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ATOMIC LATTICES OF SUBSPACES OF AN ARBITRARY VECTOR SPACE AND ASSOCIATED OPERATOR ALGEBRAS

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Abstract. We study a class of completely distributive, commutative, lattices of subspaces of an arbitrary vector space, and associated operator algebras. Our results are compared with corresponding results for commutative lattices of closed subspaces of a Hilbert space and associated algebras of bounded linear operators.

1. Introduction

Reflexive algebras of operators, i.e., algebras of operators that leave invariant given lattices of subspaces, have long had an important role in the study of non-self-adjoint operators and operator algebras acting on Hilbert space. Of particular interest are nest algebras and CSL algebras. Nest algebras are algebras of operators that leave invariant totally ordered lattices of subspaces. See, for example [2]. In [4] nest algebras of operators acting on an arbitrary vector space are examined. In [1] Arveson introduced and studied in depth CSL algebras, i.e., algebras of operators that leave invariant commutative lattices of subspaces. Here certain types of lattices of subspaces of an arbitrary vector space that are closely related to a given Hamel basis are defined. These lattices are completely distributive and commutative. We study the corresponding reflexive algebras of operators and compare them to their counterparts acting on Hilbert space.

1.1. Atomic subspace lattices

Suppose that $\mathfrak{X}$ is a vector space over a field $\mathbb{F}$. Let $\mathcal{L}(\mathfrak{X})$ denote the algebra of all *operators* on $\mathfrak{X}$, and let $\mathcal{S}(\mathfrak{X})$ denote the lattice of all *subspaces* of $\mathfrak{X}$. Here operators are linear functions that map $\mathfrak{X}$ into itself, and subspaces are linear manifolds. In this context continuity is not an issue, because we are not imposing a topology on $\mathfrak{X}$ or the field $\mathbb{F}$. The lattice operations in $\mathcal{S}(\mathfrak{X})$ are intersection and linear span, i.e., if $\{M_\alpha : \alpha \in \Omega\}$ is a subset of $\mathcal{S}(\mathfrak{X})$, then $\bigwedge_{\alpha\in\Omega} M_\alpha = \bigcap_{\alpha\in\Omega} M_\alpha$ and $\bigvee_{\alpha\in\Omega} M_\alpha = \operatorname{span}(\bigcup_{\alpha\in\Omega} M_\alpha)$. A subspace lattice is a complete sublattice (i.e., closed under intersections and spans of unions) of $\mathcal{S}(\mathfrak{X})$ that includes the trivial subspaces $\{0\}$ and $\mathfrak{X}$.

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Suppose that $\mathfrak{B}$ is a Hamel basis of $\mathfrak{X}$. Suppose also that $\preceq$ is a quasi-order, i.e., a reflexive and transitive relation, on $\mathfrak{B}$. An *initial segment* of $\mathfrak{B}$ (with respect to the quasi-order $\preceq$) is a subset Φ of $\mathfrak{B}$ with the property that $e \preceq f$ and $f \in \Phi \Longrightarrow e \in \Phi$. Let $\mathfrak{B}(\preceq)$ denote the set of all initial segments of $\mathfrak{B}$. The set $\mathfrak{B}(\preceq)$ is a complete, completely distributive, sublattice of $2^{\mathfrak{B}}$, the lattice of all subsets of $\mathfrak{B}$. For each subset Φ of $\mathfrak{B}$, let $\mathfrak{X}_\Phi = \operatorname{span}\{e : e \in \Phi\}$ and let $\mathfrak{X}_\varnothing = \{0\}$.

DEFINITION 1. $\mathfrak{L}(\preceq) = \{\mathfrak{X}_\Phi : \Phi \in \Omega(\preceq)\}$

Clearly $\mathfrak{L}(\preceq)$ is a subspace lattice that is lattice isomorphic to $\mathfrak{B}(\preceq)$. It is also easy to see that if $\mathcal{M}$ is any non-zero subspace in $\mathfrak{L}(\preceq)$ then $\mathfrak{B} \cap \mathcal{M}$ is a basis of $\mathcal{M}$.

DEFINITION 2. A subspace lattice $\mathfrak{L}$ is *atomic* if there is a Hamel basis $\mathfrak{B}$ of $\mathfrak{X}$ with the property that for each non-zero subspace $\mathcal{M}$ in $\mathfrak{L}$, $\mathfrak{B} \cap \mathcal{M}$ is a basis of $\mathcal{M}$.

Such a basis $\mathfrak{B}$ is called a *generating basis.*

The subspace lattice $\mathfrak{L}(\preceq)$ is atomic. We shall show that any atomic subspace lattice is of the form $\mathfrak{L}(\preceq)$ for some quasi-order $\preceq$ of the basis.

Suppose that $\mathfrak{L}$ is an atomic subspace lattice with generating basis $\mathfrak{B}$. For each vector $x \in \mathfrak{X}$ let $\mathfrak{L}(x) = \bigcap\{\mathcal{M} \in \mathfrak{L} \text{ and } x \in \mathcal{M}\}$. For any two basis vectors e and f write

$$e \preceq f \Longleftrightarrow \mathfrak{L}(e) \subseteq \mathfrak{L}(f).$$

It is clear that $\preceq$ is a quasi-order on $\mathfrak{B}$. We need to show that $\mathfrak{L} = \mathfrak{L}(\preceq)$. First observe that $e \preceq f \Longleftrightarrow e \in \mathfrak{L}(f)$. So for each $f \in \mathfrak{B}$, $\mathfrak{B} \cap \mathfrak{L}(f)$ is an initial segment of $\mathfrak{B}$, and $\mathfrak{L}(f) \in \mathfrak{L}(\preceq)$. Since $\mathcal{M} = \vee\{\mathfrak{L}(f) : f \in \mathfrak{B} \cap \mathcal{M}\}$. for each non-zero subspace in $\mathfrak{L}$, it follows that $\mathfrak{L} \subseteq \mathfrak{L}(\preceq)$.

For the reverse inclusion, observe that $\mathcal{M} = \vee\{\mathfrak{L}(e) : e \in \mathcal{M}\}$ for each non-zero subspace $\mathcal{M} \in \mathfrak{L}(\preceq)$. Since $\mathfrak{L}(e) \in \mathfrak{L}$ for each $e \in \mathfrak{B}$ and since $\mathfrak{L}$ is join-closed, $\mathcal{M} \in \mathfrak{L}$. So $\mathfrak{L}(\preceq) \subseteq \mathfrak{L}$.

For the remainder of this section $\mathfrak{L}(\preceq)$ is the atomic subspace lattice corresponding to the basis $\mathfrak{B}$ with quasi-order $\preceq$. For $e, f \in \mathfrak{B}$ write $e \approx f$ if $\mathfrak{L}(e) = \mathfrak{L}(f)$, i.e., $e \preceq f$ and $f \preceq e$, and write $e \prec f$ if $\mathfrak{L}(e) \subset \mathfrak{L}(f)$, where $\subset$ denotes proper inclusion, i.e., $\mathfrak{L}(e) \subseteq \mathfrak{L}(f)$ and $\mathfrak{L}(e) \neq \mathfrak{L}(f)$. Let

$$[\approx e] = \{f \in \mathfrak{B} : f \approx e\} \qquad \text{and let} \qquad [\prec e] = \{f \in \mathfrak{B} : f \prec e\}.$$

Clearly $[\approx e] \cup [\prec e] = [\preceq e]$ and $[\approx e] \cap [\prec e] = \varnothing$.

