + x^2 \frac{2}{n} + \frac{5x}{n} + \frac{4}{n^2} \right)^{\frac{1}{2}} \\
= \sqrt{\frac{x(1+x)}{n} + \frac{4}{n^2}} \\
= \frac{(1+x)}{\sqrt{n}} \sqrt{\left( 1 - \frac{1}{1+x} \right) + \frac{4}{n} \left( \frac{1}{1+x} \right)^2} \\
\leq \frac{1+x}{\sqrt{n}}.
\]

Similarly,

\[
\sum_{k=0}^{\infty} P_{n-1,k-1}(x) \left| \frac{k}{n} - x \right| \leq \left( \sum_{k=0}^{\infty} P_{n-1,k-1}(x) \left( \frac{k}{n} - x \right)^2 \right)^{\frac{1}{2}} \left( \sum_{k=0}^{\infty} P_{n-1,k-1}(x) \right)^{\frac{1}{2}} \\
= \left( x^2 - 2x \left( x - \frac{x}{n} + \frac{1}{n} \right) + x^2 \frac{2}{n} - \frac{3x}{n} + \frac{3x}{n^2} + \frac{1}{n^2} \right)^{\frac{1}{2}} \\
= \sqrt{\frac{x(1+x)}{n} + \frac{2x}{n} - \frac{5x}{n^2} + \frac{1}{n^2}} \\
\leq \frac{1+x}{\sqrt{n}}
\]

and

\[
\sum P_{n-2,k}(x) \left| \frac{k}{n} - x \right| \leq \frac{(1+x)}{\sqrt{n}}.
\]
Finally, combining these results we obtain,

\[ |V^{M,2}_n(f;x) - f(x)| \leq (|S(\varphi(x),n)| + |R(\varphi,n)| + |S(1-\varphi(x),n)|) (1 + x) \omega \left( \frac{1}{\sqrt{n}} \right) \]

\[ \leq \phi(\varphi,n)(x+2) \omega \left( \frac{1}{\sqrt{n}} \right). \]

Hence the proof is completed. □

**Remark 4.** Although the degree of approximation in Theorem 8 is of order \( O \left( \frac{1}{\sqrt{n}} \right) \), it is significantly improved to \( O \left( \frac{1}{n^2} \right) \) for smoother functions as observed in Theorem 7.

### 5. Third order operator \( V^{M,3}_n(f;x) \)

For further improvement in the degree of approximation we define the third order modified Baskakov operator by

\[ V^{M,3}_n(f;x) = \sum_{k=0}^{\infty} P^{M,3}_{n,k}(x) f \left( \frac{k}{n} \right) \]  

(5.1)

where,

\[ P^{M,3}_{n,k}(x) = S(\varphi(x),n) P_{n,k-4}(x) + R(\varphi,n) P_{n-1,k-3}(x) + V(\varphi,n) P_{n-2,k-2}(x) + R(1-\varphi,n) P_{n-3,k-1}(x) + S(1-\varphi(x),n) P_{n-4,k}(x) \]  

(5.2)

and

\[ S(\varphi(x),n) = s_4 \varphi^4 + s_3 \varphi^3 + s_2 \varphi^2 + s_1 \varphi + s_0, \]

\[ R(\varphi,n) = r_4 \varphi^4 + r_3 \varphi^3 + r_2 \varphi^2 + r_1 \varphi + r_0, \]

\[ V(\varphi,n) = v_0(\varphi(1-\varphi)^2). \]

**Remark 5.** For the operator (5.1) we note that if we put \( s_4 = 1, s_3 = -4, s_2 = 6, s_1 = -4, s_0 = 1, r_4 = -4, r_3 = 12, r_2 = -12, r_1 = 4, r_0 = 0 \) and \( v_0 = 6 \) then it reduces in the classical Baskakov operator (2.1).

