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UNITARY, SELF–ADJOINT AND $\mathscr{J}$ – SYMMETRIC WEIGHTED COMPOSITION OPERATORS ON FOCK–SOBOLEV SPACES

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Abstract. In this paper, we characterize the boundedness and compactness for weighted composition operators on the Fock-Sobolev space $\mathscr{F}^{p,m}(\mathbb{C}^n)$, $0<p<\infty$. We prove that no nontrivial unitary or self-adjoint weighted composition operators exist on $\mathscr{F}^{2,m}(\mathbb{C}^n)$ when $m \geqslant 1$. As an application, we also prove that there exist only trivial $\mathscr{J}$-symmetric weighted composition operators on $\mathscr{F}^{2,m}(\mathbb{C}^n)$ when $m \geqslant 1$.

1. Introduction

Let $\mathbb{C}^n$ be the n-dimensional complex Euclidean space. For any two points $z=(z_1,\cdots,z_n)$ and $w=(w_1,\cdots,w_n)$ in $\mathbb{C}^n$, the inner product of z,w is given by $\langle z,w\rangle = \sum_{j=1}^n z_j\overline{w_j}$. We also write $|z| = \langle z,z\rangle^{\frac{1}{2}}$. Denote by $H(\mathbb{C}^n)$ the space of all entire functions on $\mathbb{C}^n$. Given a function $\psi \in H(\mathbb{C}^n)$ and an entire map $\varphi : \mathbb{C}^n \to \mathbb{C}^n$, the weighted composition operator $W_{\psi,\varphi}$ is defined on $H(\mathbb{C}^n)$ by $W_{\psi,\varphi}f = \psi(f\circ\varphi)$. When the function ψ is identical to 1, $W_{\psi,\varphi}$ reduces to the composition operator C_φ.

For $0<p<\infty$, we denote by $\mathscr{F}^p(\mathbb{C}^n)$ the classical Fock space over $\mathbb{C}^n$, which consists of all functions $f \in H(\mathbb{C}^n)$ such that

$$\|f\|_p := \left[\left(\frac{p}{2\pi}\right)^n \int_{\mathbb{C}^n} |f(z)|^p e^{-\frac{p}{2}|z|^2} dV_n(z)\right]^{\frac{1}{p}} < \infty,$$

where dV_n is the ordinary volume measure on $\mathbb{C}^n$. Then for any non-negative integer m, the Fock-Sobolev space $\mathscr{F}^{p,m}(\mathbb{C}^n)$ consists of all functions $f \in H(\mathbb{C}^n)$ such that $\sum_{|\alpha|\leqslant m}\|\partial^\alpha f\|_p < \infty$, where $\alpha = (\alpha_1,\cdots,\alpha_n)$ is a multi-index of non-negative integers and we write $|\alpha| = \alpha_1+\cdots+\alpha_n$ and $\partial^\alpha = \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$, where $\partial_k (k=1,2,\ldots,n)$ denotes partial differentiation with respect to the k-th component. The space $\mathscr{F}^{p,m}(\mathbb{C}^n)$ was first studied by Cho and Zhu in [3] and they obtained an equivalent characterization, which showed that $f \in \mathscr{F}^{p,m}(\mathbb{C}^n)$ if and only if every function $z^\alpha f(z)$ with

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$|\alpha| = m$ is in $\mathscr{F}^p(\mathbb{C}^n)$, here $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. Then the "norm" on $\mathscr{F}^{p,m}(\mathbb{C}^n)$ can be defined as follows:

$$\|f\|_{p,m}^p := \omega(n,p,m) \int_{\mathbb{C}^n} |f(z)|^p |z|^{mp} e^{-\frac{p}{2}|z|^2} dV_n(z),$$

where

$$\omega(n,p,m) = \left(\frac{p}{2}\right)^{\frac{mp}{2}+n} \frac{(n-1)!}{\pi^n \Gamma(\frac{mp}{2}+n)}$$

is the normalized constant so that the constant function 1 has norm 1 in $\mathscr{F}^{p,m}(\mathbb{C}^n)$. According to [8, Proposition 2.1] or [3, Lemma 3], for any $f \in \mathscr{F}^{p,m}(\mathbb{C}^n)$ and $z \in \mathbb{C}^n$, we have

$$|f(z)| \lesssim \frac{e^{\frac{|z|^2}{2}}}{1+|z|^m} \|f\|_{p,m}. \tag{1.1}$$

Here, for convenience, we write $U \lesssim V$ (or equivalently $V \gtrsim U$) for two real valued non-negative quantities U and V, whenever there is a constant $c > 0$ independent of the argument such that $U \leqslant cV$. We write $U \simeq V$ if both $U \lesssim V$ and $V \lesssim U$.