Let $\mathfrak{L}(e)_- = \operatorname{span}[\prec e]$, for each $e \in \mathfrak{B}$. Clearly $\mathfrak{L}(e)_-$ is the unique immediate predecessor of $\mathfrak{L}(e)$ in $\mathfrak{L}$, and $\mathfrak{L}(e)_- = \bigvee\{\mathcal{M} \in \mathfrak{L} : \mathcal{M} \subset \mathfrak{L}(e)\}$.

The subspaces of $\mathfrak{X}$ of the form $\operatorname{span}[\approx e]$ are called the *atoms* of $\mathfrak{L}(\preceq)$. The atoms of $\mathfrak{L}(\preceq)$ span $\mathfrak{X}$, and each subspace in $\mathfrak{L}(\preceq)$ is a join of atoms. The atoms of $\mathfrak{L}(\preceq)$ are not necessarily subspaces in $\mathfrak{L}(\preceq)$. In fact $\operatorname{span}[\approx e] \in \mathfrak{L}(\preceq)$ if and only if $\mathfrak{L}(e)_- = \{0\}$, and $\mathfrak{L}(\preceq)$ contains each of its atoms if and only if $\mathfrak{L}(\preceq)$ is complemented, i.e., a Boolean algebra.

For each subset Φ of $\mathfrak{B}$ let P_Φ denote the unique operator in $\mathcal{L}(\mathfrak{X})$ that satisfies

$$P_\Phi x = \begin{cases} x & \text{if } x \in \mathfrak{X}_\Phi \\ 0 & \text{if } x \in \mathfrak{X}_{\mathfrak{B}\setminus\Phi} \end{cases}$$

Thus P_Φ is a projection with range $\mathfrak{X}_\Phi$ and kernel $\mathfrak{X}_{\mathfrak{B}\setminus\Phi}$. The projections $P_\Phi : \Phi \subseteq \mathfrak{B}$ are a commuting family satisfying

$$P_\Phi P_\Psi = P_\Psi P_\Phi = P_{\Phi\cap\Psi},\ P_\Phi + P_\Psi = P_{\Phi\cap\Psi} + P_{\Phi\cup\Psi},\ P_\Phi = 0 \text{ and } P_{\mathfrak{B}} = I$$

for any $\Phi, \Psi \subseteq \mathfrak{B}$.

It is convenient to identify $\mathfrak{L}(\preceq)$ with its *projection lattice*, i.e., the projections $P_\Phi : \Phi \in \mathfrak{B}$. We say that $\mathfrak{L}(\preceq)$ is commutative because its projection lattice is commutative. Thus the atomic subspace lattice $\mathfrak{L}(\preceq)$ is commutative, completely distributive and isomorphic to $\mathfrak{B}(\preceq)$.

Following Ringrose [8] we say that $\mathfrak{L}(\preceq)$ is *simple* if its atoms are 1-dimensional, i.e., $[\approx e] = \{e\}$ for all $e \in \mathfrak{B}$.

EXAMPLE 1. If $\preceq$ is a total order on the equivalence classes of $\mathfrak{B}$, i.e., $e \preceq f$ or $f \preceq e$ for all $e, f \in \mathfrak{B}$, then $\mathfrak{L}(\preceq)$ is totally ordered and is called a *nest*.

EXAMPLE 2. If $\preceq$ is an inorder on the equivalence classes of $\mathfrak{B}$, i.e., $e \preceq f$ if and only if $f \preceq e$, then $\mathfrak{L}(\preceq)$ is a Boolean algebra of subspaces of $\mathfrak{X}$.

It is useful to introduce a simple topology on $\mathcal{L}(\mathfrak{X})$.

DEFINITION 3. The strict operator topology is the topology on $\mathcal{L}(\mathfrak{X})$ whose subbasic open sets are subsets of the form

$$\mathcal{U}(T,x) = \{S \in \mathcal{L}(\mathcal{X}) : Sx = Tx\},$$

where $T \in \mathcal{L}(\mathfrak{X})$ and $x \in \mathfrak{X}$. It is easy to verify that addition and multiplication are both jointly continuous operations in the strict topology on the algebra $\mathcal{L}(\mathfrak{X})$

LEMMA 1. $\mathfrak{L}(\preceq)$ *is strictly closed.*

Proof. Suppose that T is in the strict closure of $\{P_\Phi : \Phi \in \mathfrak{B}(\preceq)\}$, and that $e \in \mathfrak{B}$. Then $Te = P_\Phi e$ for some $\Phi \in \mathfrak{B}(\preceq)$. So $Te = e$ or $Te = 0$. Let $\Psi = \{f \in \mathfrak{B} : Tf = f\}$. Then clearly $T = P_\Psi$. It remains to be shown that $\Psi \in \mathfrak{B}(\preceq)$. So suppose that $f \in \Psi$ and $e \preceq f$. Then $Te = P_\Phi e$ and $Tf = P_\Phi f$ for some $\Phi \in \mathfrak{B}(\preceq)$. Now $Tf = f$ and so $f \in \Phi$, hence $e \in \Phi$. So $Te = P_\Phi e = e$. Hence $e \in \Psi$, and so $\Psi \in \mathfrak{B}(\preceq)$, as required. $\square$

1.2. The algebra $\mathfrak{A}(\preceq)$

Suppose that $T \in \mathcal{L}(\mathfrak{X})$. The *representing matrix* for T relative to the basis $\mathfrak{B}$ is the array $(T_{e,f})_{(e,f)\in\mathfrak{B}\times\mathfrak{B}}$, where each $T_{e,f}$ is a scalar, i.e., an element of $\mathbb{F}$, and

$$Tf = \sum_{e\in\mathfrak{B}} T_{e,f}e \qquad \text{for each} \qquad f \in \mathfrak{B}.$$

The array $(T_{e,f})$ is column-finite, in the sense that for each $f \in \mathfrak{B}$, $T_{e,f} = 0$ for all but finitely many e. Conversely, any column-finite array of scalars is the representing matrix of an operator in $\mathcal{L}(\mathfrak{X})$.

DEFINITION 4. The support, $\operatorname{supp} x$, of any vector $x \in \mathfrak{X}$ is defined by

$$\operatorname{supp} x = \{e \in \mathfrak{B} : P_e x \neq 0\}$$

where $P_e = P_{\{e\}}$. The support, $\operatorname{supp} T$, of any operator $T \in \mathcal{L}(\mathfrak{X})$ is defined by

$$\operatorname{supp} T = \{(e,f) \in \mathfrak{B}\times\mathfrak{B} : T_{e,f} \neq 0\}.$$

For each $\Gamma \subseteq B\times B$, the *incidence space* $\mathcal{L}(\mathfrak{X},\Gamma)$ is defined by

$$\mathcal{L}(\mathfrak{X},\Gamma) = \{T \in \mathcal{L}(\mathfrak{X}) : \operatorname{supp} T \subseteq \Gamma\}$$

The incidence space $\mathcal{L}(\mathfrak{X},\Gamma)$ is the linear subspace of $\mathcal{L}(\mathfrak{X})$ consisting of all operators in $\mathcal{L}(\mathfrak{X})$ that are supported on Γ.

The *graph* $\mathfrak{G}(\preceq)$ of the quasi-order $\preceq$ is defined by

$$\mathfrak{G}(\preceq) = \{(e,f) \in \mathfrak{B}\times\mathfrak{B} : e \preceq f\}.$$

DEFINITION 5. $\mathfrak{A}(\preceq) = \mathcal{L}(\mathfrak{X},\mathfrak{G}(\preceq))$.