**Lemma 6.** For \( r = 0, 1, 2, 3, 4 \) we have

1. \( \sum_{k=0}^{\infty} P_{n+r-4,k-r}(x) = 1 \)
2. \( \sum_{k=0}^{\infty} P_{n+r-4,k-r} \left( \frac{k}{n} \right) = \frac{n}{n} + \frac{(n+r-4)x}{n} \)
3. \( \sum_{k=0}^{\infty} P_{n+r-4,k-r} \left( \frac{k}{n} \right)^2 = \frac{r^2}{n^2} + \frac{(2r+1)(n+r-4)x}{n^2} + \frac{(n+r-4)(n+r-3)x^2}{n^2} \)
4. \[ \sum_{k=0}^{\infty} P_{n+r-4,k-r}(x) \left( \frac{k}{n} \right)^3 = \frac{r^3}{n^3} + \frac{(3r^2 + 3r + 1)}{n^3} \frac{(n+r-4)x}{n^3} + 3(n+r-4)(n+r-3)(n+r-2)x^3, \]

Using the technique as for the operator \( V_n^{M.2}(f; x) \), we find following sequences

\[ s_4 = 1 + \frac{23n(1+x)}{12} + \frac{n^2(1+x)^2}{8}, \]
\[ s_3 = -4 - \frac{14n(1+x)}{3} - \frac{n^2(1+x)^2}{4}, \]
\[ s_2 = 6 + \frac{10n(1+x)}{3} + \frac{n^2(1+x)^2}{8}, \]
\[ s_1 = -4 - \frac{7n(1+x)}{12}, \]
\[ s_0 = 1, \]
\[ r_4 = -4 - \frac{23n(1+x)}{3} - \frac{n^2(1+x)^2}{2}, \]
\[ r_3 = 12 + 17n(1+x) + n^2(1+x)^2, \]
\[ r_2 = -12 - \frac{31n(1+x)}{3} - \frac{n^2(1+x)^2}{2}, \]
\[ r_1 = 4 + n(1+x), \]
\[ r_0 = 0, \]
\[ v_0 = 6 + \frac{23n(1+x)}{2} + \frac{3n^2(1+x)^2}{4}. \]

Using the values of sequences \( s_i, r_i \) where \( i = 0, 1, 2, 3, 4 \) and \( v_0 \) we find the moments of operator (5.1) in the next lemma.

**Lemma 7.** The operator \( V_n^{M.3}(f; x) \) verifies

1. \( V_n^{M.3}(e_0; x) = 1, \)
2. \( V_n^{M.3}(e_1; x) = x, \)
3. \( V_n^{M.3}(e_2; x) = x^2, \)
4. \( V_n^{M.3}(e_3; x) = x^3, \)
5. \( V_n^{M.3}(e_4; x) = x^4 + \frac{12x(1+x)^3}{n^3}. \)

Finally we have that

**Corollary 2.**

\( V_n^{M.3}((e_1 - x)^j; x) = 0, \ j = 0, 1, 2, 3 \)
and
\[ V_n^{M,3}((t-x)^4;x) = \frac{12x(1+x)^3}{n^3} = O\left(\frac{1}{n^3}\right), \quad x \in [0,\infty). \]

**THEOREM 9.** Let \( f \in \overline{C}[0,\infty) \). Then the operator (5.1) together with the values of sequences \( v_0, s_i, r_i, i = 0, \ldots, 4 \) converges uniformly to \( f(x) \).

**THEOREM 10.** Let \( f \) be a function such that \( f^{(iv)}(x) \in \overline{C}[0,\infty) \). Then,
\[
\lim_{n \to \infty} n^3 \left( V_n^{M,3}(f;x) - f(x) \right) = \frac{1}{2} x(x+1)^3 f^{(iv)}(x).
\]

**REMARK 6.** We observe that \( s_i(n), r_i(n) \) and \( v_0(n) \) are the sequences of functions i.e. depend on \( x \) while in the case of Bernstein operators (see [20]), the unknown sequences are numerical sequences only. The degree of approximation by the operators \( V_n^{M,3}(f;x) \) is \( O(n^{-3}) \) for sufficiently smooth functions.