When $p = 2$, the space $\mathscr{F}^{2,m}(\mathbb{C}^n)$ is a Hilbert space with the following inner product

$$\langle f, g \rangle_m = \frac{(n-1)!}{\pi^n \Gamma(m+n)} \int_{\mathbb{C}^n} f(z)\overline{g(z)} |z|^{2m} e^{-|z|^2} dV_n(z), \quad f,\ g \in \mathscr{F}^{2,m}(\mathbb{C}^n).$$

(1.1) tells us that the point evaluations are bounded linear functionals on $\mathscr{F}^{p,m}(\mathbb{C}^n)$. It follows the Riesz representation theorem in Hilbert space theory that for each $w \in \mathbb{C}^n$, there exists a unique function $K_{m,w}$ in $\mathscr{F}^{2,m}(\mathbb{C}^n)$ such that

$$f(w) = \langle f, K_{m,w} \rangle_m$$

for any $f \in \mathscr{F}^{2,m}(\mathbb{C}^n)$. By [3, Theorem 12] (see also [8]), we have

$$K_{m,w}(z) = \frac{(n+m-1)!}{(n-1)!} \sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n+m+k-1)!} \frac{\langle z, w \rangle^k}{k!}, \quad z,\ w \in \mathbb{C}^n.$$

(1.1), along with [8, Proposition 2.2], implies that

$$\|K_{m,w}\|_{p,m} \simeq \frac{e^{\frac{1}{2}|w|^2}}{1+|w|^m} \tag{1.2}$$

for any $w \in \mathbb{C}^n$.

The reasearch of (weighted) composition operators reflects deep relationship between operator theory and function theory. Recently, much progress was made in the study of (weighted) composition operators on the Fock spaces. Le [11] obtained some characterizations for the boundedness and compactness of $W_{\psi,\varphi}$ on the classical Fock space $\mathscr{F}^2(\mathbb{C})$. And then, in [15], Tien and Khoi generalized Le's results to $W_{\psi,\varphi}$ acting between different Fock spaces. Later, following the same route, they extended their results to several variables in [17]. In [4], Choe, Izuchi and Koo showed that a linear sum of two composition operators is bounded (compact, resp.) on $\mathscr{F}^2(\mathbb{C}^n)$ if and only if both composition operators in the sum are bounded (compact, resp.). And then, in [16], Tien and Khoi studied the differences of weighted composition operators between different Fock spaces. In [1], Cho, Choe and Koo solved the problem raised in [4]. They studied linear combination of composition operators acting on the Fock-Sobolev space $\mathscr{F}^{2,m}(\mathbb{C}^n)$, and showed that such an operator is bounded only when all the composition operators in the combination are bounded individually. Mengetie [13] studied the weighted composition operators from $\mathscr{F}^{p,m}(\mathbb{C}^n)$ to $\mathscr{F}^{q,m}(\mathbb{C}^n)$ via (p,q) Fock-Carleson measures and obtained some characterizations for the boundedness and compactness of $W_{\psi,\varphi}$. For more information about Fock space and Fock-Sobolev space, one can see [2], [8] and [22].

This paper is organized as follows. In section 2, inspired by [17], we investigate the boundedness and compactness of $W_{\psi,\varphi}$ on $\mathscr{F}^{p,m}(\mathbb{C}^n)$ by singular value decomposition of an $n\times n$ matrix. In section 3, we study unitary weighted composition operators on $\mathscr{F}^{2,m}(\mathbb{C}^n)$, and prove that there are only trivial unitary weighted composition operators on $\mathscr{F}^{2,m}(\mathbb{C}^n)$ when $m\geqslant 1$. More precisely, we show that $W_{\psi,\varphi}$ is unitary on $\mathscr{F}^{2,m}(\mathbb{C}^n)$ when $m\geqslant 1$ if and only if $\varphi(z)=Az$ for some unitary $n\times n$ matrix A and ψ is a constant function of unimodule. Self-adjoint and $\mathscr{J}$-symmetric weighted composition operators on $\mathscr{F}^{2,m}(\mathbb{C}^n)$ with $m\geqslant 1$ are described in section 4. We also show that no nontrivial self-adjoint or $\mathscr{J}$-symmetric weighted composition operators exist on $\mathscr{F}^{2,m}(\mathbb{C}^n)$ when $m\geqslant 1$.

2. Boundedness and compactness

In this section we study the boundedness and compactness for weighted composition operators on $\mathscr{F}^{p,m}(\mathbb{C}^n)$. The idea derives from [17].

Firstly we need the following lemma.

LEMMA 2.1. *Suppose m is a non-negative integer, then for any $R>0$, there exists a constant $C=C_R>0$ such that*

$$\frac{x}{1+R}-\log\left(1+R^m\right)+C_R>0$$

for all $x\geqslant 0$. Moreover, $\lim\limits_{R\to\infty}\frac{C_R}{R}=0$.