Clearly $\mathfrak{A}(\preceq)$ is closed under addition and multiplication by scalars. Since $\preceq$ is reflexive and transitive, $\mathfrak{A}(\preceq)$ contains the identity operator I and is closed under multiplication. It is also easy to show that $\mathfrak{A}(\preceq)$ is strictly closed. So $\mathfrak{A}(\preceq)$ is a strictly closed subalgebra of $\mathcal{L}(\mathfrak{X})$ that contains I

In the latter parts of this paper we examine properties of the algebra $\mathfrak{A}(\preceq)$ and compare these to the properties of CSL (commutative subspace lattice) algebras in a Hilbert space context. First we establish an important relationship between $\mathfrak{A}(\preceq)$ and $\mathfrak{L}(\preceq)$.

1.3. Reflexivity

For any $\mathcal{L} \subseteq \mathcal{S}(\mathfrak{X})$ and any $\mathcal{F} \subseteq \mathcal{L}(\mathfrak{X})$ we define

$$\begin{aligned}
\operatorname{Alg}\mathcal{L} &= \{T \in \mathcal{L}(\mathcal{X}) : T(\mathcal{M}) \subseteq \mathcal{M} \text{ for all } \mathcal{M} \in \mathcal{L}\}, \text{ and}\\
\operatorname{Lat}\mathcal{F} &= \{\mathcal{M} \in \mathcal{S}(\mathcal{X}) : T(\mathcal{M}) \subseteq \mathcal{M} \text{ for all } T \in \mathcal{F}\}.
\end{aligned}$$

That is, $\operatorname{Alg}\mathcal{L}$ is the set of operators on $\mathcal{X}$ that leave each subspace in $\mathcal{L}$ invariant, and $\operatorname{Lat}\mathcal{F}$ is the set of all subspaces of $\mathcal{X}$ that are invariant under each operator in $\mathcal{F}$.

We say a subset $\mathcal{L}$ of $\mathcal{S}(\mathcal{X})$ is *reflexive* if $\mathcal{L}=\operatorname{Lat}\operatorname{Alg}\mathcal{L}$. Since $\operatorname{Lat}\mathcal{F}=\operatorname{Lat}\operatorname{Alg}\operatorname{Lat}\mathcal{F}$ for any $\mathcal{F}\subseteq\mathcal{L}(\mathcal{X})$, $\mathcal{L}$ is reflexive if and only if $\mathcal{L}=\operatorname{Lat}\mathcal{F}$ for some $\mathcal{F}\subseteq\mathcal{L}(\mathcal{X})$. Similarly, a subset $\mathcal{F}$ of $\mathcal{L}(\mathcal{X})$ is *reflexive* if $\mathcal{F}=\operatorname{Alg}\operatorname{Lat}\mathcal{F}$. Since $\operatorname{Alg}\mathcal{L}=\operatorname{Alg}\operatorname{Lat}\operatorname{Alg}\mathcal{L}$ for any $\mathcal{L}\subseteq\mathcal{S}(\mathcal{X})$, $\mathcal{F}$ is reflexive if and only if $\mathcal{F}=\operatorname{Alg}\mathcal{L}$ for some $\mathcal{L}\subseteq\mathcal{S}(\mathcal{X})$.

A reflexive family of subspaces of $\mathcal{X}$ is necessarily of the form $\operatorname{Lat}\mathcal{F}$ for some family $\mathcal{F}$ of operators on $\mathcal{X}$, and is a subspace lattice, i.e., a complete sublattice of $\mathcal{S}(\mathcal{X})$ containing the trivial subspaces $\{0\}$ and $\mathfrak{X}$, and a reflexive family of operators is necessarily of the form $\operatorname{Alg}\mathcal{L}$ for some family $\mathcal{L}$ of subspaces of $\mathcal{X}$, and is a strictly closed subalgebra of $\mathcal{L}(\mathcal{X})$ containing the identity operator I.

We shall demonstrate the reflexivity of $\mathfrak{A}(\preceq)$ and $\mathfrak{L}(\preceq)$, but first we need some notational preliminaries. Let $\mathfrak{X}'$ denote the algebraic dual of $\mathfrak{X}$, i.e., the set of all scalar-valued functions defined on $\mathfrak{X}$. For each $x\in\mathfrak{X}$ and each $\varphi\in\mathfrak{X}'$, let $x\otimes\varphi$ denote the rank one operator on $\mathfrak{X}$ defined by $(x\otimes\varphi)(y)=\varphi(y)x$ for each $y\in\mathfrak{X}$. For each $f\in\mathfrak{B}$ let f' denote the element of $\mathfrak{X}'$ which has the properties

$$f'(f)=1 \qquad \text{and} \qquad f'(g)=0 \qquad \text{if} \qquad g\in\mathfrak{B}\setminus\{f\}.$$

Suppose that $e\in\mathfrak{B}$ and $f\in\mathfrak{B}$. It is easy to see that $\operatorname{supp}(e\otimes f')=\{(e,f)\}$ and $e\otimes f'\in\mathfrak{A}(\preceq)$ if and only if $e\preceq f$.

THEOREM 1. $\operatorname{Lat}\mathfrak{A}(\preceq)=\mathfrak{L}(\preceq)$ *and* $\operatorname{Alg}\mathfrak{L}(\preceq)=\mathfrak{A}(\preceq)$.

Proof. Clearly $\mathfrak{L}(\preceq)\subseteq\operatorname{Lat}\mathfrak{A}(\preceq)$. Suppose that $\mathcal{M}\in\operatorname{Lat}\mathfrak{A}(\preceq)$. Then $\mathcal{M}\subseteq\mathcal{X}_\Phi$, where $\Phi=\bigcup\{\operatorname{supp}x : x\in\mathcal{M}\}$. Since $f\otimes f'\in\mathfrak{A}(\preceq)$ for all $f\in\mathfrak{B}$, it follows that $f\in\mathcal{M}$ for all $f\in\Phi$. So $\mathcal{M}=\mathfrak{X}_\Phi$. It remains to be shown that $\Phi\in\mathfrak{B}(\preceq)$. So suppose that $e\preceq f$ and $f\in\Phi$. Then $f\in\mathcal{M}$, and $e\otimes f'\in\mathfrak{A}(\preceq)$, and so $e=(e\otimes f')f\in\mathcal{M}$. So Φ is an initial segment, and hence $\mathcal{M}\in\mathfrak{L}(\preceq)$. So $\operatorname{Lat}\mathfrak{A}(\preceq)=\mathfrak{L}(\preceq)$.

It is also clear that $\mathfrak{A}(\preceq)\subseteq\operatorname{Alg}\mathfrak{L}(\preceq)$. Suppose that $T\in\operatorname{Alg}\mathfrak{L}(\preceq)$ and that $(e,f)\in\operatorname{supp}T$. Then $P_eTP_fx\neq 0$ for some $x\in\mathcal{X}$. Now $P_fx\in\mathcal{X}_{[\preceq f]}\in\mathfrak{L}(\preceq)$, and so $TP_fx\in\mathcal{X}_{[\preceq f]}$. Since $P_eTP_fx\neq 0$ it follows that $e\in[\preceq f]$, i.e., $e\preceq f$. So $\operatorname{supp}T\subseteq\mathfrak{G}(\preceq)$, i.e., $T\in\mathfrak{A}(\preceq)$. So $\operatorname{Alg}\mathfrak{L}(\preceq)=\mathfrak{A}(\preceq)$. $\square$

The following corollary is a simple consequence of Theorem 1.

COROLLARY 1. $\mathfrak{A}(\preceq)$ *and* $\mathfrak{L}(\preceq)$ *are reflexive.*

2. The radical

Suppose that Ω is a subset of $\mathfrak{B}$ with the property that $|\Omega\cap[\approx e]|=1$ for each equivalence class $[\approx e]$. That is, the atoms $[\approx e]$, $e\in\Omega$ are all distinct, and $\bigcup\{[\approx e] : e\in\Omega\}=\mathfrak{B}\}$. For each T in $\mathcal{L}(\mathfrak{X})$ let

$$\delta(T)=\sum_{e\in\Omega}P_{[\approx e]}TP_{[\approx e]} \tag{2.1}$$

Note that $\delta^2 = \delta$ and $\delta(I) = I$, and that the sum in (2.1) converges in the strict topology.