## 6. Numerical verification

Having been establishing the orders \( O(n^{-2}) \) and \( O(n^{-3}) \) we apply the operators \( V_n^{M,i}(f;x), \; k = 2, 3 \) to the various smooth and non-smooth functions. First, we choose \( \exp(-x/10), \; 0 \leq x \leq 30 \) for approximation by \( V_n^{M,2}(f;x) \) for the degrees \( n = 5 \) and \( n = 10 \).

![Figure 1: Comparison of \( V_n^{M,2}(f;x) \) and \( V_n^{M,3}(f;x) \) for \( \exp(-x/10) \)](image)

The large error are obtained near the end \( x = 30 \) which is due to the small values of \( n \) and truncation of the series for \( V_n^{M,2}(f;x) \) at \( k = 400 \). For approximation in large
interval we need to enhance both the degree of polynomial as well as degree of partial sums in $V^M_n(f;x)$. The tables 1, 2 provide the absolute errors, values of $f(x)$ and the polynomials $V^M_n(f;x)$, $n = 5, 10$.

Table 1: Comparison of $V^M_5(f;x)$ and $V^M_{10}(f;x)$ with respect to the function $f(x) = \exp(-x/10)$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$V^M_5(f;x)$</th>
<th>$f(x) - V^M_5(f;x)$</th>
<th>$V^M_{10}(f;x)$</th>
<th>$f(x) - V^M_{10}(f;x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.740818</td>
<td>0.741429</td>
<td>0.00061227</td>
<td>0.74097</td>
<td>0.000151318</td>
</tr>
<tr>
<td>6</td>
<td>0.548812</td>
<td>0.548313</td>
<td>0.000498911</td>
<td>0.548662</td>
<td>0.00014917</td>
</tr>
<tr>
<td>9</td>
<td>0.40657</td>
<td>0.403135</td>
<td>0.00343497</td>
<td>0.405634</td>
<td>0.000935989</td>
</tr>
<tr>
<td>12</td>
<td>0.301194</td>
<td>0.293361</td>
<td>0.00783306</td>
<td>0.299085</td>
<td>0.0021096</td>
</tr>
<tr>
<td>15</td>
<td>0.22313</td>
<td>0.209947</td>
<td>0.0131828</td>
<td>0.219608</td>
<td>0.00352233</td>
</tr>
<tr>
<td>18</td>
<td>0.165299</td>
<td>0.14631</td>
<td>0.01898883</td>
<td>0.160272</td>
<td>0.00502694</td>
</tr>
<tr>
<td>21</td>
<td>0.122456</td>
<td>0.0976175</td>
<td>0.02483893</td>
<td>0.115956</td>
<td>0.00650006</td>
</tr>
<tr>
<td>24</td>
<td>0.090718</td>
<td>0.0602921</td>
<td>0.03042584</td>
<td>0.0828675</td>
<td>0.00785048</td>
</tr>
<tr>
<td>27</td>
<td>0.0672055</td>
<td>0.0316653</td>
<td>0.03554032</td>
<td>0.0581868</td>
<td>0.00901875</td>
</tr>
<tr>
<td>30</td>
<td>0.0497871</td>
<td>0.00973072</td>
<td>0.04005643</td>
<td>0.0398142</td>
<td>0.00997283</td>
</tr>
</tbody>
</table>

The maximum absolute errors by $V^M_5(f;x)$ and $V^M_{10}(f;x)$ are 0.0400564 and 0.0097283 at $x = 30$, while the errors by $V^M_5(f;x)$ and $V^M_{10}(f;x)$ are 0.051334 and 0.00584135 obtained at $x = 30$ again. It should be noted that there is slight difference between the values of error shown in table and the figure near end point because in software calculations some values are too small to represent as a normalized machine number so there is loss of precision near end point.