LEMMA 2. *The map $T \to \delta(T)$ is a homomorphism of $\mathfrak{A}(\preceq)$ into $\mathfrak{A}(\preceq)$.*

Proof. Clearly the map δ is linear. Suppose that $T \in \mathfrak{A}(\preceq)$. For each $e \in \mathfrak{B}$, $P_{[\approx e]} \in \mathfrak{A}(\preceq)$, and so $\delta(T) \in \mathfrak{A}(\preceq)$ by (2.1).

Suppose also that $S \in \mathfrak{A}(\preceq)$. It is easy to see that

$$\delta(ST) = \sum_{e \in \Omega} P_{[\approx e]} ST P_{[\approx e]} \qquad \text{and} \qquad \delta(S)\delta(T) = \sum_{e \in \Omega} P_{[\approx e]} S P_{[\approx e]} T P_{[\approx e]}$$

So to show that the map δ is an algebra homomorphism, it suffices to show that $P_{[\approx e]} ST P_{[\approx e]} = P_{[\approx e]} S P_{[\approx e]} T P_{[\approx e]}$.

Note that $P_{[\preceq e]} = P_{[\approx e]} + P_{[\prec e]}$ and that $P_{[\approx e]} P_{[\prec e]} = 0$. Since $[\prec e]$ is an initial segment of Ω, $P_{[\approx e]} S P_{[\prec e]} = P_{[\approx e]} P_{[\prec e]} S P_{[\prec e]} = 0$, and so

$$P_{[\approx e]} S P_{[\approx e]} = P_{[\approx e]} S P_{[\preceq e]} - P_{[\approx e]} S P_{[\prec e]} = P_{[\approx e]} S P_{[\preceq e]}$$

Furthermore,

$$P_{[\preceq e]} T P_{[\approx e]} = P_{[\preceq e]} T P_{[\preceq e]} P_{[\approx e]} = T P_{[\preceq e]} P_{[\approx e]} = T P_{[\approx e]}$$

and so

$$P_{[\approx e]} S P_{[\approx e]} T P_{[\approx e]} = P_{[\approx e]} S P_{[\preceq e]} T P_{[\approx e]} = P_{[\approx e]} ST P_{[\approx e]}$$

as required. $\square$

DEFINITION 6. $\mathfrak{A}(\approx)$, the *diagonal subalgebra* of $\mathfrak{A}(\preceq)$, is defined by

$$\mathfrak{A}(\approx) = \mathcal{L}(\mathfrak{X}, \mathfrak{G}(\approx)), \qquad \text{where} \qquad \mathfrak{G}(\approx) = \{(e,f) \in \mathfrak{B} \times \mathfrak{B} : e \approx f\}$$

and $\mathfrak{A}(\prec)$, the *strictly triangular ideal* of $\mathfrak{A}(\preceq)$, is defined by

$$\mathfrak{A}(\prec) = \mathcal{L}(\mathfrak{X}, \mathfrak{G}(\prec)), \qquad \text{where} \qquad \mathfrak{G}(\prec) = \{(e,f) \in \mathfrak{B} \times \mathfrak{B} : e \prec f\}$$

Note that $\mathfrak{A}(\approx) = \operatorname{ran} \delta$ and that $\mathfrak{A}(\prec) = \ker \delta$, if the domain of δ is restricted to $\mathfrak{A}(\preceq)$. Note also since δ is a homomorphism on $\mathfrak{A}(\preceq)$, $\mathfrak{A}(\prec)$ is a two-sided ideal in $\mathfrak{A}(\preceq)$.

EXAMPLE 3. If $\preceq$ is an inorder, $\mathfrak{A}(\preceq) = \mathfrak{A}(\approx)$.

EXAMPLE 4. If $\mathfrak{B}$ is finite then the diagonal algebra is the set of all 'block diagonal' matrices. After a suitable ordering of the elements of $\mathfrak{B}$, $\mathfrak{A}(\preceq)$ is a subalgebra of the set of all 'block upper triangular' matrices, and $\mathfrak{A}(\prec)$ consists of all 'block strictly upper triangular' matrices in $\mathfrak{A}(\preceq)$. If $\preceq$ is a total order, $\mathfrak{A}(\preceq)$ is the set of all 'block upper triangular' matrices.

REMARK 1. As remarked earlier, if $\preceq$ is a total order $\mathfrak{L}(\preceq)$ is a nest. As in [4] we can define, for each non-zero $x \in \mathfrak{X}$,

$$\mathcal{M}(x) = \bigcap\{M : x \in M \text{ and } M \in \mathfrak{L}(\preceq)\}$$

and

$$\mathcal{M}(x)_- = \bigcup\{M : x \notin M \text{ and } M \in \mathfrak{L}(\preceq)\}.$$

Suppose that $T \in \mathcal{L}(\mathfrak{X})$. It is easy to check that $\operatorname{supp} T \subseteq \mathfrak{G}(\prec) \Longleftrightarrow T\mathcal{M}(x) \subseteq \mathcal{M}(x)_-$ for all non-zero $x \in \mathfrak{X}$. So the definition of the strictly triangular ideal given here agrees with that given in [4].

LEMMA 3. *The diagonal algebra $\mathfrak{A}(\approx)$ is abelian if and only if $\mathfrak{L}(\preceq)$ is simple.*

Proof. Suppose that $\mathfrak{L}(\preceq)$ is simple, i.e., $[\approx e] = \{e\}$ for each $e \in \mathfrak{B}$. So for each $T \in \mathfrak{A}(\preceq)$, there exists a set of scalars $\{t_e : e \in \mathfrak{B}\}$ such that $\delta(T) = \sum_{e\in\mathfrak{B}} t_e Pe$. It is easy to check that

$$\delta(S)\delta(T) = \sum_{e\in\mathfrak{B}} s_e t_e Pe = \sum_{e\in\mathfrak{B}} t_e s_e Pe = \delta(T)\delta(S)$$

If $\mathfrak{L}(\preceq)$ is not simple, $\dim \mathfrak{X}_{[\approx e]} > 1$ for some $e \in \mathfrak{B}$. Choose $x \in \mathfrak{X}_{[\approx e]}$ and $S, T \in \mathcal{L}(\mathfrak{X}_{[\approx e]})$ such that $STx \neq TSx$. Let $S' = P_{[\approx e]} S P_{[\approx e]}$ and $T' = P_{[\approx e]} T P_{[\approx e]}$. Then $S', T' \in \mathfrak{A}(\approx)$ and

$$\delta(S')\delta(T')x = P_{[\approx e]} S P_{[\approx e]} T P_{[\approx e]} x = STx \neq TSx = P_{[\approx e]} T P_{[\approx e]} S P_{[\approx e]} x = \delta(T')\delta(S')x.$$

So $\mathfrak{A}(\approx)$ is not abelian. $\square$

DEFINITION 7. The radical $\operatorname{Rad}\mathfrak{A}(\preceq)$ of $\mathfrak{A}(\preceq)$ is defined by:

$$\operatorname{Rad}\mathfrak{A}(\preceq) = \{T \in \mathfrak{A}(\preceq) : (I - AT) \text{ has an inverse in } \mathfrak{A}(\preceq) \text{ for all } A \in \mathfrak{A}(\preceq)\}$$

It is easy to check that $I + TBA = (I - TA)^{-1}$ if $B = (I - AT)^{-1}$. Similarly, $I + ACT = (I - AT)^{-1}$ if $C = (I - TA)^{-1}$. So $(I - AT)$ has an inverse in $\mathfrak{A}(\preceq)$ if and only if $(I - TA)$ has an inverse in $\mathfrak{A}(\preceq)$. So

$$\operatorname{Rad}\mathfrak{A}(\preceq) = \{T \in \mathfrak{A}(\preceq) : (I - TA) \text{ has an inverse in } \mathfrak{A}(\preceq) \text{ for all } A \in \mathfrak{A}(\preceq)\}$$

The radical is also a two sided ideal in $\mathfrak{A}(\preceq)$.

LEMMA 4. $\operatorname{Rad}\mathfrak{A}(\preceq) \subseteq \mathfrak{A}(\prec)$.

Proof. Suppose that $T \in \mathfrak{A}(\preceq) \setminus \mathfrak{A}(\prec)$. Then $P_{[\approx e]} T P_{[\approx e]} x \neq 0$ for some $e \in \mathfrak{B}$ and some $x \in \mathfrak{X}$. Choose $S \in \mathcal{L}(\mathfrak{X})$ such that $SP[\approx e]TP[\approx e]x = P_{[e]}x$, and let $A = P_{[\approx e]} S P_{[\approx e]}$. Then $A \in \mathfrak{A}(\preceq)$, and

$$ATP_{[\approx e]}x = P_{[\approx e]} S P_{[\approx e]} T P_{[\approx e]} x = P_{[\approx e]} x.$$

So $(I-AT)P_{[\approx e]}x=0$. Since $P_{[\approx e]}x\neq 0$, it follows that $I-AT$ is not invertible and so $T\notin \operatorname{Rad}\mathfrak{A}(\preceq)$. $\square$

We now seek conditions on the quasi-order $\preceq$ which are necessary and sufficient for the equality of the radical $\operatorname{Rad}\mathcal{A}(\preceq)$ and the ideal $\mathfrak{A}(\prec)$. The notion of local nilpotence will be useful.

DEFINITION 8. We say that $T\in\mathcal{L}(\mathfrak{X})$ is nilpotent at $x\in\mathfrak{X}$ if $T^n x=0$ for sufficiently large n. We say that T is locally nilpotent if it is nilpotent at each $x\in\mathfrak{X}$.

Clearly T is locally nilpotent if and only if $T^n\to 0$ as $n\to\infty$ in the strict operator topology.

LEMMA 5. *Suppose that $T\in\mathcal{L}(\mathfrak{X})$, and that T is not nilpotent at x. Then T is not nilpotent at Tx and T is not nilpotent at e for some $e\in\operatorname{supp}x$.*

Proof. The first conclusion is obvious. The second follows from the fact that x is a finite linear combination of basis vectors. $\square$

LEMMA 6. *Suppose that $T\in\mathfrak{A}(\preceq)$ and is locally nilpotent, then $(I-T)^{-1}$ exists and is in $\mathfrak{A}(\preceq)$. Furthermore $(I-T)^{-1}$ is the strict limit of the partial sums of $\sum\limits_{n=0}^{\infty}T^n$.*

Proof. Suppose that $x\in\mathfrak{X}$. Then $T^{N+1}x=0$ for some $N\in\mathbb{N}$. So the partial sums of $\sum_{n=0}^{\infty}T^n$ converge strictly. Furthermore,

$$\left(\sum_{n=0}^{\infty}T^n\right)(I-T)x=(I-T)\sum_{n=0}^{\infty}T^n x=(I-T)\sum_{n=0}^{N}T^n x=x-T^{N+1}x=x$$

i.e., $(I-T)^{-1}=\sum\limits_{n=0}^{\infty}T^n$. Since $\mathfrak{A}(\preceq)$ is strictly closed, $\sum\limits_{n=0}^{\infty}T^n\in\mathfrak{A}(\preceq)$. $\square$

COROLLARY 2. *Suppose that $T\in\mathfrak{A}(\prec)$ and $I-T$ has no inverse in $\mathfrak{A}(\preceq)$. Then there exists an infinite sequence $(e_j)_{j=1}^{\infty}$ in $\mathfrak{B}$ such that $(e_{j+1},e_j)\in\operatorname{supp}T$ for $j=1,2,3,\cdots$.*

Proof. Since $I-T\in\mathcal{A}(\preceq)$, it follows from Lemma 6 that T is not locally nilpotent. Choose $x\in\mathfrak{X}$ such that T is not nilpotent at x. By Lemma 5 there exist $e_1\in\operatorname{supp}x$ such that T is not nilpotent at e_1. Lemma 5 can also be used for the recursive steps of construction of the sequence $(e_j)_{j=1}^{\infty}$ which has the desired properties: if T is not nilpotent at e_j, then by Lemma 5 T is not nilpotent at e_{j+1} for some $e_{j+1}\in\operatorname{supp}Te_j$. Clearly $(e_{j+1},e_j)\in\operatorname{supp}T$ for $j=1,2,3,\cdots$. $\square$

Note that any sequence $(e_j)_{j=1}^{\infty}$ in $\mathfrak{B}$ with the property $(e_{j+1},e_j)\in\operatorname{supp}T$ for $j=1,2,3,\cdots$, and some $T\in\mathfrak{A}(\prec)$, is necessarily *strictly deceasing*, i.e., $e_{j+1}\prec e_j$ for $j=1,2,3,\cdots$.

LEMMA 7. *Suppose that* $T \in \mathfrak{A}(\prec) \setminus \mathrm{Rad}\,\mathfrak{A}(\preceq)$. *Then there exist infinite sequences* $(e_j)_{j=1}^{\infty}$ *and* $(f_j)_{j=1}^{\infty}$ *in* $\mathfrak{B}$ *which have the following properties:*

$$(e_j, f_j) \in \mathrm{supp}\, T, \qquad \text{and} \qquad f_{j+1} \preceq e_j \prec f_j \qquad \text{for} \qquad j = 1, 2, 3, \cdots. \tag{2.2}$$

Proof. Since $T \notin \mathrm{Rad}\,\mathfrak{A}(\preceq)$, there exists $A \in \mathfrak{A}(\preceq)$ such that $I - AT$ has no inverse in $\mathfrak{A}(\preceq)$. Now $AT \in \mathfrak{A}(\prec)$, and so by Corollary 2 there exists an infinite sequence $(f_j)_{j=1}^{\infty}$ in $\mathfrak{B}$ such that $(f_{j+1}, f_j) \in \mathrm{supp}\, AT$ for each $j \in \mathbb{N}$. Since $(f_{j+1}, f_j) \in \mathrm{supp}\, AT$, , there exists $e_j \in \mathfrak{B}$ such that $(e_j, f_j) \in \mathrm{supp}\, T$ and $(f_{j+1}, e_j) \in \mathrm{supp}\, A$. Since $T \in \mathfrak{A}(\prec)$, $e_j \prec f_j$, and since $A \in \mathfrak{A}(\preceq)$, $f_{j+1} \preceq e_j$. The sequences $(e_j)_{j=1}^{\infty}$ and $(f_j)_{j=1}^{\infty}$ are strictly decreasing because $e_{j+1} \prec f_{j+1} \preceq e_j \prec f_j$ for $j = 1, 2, 3, \cdots$. □

COROLLARY 3. $\mathrm{Rad}\,\mathfrak{A}(\preceq)$ *is strictly dense in* $\mathfrak{A}(\prec)$.

Proof. Suppose that $S \in \mathfrak{A}(\prec)$ and that $\mathcal{F}$ is a finite subset of $\mathfrak{X}$. We need to show that there exists $T \in \mathrm{Rad}\,\mathfrak{A}(\preceq)$ such that $Sx = Tx$ for all $x \in \mathcal{F}$.