Table 2: Comparison of $V^M_5(f;x)$ and $V^M_{10}(f;x)$ with respect to the function $f(x) = \exp(-x/10)$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$V^M_5(f;x)$</th>
<th>$f(x) - V^M_5(f;x)$</th>
<th>$V^M_{10}(f;x)$</th>
<th>$f(x) - V^M_{10}(f;x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.740818</td>
<td>0.740826</td>
<td>0.000057533</td>
<td>0.740826</td>
<td>0.00000734</td>
</tr>
<tr>
<td>6</td>
<td>0.548812</td>
<td>0.548876</td>
<td>0.000501069</td>
<td>0.548876</td>
<td>0.000063982</td>
</tr>
<tr>
<td>9</td>
<td>0.40657</td>
<td>0.406798</td>
<td>0.00179559</td>
<td>0.406798</td>
<td>0.000228654</td>
</tr>
<tr>
<td>12</td>
<td>0.301194</td>
<td>0.301743</td>
<td>0.00838108</td>
<td>0.301743</td>
<td>0.00104858</td>
</tr>
<tr>
<td>15</td>
<td>0.22313</td>
<td>0.224179</td>
<td>0.0140365</td>
<td>0.224179</td>
<td>0.00173079</td>
</tr>
<tr>
<td>18</td>
<td>0.165299</td>
<td>0.167033</td>
<td>0.0212904</td>
<td>0.167033</td>
<td>0.00259683</td>
</tr>
<tr>
<td>21</td>
<td>0.122456</td>
<td>0.125053</td>
<td>0.0212904</td>
<td>0.125053</td>
<td>0.00259683</td>
</tr>
<tr>
<td>24</td>
<td>0.090718</td>
<td>0.0943438</td>
<td>0.0300393</td>
<td>0.0943438</td>
<td>0.00362582</td>
</tr>
<tr>
<td>27</td>
<td>0.0672055</td>
<td>0.0718925</td>
<td>0.0401196</td>
<td>0.0718925</td>
<td>0.00468703</td>
</tr>
<tr>
<td>30</td>
<td>0.0497871</td>
<td>0.0553377</td>
<td>0.051334</td>
<td>0.0553377</td>
<td>0.0055506</td>
</tr>
</tbody>
</table>
Our next example discusses convergence of $V_{n}^{M,2}(f;x)$ and $V_{n}^{M,3}(f;x)$ for $n = 15$ to the function

$$f(x) = \frac{x}{x^2 + 1}. \quad (6.1)$$

It is clear that $f \in C[0,\infty)$. Table 3 provides a comparison of the error bounds by $V_{15}^{M,2}(f;x)$ and $V_{15}^{M,3}(f;x)$ at different points.

Fig. 3 verifies that the absolute error $|f(x) - V_{n}^{M,i}(f;x)|$ by third order operator is less than the corresponding error $|f(x) - V_{n}^{M,2}(f;x)|$ by the second order operator $V_{n}^{M,2}(f;x)$. 

Figure 2: Approximation of a smooth functions by $V_{n}^{M,i}(f;x)$

Figure 3: Comparison of absolute errors by $|f(x) - V_{n}^{M,i}(f;x)|$ for $i = 2, i = 3$. 
Table 3: Comparison of $V^{M,2}_{15}(f;x)$ and $V^{M,3}_{15}(f;x)$ with respect to the function 6.1.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$i = 2$</th>
<th>$i = 3$</th>
<th>$i = 2$</th>
<th>$i = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.501087</td>
<td>0.499991</td>
<td>0.00108733</td>
<td>0.000008679</td>
</tr>
<tr>
<td>2</td>
<td>0.4</td>
<td>0.408345</td>
<td>0.392403</td>
<td>0.00834536</td>
<td>0.00759716</td>
</tr>
<tr>
<td>3</td>
<td>0.3</td>
<td>0.303171</td>
<td>0.292277</td>
<td>0.00317085</td>
<td>0.00772254</td>
</tr>
<tr>
<td>4</td>
<td>0.235294</td>
<td>0.234811</td>
<td>0.230935</td>
<td>0.000483567</td>
<td>0.00435922</td>
</tr>
<tr>
<td>5</td>
<td>0.192308</td>
<td>0.190146</td>
<td>0.190814</td>
<td>0.00216195</td>
<td>0.00149363</td>
</tr>
<tr>
<td>6</td>
<td>0.162162</td>
<td>0.159344</td>
<td>0.162493</td>
<td>0.00281795</td>
<td>0.00330981</td>
</tr>
<tr>
<td>7</td>
<td>0.14</td>
<td>0.136995</td>
<td>0.141383</td>
<td>0.00300469</td>
<td>0.00138254</td>
</tr>
<tr>
<td>8</td>
<td>0.123077</td>
<td>0.120094</td>
<td>0.125029</td>
<td>0.0029827</td>
<td>0.00195226</td>
</tr>
<tr>
<td>9</td>
<td>0.109756</td>
<td>0.106885</td>
<td>0.111993</td>
<td>0.0028715</td>
<td>0.00223698</td>
</tr>
<tr>
<td>10</td>
<td>0.0990099</td>
<td>0.0962838</td>
<td>0.101366</td>
<td>0.00272612</td>
<td>0.00235607</td>
</tr>
</tbody>
</table>