Choose a finite subset Φ of $\mathfrak{B}$ with the properties $x = P_\Phi x$ and $Sx = P_\Phi Sx$ for each $x \in \mathcal{F}$. Such a subset Φ exists because $\mathcal{F}$ is finite and $\mathrm{supp}\, x$ and $\mathrm{supp}\, Sx$ are finite for all $x \in \mathfrak{X}$. Let $T = P_\Phi S P_\Phi$. Then $Sx = P_\Phi Sx = P_\Phi S P_\Phi x = Tx$ for each $x \in \mathcal{F}$. Furthermore, $T \in \ker \delta \cap \mathcal{A}(\preceq)$, since $P_\Phi \in \mathfrak{A}(\preceq)$ and $S \in \mathfrak{A}(\prec)$.

Now $\mathrm{supp}\, T \subseteq \Phi \times \Phi$ and is finite. So there are no infinite sequences $(e_j)_{j=1}^{\infty}$ and $(f_j)_{j=1}^{\infty}$ in $\mathfrak{B}$ which satisfy (2.2). So $T \in \mathrm{Rad}\,\mathfrak{A}(\preceq)$. □

The following result is a converse of Lemma 7.

LEMMA 8. *Suppose that* $T \in \mathfrak{A}(\prec)$, *and that* $(e_j)_{j=1}^{\infty}$ *and* $(f_j)_{j=1}^{\infty}$ *are infinite sequences in* $\mathfrak{B}$ *with properties* (2.2). *Then* $T \notin \mathrm{Rad}\,\mathfrak{A}(\preceq)$.

Proof. By taking subsequences of $(e_j)_{j=1}^{\infty}$ and $(f_j)_{j=1}^{\infty}$ if necessary, we construct a sequence $(\lambda_j)_{j=1}^{\infty}$ of non-zero scalars and a sequence of operators $(A_j)_{j=1}^{\infty}$ of operators in $\mathcal{L}(\mathfrak{X})$ with the properties

$$P_{e_j} T f_f = \lambda_j e_j \qquad \text{and} \qquad A_j = \lambda_j^{-1}(f_{j+1} \otimes e'_j)$$

for each $j = 1, 2, 3, \cdots$. To do this we start by noting that since $(e_1, f_1) \in \mathrm{supp}\, T$, $P_{e_1} T f_1 = \lambda_1 e_1$ where $\lambda_1 \neq 0$. By eliminating finitely many of the pairs (e_j, f_j), $j > 1$, and relabelling if necessary, we may assume that $e_j \notin \mathrm{supp}\, T f_1$ if $j > 1$. This is possible since $\mathrm{supp}\, T f$ is finite. Now let $A_1 = \lambda_1^{-1}(f_2 \otimes e'_1)$. Then $A_1 \in \mathfrak{A}(\preceq)$ and $A_1 T f_1 = f_2$.

Since $(e_2, f_2) \in \mathrm{supp}\, T$, $P_{e_2} T f_2 = \lambda_2 e_2$ where $\lambda_2 \neq 0$. By eliminating finitely many pairs (e_j, f_j), $j = 2, 3, 4, \cdots$, and relabelling if necessary, we may assume that $e_j \notin \mathrm{supp}\, T f_2$ if $j > 2$. Again this is possible since $\mathrm{supp}\, T f_2$ is finite. Now let $A_2 = \lambda_2^{-1}(f_j \otimes e'_2)$. Then $A_2 \in \mathfrak{A}(\preceq)$ and $A_2 T f_2 = f_3$. By continuing this process inductively, we obtained sequences $(\lambda_j)_{j=1}^{\infty}$ and $(A_j)_{j=1}^{\infty}$ with the desired properties.

Let $A = \sum_{j=1}^{\infty} A_j$. This sum converges strictly, $A \in \mathfrak{A}(\preceq)$, and $ATf_j = f_{j+1}$ for $j = 1,2,3,\cdots$. Suppose that $S = (I - AT)^{-1} \in \mathfrak{A}(\preceq)$. Then

$$\begin{aligned} S(I-(AT)^n)f_1 &= S(I-AT)(I+AT+(AT)^2+\cdots+(AT)^{n-1})f_1 \\ &= f_1+f_2+\cdots+f_n \end{aligned}$$

and so $Sf_1 = f_1 + f_2 + \cdots + f_n + Sf_{n+1}$.

Now $f_{n+1} \in \operatorname{span}[\preceq f_{n+1}]$, and since $S \in \mathfrak{A}(\preceq)$, $Sf_{n+1} \in [\preceq f_{n+1}]$. So $\{f_1, f_2, \cdots, f_n\} \cap [\preceq f_{n+1}] = \varnothing$, it follows that $\{f_1, f_2, \cdots, f_n\} \subseteq \operatorname{supp} Sf_1$ for each $n \geqslant 1$. But this is impossible since $\operatorname{supp} Sf_1$ is finite. So $I - AT$ has no inverse in $\mathfrak{A}(\preceq)$, and $T \notin \operatorname{Rad}\mathfrak{A}(\preceq)$. $\square$

Lemmas 7 and 8 together provide the proof of the following:

THEOREM 2. *Suppose that* $T \in \mathcal{A}(\prec)$. *Then* $T \notin \operatorname{Rad}\mathfrak{A}(\preceq)$ *if and only if there are infinite sequences* $(e_j)_{j=1}^{\infty}$ *and* $(f_j)_{j=1}^{\infty}$ *in* $\mathfrak{B}$ *satisfying property* (2.2).

DEFINITION 9. We say that an infinite sequence $(f_j)_{j=1}^{\infty}$ in $\mathfrak{B}(\preceq)$ is decreasing if $f_{j+1} \preceq f_j$ for all $j \in \mathbb{N}$.

DEFINITION 10. We say that $\mathfrak{B}(\preceq)$ satisfies the descending chain condition if every decreasing infinite sequence in $\mathfrak{B}(\preceq)$ is eventually constant.

LEMMA 9. *Suppose that* $\mathfrak{B}(\preceq)$ *does not satisfy the descending chain condition. Then* $\operatorname{Rad}\mathfrak{A}(\preceq) \neq \mathfrak{A}(\prec)$.

Proof. Suppose that $\mathfrak{B}(\preceq)$ does not satisfy the descending chain condition. Then there exists a strictly decreasing infinite sequence $(f_j)_{j=1}^{\infty}$ in $\mathfrak{B}(\preceq)$. Let $T = \sum_{j=1}^{\infty}(f_{j+1} \otimes f_j')$. This sum converges strictly and so T is well defined. Furthermore

$$\operatorname{supp} T = \bigcup_{j=1}^{\infty} \operatorname{supp}(f_{j+1} \otimes f_j') = \bigcup_{j=1}^{\infty} (f_{j+1}, f_j) \subseteq \mathfrak{G}(\preceq)$$

and so $T \in \mathfrak{A}(\prec)$. It follows from Lemma 8 (with $e_j = f_{j+1}$ for each $j \in \mathbb{N}$), that $T \notin \operatorname{Rad}\mathfrak{A}(\preceq)$. $\square$

Lemmas 7 and 9 provide a proof of the following:

THEOREM 3. $\operatorname{Rad}\mathfrak{A}(\preceq) = \mathfrak{A}(\prec)$ *if and only if* $\mathfrak{B}(\preceq)$ *satisfies the descending chain condition.*

EXAMPLE 5. Suppose that $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are vector spaces over the same field $\mathbb{F}$, and that $\mathfrak{L}_k(\preceq_k)$ is an atomic lattice of subspaces of $\mathfrak{X}_k$, for $k \in \{1,2\}$. The direct sum $\mathfrak{L}_1(\preceq_1) \oplus \mathfrak{L}_2(\preceq_2)$ is the lattice of subspaces of $\mathfrak{X} = \mathfrak{X}_1 \oplus \mathfrak{X}_2$ defined by

$$\mathfrak{L}_1(\preceq_1) \oplus \mathfrak{L}_2(\preceq_2) = \{\mathcal{N}_1 \oplus \mathcal{N}_2 : \mathcal{N}_k \in \mathfrak{L}_k(\preceq_k) \text{ for } k \in \{1,2\}\},$$

where $\mathcal{N}_1 \oplus \mathcal{N}_2 = \{x_1 \oplus x_2 : x_k \in \mathcal{N}_k \quad \text{for} \quad k \in \{1,2\}\}$.