Next, we discuss convergence of $V^{M,3}_{n}(f;x)$ for a non-differentiable function $f(x)$ defined by

$$f(x) = \begin{cases} 
  x^2, & \text{if } 0 \leq x < 2 \\
  4, & \text{if } 2 \leq x < 6, \\
  (x-8)^2, & \text{if } 6 \leq x \leq 8, \\
  0, & \text{if } x > 8.
\end{cases} \quad (6.2)$$

Figure 4: Approximation of Discontinuous functions by $V^{M,3}_{n}(f;x)$
Table 4 discusses the comparison of values of the function with respect to the value by $V_n^{M,3}(f;x)$ for $n = 20$ and $n = 40$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$n = 20$</th>
<th>$n = 40$</th>
<th>$n = 20$</th>
<th>$n = 40$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>0.994772</td>
<td>0.999724</td>
<td>0.00522774</td>
<td>0.00027647</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>3.70739</td>
<td>3.78039</td>
<td>0.292613</td>
<td>0.219613</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>4.09573</td>
<td>3.97765</td>
<td>0.0957321</td>
<td>0.0223536</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>3.91882</td>
<td>3.98593</td>
<td>0.0395879</td>
<td>0.0534708</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>3.96041</td>
<td>4.05347</td>
<td>0.0395879</td>
<td>0.0534708</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>3.29052</td>
<td>3.4432</td>
<td>0.709483</td>
<td>0.556804</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>1.72108</td>
<td>1.39419</td>
<td>0.721076</td>
<td>0.39419</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>0.263065</td>
<td>-0.0637303</td>
<td>0.263065</td>
<td>0.0637303</td>
</tr>
</tbody>
</table>

Conclusions

It is shown that the absolute error, $|f(x) - V_n^{M,i}(f;x)|$ for a sufficiently smooth function $f$ is of order $n^{-i}$. It is worth pointing out that in order to achieve higher rate of approximation, it is better to use a suitable modification $V_n^{M,2}(f;x)$ rather than increasing the degree of ordinary operator $V_n(f;x)$. It is observed that our method is applicable to approximate non-smooth functions too. However, the choice of the degree of polynomials $V_n^{M,i}(f;x)$ depends on the length of the interval taken for the approximation. The approximation of unbounded functions on $[0, \infty)$ with certain growth conditions i.e. functions $f$ satisfying the inequalities $|f(x)| \leq C\phi(x)$ for given well-behaved functions $\phi(x)$ can be interesting and significant topic for future investigations. For weighted approximation and numerical quadrature such functions are quite suitable so that the interested readers can consider this as open area of study.

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Asha Ram Gairola  
Department of Mathematics  
Doon University  
Dehradun-248001 (Uttarakhand), India  
e-mail: ashagairola@gmail.com

Amrita Singh  
Department of Mathematics  
Doon University  
Dehradun-248001 (Uttarakhand), India  
e-mail: ami449628@gmail.com

Laxmi Rathour  
Ward Number – 16, Bhagatbandh, Anuppur 484 224  
Madhya Pradesh, India  
e-mail: laxmirathour817@gmail.com

Vishnu Narayan Mishra  
Department of Mathematics  
Indira Gandhi National Tribal University  
Lalpur, Amarkantak-484 887, M.P., India  
e-mail: vishnunarayann Mishra@gmail.com  
vnm@igntu.ac.in