Clearly $\mathfrak{L}_1(\preceq_1) \oplus \mathfrak{L}_2(\preceq_2)$ is atomic, with generating basis $\mathfrak{B}$ given by

$$\mathfrak{B} = \{e \oplus 0 : e \in \mathfrak{B}_1\} \cup \{0 \oplus f : f \in \mathfrak{B}_2\},$$

where $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are generating bases of $\mathfrak{L}_1(\preceq_1)$ and $\mathfrak{L}_2(\preceq_2)$ respectively. The quasi-order $\preceq$ on $\mathfrak{B}$ is defined by:

$$e \oplus 0 \preceq e^* \oplus 0 \Leftrightarrow e \preceq_1 e^* \qquad \text{and} \qquad 0 \oplus f \preceq 0 \oplus f^* \Leftrightarrow f \preceq_2 f^*.$$

Let $\mathcal{A} = \mathrm{Alg}(\mathfrak{L}_1(\preceq_1) \oplus \mathfrak{L}_2(\preceq_2))$, and let $\mathcal{A}_k = \mathrm{Alg}\,\mathfrak{L}_k(\preceq_k)$, for $k \in \{1,2\}$, It is easy to see that $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$, i.e., if $T = \begin{pmatrix} A_1 & B \\ C & A_2 \end{pmatrix} \in \mathcal{L}(\mathfrak{X})$, then $T \in \mathcal{A}$ if and only if $B = C = 0$ and $A_k \in \mathcal{A}_k$ for $k \in \{1,2\}$. Similarly, $\mathcal{A}_- = (\mathcal{A}_1)_- \oplus (\mathcal{A}_2)_-$ and $\mathrm{Rad}\,\mathcal{A} = \mathrm{Rad}\,\mathcal{A}_1 \oplus \mathrm{Rad}\,\mathcal{A}_2$. So $\mathrm{Rad}\,\mathcal{A} = \mathcal{A}_-$ if and only if $\mathrm{Rad}\,\mathcal{A}_k = (\mathcal{A}_-)_k$ for $k \in \{1,2\}$.

It is also easy to verify that $\mathfrak{B}(\preceq)$ satisfies the descending chain condition if and only $\mathfrak{B}_1(\preceq_1)$ and $\mathfrak{B}_1(\preceq_1)$ both satisfy the descending chain condition.

EXAMPLE 6. Suppose that $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are vector spaces over the same field $\mathbb{F}$, and that $\mathfrak{L}_k(\preceq_k)$ is an atomic lattice of subspaces of $\mathfrak{X}_k$ with generating basis $\mathfrak{B}_k$, for $k \in \{1,2\}$. A Hamel basis for the tensor product vector space $\mathfrak{X}_1 \otimes \mathfrak{X}_2$ is the set $\mathfrak{B}_1 \otimes \mathfrak{B}_2 = \{e \otimes f : e \in \mathfrak{B}_1 \text{ and } f \in \mathfrak{B}_2\}$. The tensor product $\mathfrak{L}_1(\preceq_1) \otimes \mathfrak{L}_2(\preceq_2)$ is the atomic subspace lattice of subspaces of $\mathfrak{X}_1 \otimes \mathfrak{X}_2$ for which $\mathfrak{B}$ is a generating basis, and where the quasi-order on $\mathfrak{B}$ is the 'product order' defined by:

$$e \otimes f \preceq e^* \otimes f^* \Leftrightarrow e \preceq_1 e^* \text{and } f \;\preceq_2 f^*.$$

The subspaces in $\mathfrak{L}_1(\preceq_1) \otimes \mathfrak{L}_2(\preceq_2)$ correspond to subsets of $\mathfrak{B}_1 \otimes \mathfrak{B}_2$ which are initial segments with respect to the quasi-order $\preceq$. Let Λ denote a map from $\mathfrak{B}_1$ into $2^{\mathfrak{B}_2}$, the set of all subsets of $\mathfrak{B}_2$. and let $D_\Lambda = \{e \otimes f : e \in \mathfrak{B}_1 \text{ and } f \in \Lambda(e)\}$. The subset $D_\Lambda \in \mathfrak{B}(\preceq)$ if and only if the 'sectional map' Λ has the properties:

$$\begin{array}{lll} \Lambda(e) \in \mathfrak{B}_2(\preceq_2) & \text{for each} \quad e \in \mathfrak{B}_1 & \text{and} \\ \Lambda & \text{is decreasing, in the sense that} & e \preceq_1 e^* \Rightarrow \Lambda(e^*) \subseteq \Lambda(e) \end{array}$$

It is also easy to verify that $\mathfrak{B}(\preceq)$ satisfies the descending chain condition if and only $\mathfrak{B}_1(\preceq_1)$ and $\mathfrak{B}_1(\preceq_1)$ both satisfy the descending chain condition. Let $\mathcal{A} = \mathrm{Alg}(\mathfrak{L}_1(\preceq_1) \otimes \mathfrak{L}_2(\preceq_2))$, and let $\mathcal{A}_k = \mathrm{Alg}\,\mathfrak{L}_k(\preceq_k)$, for $k \in \{1,2\}$. It follows that $\mathrm{Rad}\,\mathcal{A} = \mathcal{A}_-$ if and only if $\mathrm{Rad}\,\mathcal{A}_k = (\mathcal{A})_-$ for $k \in \{1,2\}$.

3. Finite-rank operators in $\mathcal{A}(\preceq)$

The *rank* of an operator in $\mathcal{L}(\mathfrak{X})$ is defined as the dimension of its range. In this section we examine the properties of operators in $\mathfrak{A}(\preceq)$ whose ranks are finite. Let $\mathcal{R}$ denote the set of finite-rank operators in $\mathcal{L}(\mathfrak{X})$, and let $\mathcal{R}(\preceq) = \mathcal{R} \cap \mathfrak{A}(\preceq)$. Various authors have investigated the properties of $\mathcal{R}(\preceq)$ in the Hilbert space context. For example, Erdos proved [3] that if $\mathfrak{L}(\preceq)$ is a nest, i.e., totally ordered, then the strong closure of $\mathcal{R}(\preceq)$ is $\mathfrak{A}(\preceq)$.

Operators in $\mathfrak{A}(\preceq)$ with rank one also have an important role in the Hilbert space context. Let $\mathcal{R}_1$ denote the set of all rank one operators in $\mathcal{L}(\mathfrak{X})$, and let $\mathcal{R}_1(\preceq) = \mathcal{R}_1 \cap \mathfrak{A}(\preceq)$. Each rank one operator on $\mathfrak{X}$ has the form $x \otimes \varphi$, where $x \in \mathfrak{X}$ and $\varphi \in \mathfrak{X}'$, and $(y \otimes \varphi)(y) = \varphi(y)x$ for all $y \in \mathfrak{X}$.

3.1. Reflexivity

There is a rich supply of rank one operators in $\mathfrak{A}(\preceq)$. In particular there are enough to determine $\mathfrak{L}(\preceq)$ as the invariant subspace lattice of $\mathfrak{A}(\preceq)$. The following propositions are easy adaptations of results obtained by Longstaff [7] that apply in the Hilbert space context.

PROPOSITION 1. *Suppose that $x \in \mathfrak{X}$ and $\varphi \in \mathfrak{X}'$. Then $x \otimes \varphi \in \mathfrak{A}(\preceq)$ if and only if there exists $\mathcal{M} \in \mathfrak{L}(\preceq)$ with the properties: $x \in \mathcal{M}$ and $\mathcal{N} \subseteq \ker\varphi$ if $\mathcal{N} \in \mathfrak{L}(\preceq)$ and $\mathcal{M} \not\subseteq \mathcal{N}$.*

Proof. Suppose that $x \otimes \varphi \in \mathfrak{A}(\preceq)$. Let $\mathcal{M} = \bigcap\{\mathcal{N} \in \mathfrak{L}(\preceq) : x \in \mathcal{N}\}$. Then $x \in \mathcal{M}$. Suppose that $\mathcal{N} \in \mathfrak{L}(\preceq)$ and $\mathcal{M} \not\subseteq \mathcal{N}$. Then $x \notin \mathcal{N}$. Since $(x \otimes \varphi)(\mathcal{N}) \subseteq \operatorname{span}\{x\} \subseteq \mathcal{N}$, it follows that $(x \otimes \varphi)(\mathcal{N}) = \{0\}$.

Now suppose that $x \in \mathcal{M} \in \mathfrak{L}(\preceq)$ and $\mathcal{N} \subseteq \ker\varphi$ if $\mathcal{N} \in \mathfrak{L}(\preceq)$ and $\mathcal{M} \not\subseteq \mathcal{N}$. Suppose that $\mathcal{N} \in \mathfrak{L}(\preceq)$. If $\mathcal{M} \subseteq \mathcal{N}$, then $(x \otimes \varphi)\mathcal{N} \subseteq \operatorname{span}\{x\} \subseteq \mathcal{M} \subseteq \mathcal{N}$. If $\mathcal{M} \not\subseteq \mathcal{N}$ then $\mathcal{N} \subseteq \ker\varphi$, and so $(x \otimes \varphi)\mathcal{N} = \{0\} \subseteq \mathcal{N}$. So $x \otimes \varphi \in \operatorname{Alg}\mathfrak{L}(\preceq) = \mathfrak{A}(\preceq)$. $\square$

PROPOSITION 2. $\mathfrak{L}(\preceq) = \operatorname{Lat}\mathcal{R}_1(\preceq)$.

Proof. Since $\mathcal{R}_1(\preceq) \subseteq \mathfrak{A}(\preceq)$, it is enough to prove that $\operatorname{Lat}\mathcal{R}_1(\preceq) \subseteq \mathfrak{L}(\preceq)$.

Suppose that $\mathcal{M} \in \operatorname{Lat}\mathcal{R}_1(\preceq)$, and let $\Phi = \operatorname{supp}\mathcal{M} = \bigcup\{\operatorname{supp} x : x \in \mathcal{M}\}$. Then $\mathcal{M} \subseteq \mathfrak{X}_\Phi$. We shall show that $\mathcal{M} = \mathfrak{X}_\Phi \in \mathfrak{L}(\preceq)$. Since $P_e \in \mathcal{R}_1(\preceq)$ for each $e \in \mathfrak{B}$, it follows that $e \in \mathcal{M}$ for each $e \in \Phi$. So $\mathcal{M} = \mathfrak{X}_\Phi$. Suppose that $e \in \Phi$ and $f \preceq e$, and consider the rank one operator $R = f \otimes e'$. Since $\operatorname{supp} R = \{(f,e)\} \subseteq \mathfrak{G}(\preceq)$, $R \in \mathcal{R}_1(\preceq)$. So $f = Re \in \mathcal{M}$. So $\Phi \in \mathfrak{B}(\preceq)$ and $\mathcal{M} \in \mathfrak{L}(\preceq)$. Therefore $\mathfrak{L}(\preceq) = \operatorname{Lat}\mathcal{R}_1(\preceq)$. $\square$

3.2. Decomposability

Let $\mathcal{R}_1^*(\preceq) = \operatorname{span}\mathcal{R}_1(\preceq)$, the set of all finite sums of operators in $\mathcal{R}_1(\preceq)$. Clearly $\mathcal{R}_1^*(\preceq) \subseteq \mathcal{R}(\preceq)$, but in the Hilbert space context there are examples of subspace lattices $\mathcal{L}(\preceq)$ for which $\mathcal{R}_1^*(\preceq) \neq \mathcal{R}(\preceq)$. For example, Hopenwasser and Moore [5] showed that there exists a ‘totally atomic’, commutative subspace lattice $\mathcal{L}(\preceq)$ with the property that $\mathcal{R}(\preceq)$ contains a rank two operator that is not in $\mathcal{R}_1^*(\preceq)$. The Hopenwasser-Moore example does not have an analogue in the general linear algebra context because it uses infinite combinations of the basis vectors. The following theorem shows that in the operator algebra $\mathfrak{A}(\preceq)$, as defined in Section 1 of this paper, every finite rank operator in $\mathfrak{A}(\preceq)$ is indeed a finite sum of rank one operators in $\mathcal{A}(\preceq)$.

THEOREM 4. $\mathcal{R}_1^*(\preceq) = \mathcal{R}(\preceq)$.

Proof. Since $\mathcal{R}_1^*(\preceq) \subseteq \mathcal{R}(\preceq)$, we need to show that $\mathcal{R}(\preceq) \subseteq \mathcal{R}_1^*(\preceq)$. Suppose that $T \in \mathcal{R}(\preceq)$. Since $P_e \in \mathfrak{A}(\preceq)$, $P_eT \in \mathcal{R}_1(\preceq)$ or $P_eT = 0$ for each $e \in \mathfrak{B}$. Let $\Phi = \operatorname{supp}(\operatorname{ran} T) = \bigcup\{\operatorname{supp} Tx : x \in \mathfrak{X}\}$. Note that Φ is finite since $\operatorname{ran} T$ is finite-dimensional, and $\operatorname{supp} Tx$ is finite for all $x \in \mathfrak{X}$. Now $T = \sum_{e\in\mathfrak{B}} P_eT$, and $P_eT \in \mathcal{R}_1(\preceq)$, and so $T \in \mathcal{R}_1^*(\preceq)$. $\square$

3.3. Density

Laurie and Longstaff [6] showed that a commutative subspace lattice $\mathfrak{L}(\preceq)$ is completely distributive if and only if $\mathcal{R}_1^*(\preceq)$ is dense in $\mathfrak{A}(\preceq)$ in any of the following four topologies: the weak operator topology, the strong operator topology, the ultraweak operator topology and the ultra strong operator topology. This suggests the following possibility, namely that in the general linear algebra context as outlined in Section 1, $\mathfrak{A}(\preceq)$ is the strict closure of $\mathcal{R}_1^*(\preceq)$.

THEOREM 5. *The strict closure of $\mathcal{R}_1^*(\preceq)$ is $\mathfrak{A}(\preceq)$.*

Proof. Suppose that $\mathcal{F}$ is a finite subset of $\mathfrak{X}$ and that $T \in \mathfrak{A}(\preceq)$. Let $\Phi = \bigcup\{\operatorname{supp} x : x \in \mathcal{F}\}$, and note that Φ is finite. Note also that $Tx = \sum_{e\in\Phi} TP_ex$ for each $x \in \mathcal{F}$, each P_e has rank one, and $P_e \in \mathfrak{A}(\preceq)$. Furthermore, $Tx = \sum_{e\in\Phi} TP_ex$ for each $x \in \mathcal{F}$, and so $T \in \mathcal{R}_1^*(\preceq)$. $\square$

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