Proof. Assume $m\geqslant 1$, without loss of generality. Fix $R>0$, consider the function $f(x)=\log(1+x^m)-\frac{x}{1+R}$. Let x_R be the maximum point of f. By a simple calculation, we get $x_R\leqslant m(R+1)$. Set $C_R=f(x_R)$, then we have

$$\begin{aligned}0\leqslant \lim_{R\to\infty}\frac{C_R}{R}&=\lim_{R\to\infty}\frac{\log(1+x_R^m)-\frac{x_R}{R+1}}{R}\\&\leqslant\lim_{R\to\infty}\frac{\log\{1+[m(R+1)]^m\}}{R}+\lim_{R\to\infty}\frac{m(R+1)}{R(R+1)}=0.\end{aligned}$$

The proof is complete. $\square$

We will apply Lemma 2.1 to get the following proposition which is a modification of [11, Proposition 2.1].

PROPOSITION 2.2. *Let φ and ψ be entire functions on $\mathbb{C}$ with $\psi(0) \neq 0$. m is a non-negative integer. If there is a constant $M > 0$ such that*

$$|\psi(z)|^2 \frac{1+|z|^{2m}}{1+|\varphi(z)|^{2m}} e^{|\varphi(z)|^2-|z|^2} \leqslant M \tag{2.1}$$

for all $z \in \mathbb{C}$. Then $\varphi(z) = \varphi(0) + az$ for some $a \in \mathbb{C}$ with $|a| \leqslant 1$. If $|a| = 1$, then $\psi(z) = \psi(0)e^{-a\overline{\varphi(0)}z}$.

Furthermore, if

$$\lim_{|z|\to\infty} |\psi(z)|^2 \frac{1+|z|^{2m}}{1+|\varphi(z)|^{2m}} e^{|\varphi(z)|^2-|z|^2} = 0, \tag{2.2}$$

then $\varphi(z) = \varphi(0) + az$ for some $a \in \mathbb{C}$ with $|a| < 1$.

Proof. Taking logarithms on both sides of (2.1), we obtain

$$|\varphi(z)|^2 - |z|^2 + 2\log|\psi(z)| + \log(1+|z|^{2m}) - \log(1+|\varphi(z)|^{2m}) \leqslant \log M$$

for all $z \in \mathbb{C}$. By Lemma 2.1, for any $R > 0$, there exists a constant $C_R > 0$ with $\lim\limits_{R\to\infty} \frac{C_R}{R} = 0$, such that

$$\frac{R}{R+1}|\varphi(z)|^2 - |z|^2 + 2\log|\psi(z)| + \log(1+|z|^{2m}) - C_R \leqslant \log M$$

for all $z \in \mathbb{C}$. Taking $z = Re^{i\theta}$ and integrating with respect to θ on $[0, 2\pi]$, we obtain

$$\frac{R}{R+1}\int_0^{2\pi} |\varphi(Re^{i\theta})|^2 \frac{d\theta}{2\pi} - R^2 + 2\int_0^{2\pi} \log|\psi(Re^{i\theta})| \frac{d\theta}{2\pi} + \log(1+R^{2m}) - C_R \leqslant \log M.$$

Jensen's inequality tells us that

$$\int_0^{2\pi} \log|\psi(Re^{i\theta})| \frac{d\theta}{2\pi} \geqslant \log|\psi(0)|.$$

And we consider the power expansion $\varphi(z) = \varphi(0) + az + \sum_{j=2}^{\infty} a_j z^j$ for $z \in \mathbb{C}$. It follows that

$$\int_0^{2\pi} |\varphi(Re^{i\theta})|^2 \frac{d\theta}{2\pi} = |\varphi(0)|^2 + |a|^2R^2 + \sum_{j=2}^{\infty} |a_j|^2 R^{2j}.$$

Therefore, we get the following inequality,

$$\frac{R}{R+1}|\varphi(0)|^2 + \Big(\frac{R}{R+1}|a|^2 - 1\Big)R^2 + \frac{R}{R+1}\sum_{j=2}^{\infty}|a_j|^2R^{2j}$$
$$+2\log|\psi(0)| + \log(1+R^{2m}) - C_R \leqslant \log M.$$

Then we have

$$\limsup_{R\to\infty}\Big(\frac{R}{R+1}|a|^2-1\Big)R+\sum_{j=2}^{\infty}|a_j|^2R^{2j-1}\leqslant 0,$$

which yields that $a_j=0$ for $j\geqslant 2$ and $|a|\leqslant 1$. Thus $\varphi(z)=\varphi(0)+az$ with $|a|\leqslant 1$.

If $|a|=1$, then we have $|\varphi(z)|^2-|z|^2=|\varphi(0)|^2+2Re(a\overline{\varphi(0)}z)$ and $\lim\limits_{|z|\to\infty}\frac{1+|z|^{2m}}{1+|\varphi(z)|^{2m}}$ $=1$. The inequality (2.1) yields that

$$|\psi(z)e^{a\overline{\varphi(0)}z}|^2\lesssim Me^{-|\varphi(0)|^2}$$

for all $z\in\mathbb{C}$, which implies that $\psi(z)e^{a\overline{\varphi(0)}z}$ is a constant function by Liouville's theorem. Then $\psi(z)=\psi(0)e^{-a\overline{\varphi(0)}z}$. Moreover, when $|a|=1$, the limit in (2.2) as $|z|\to\infty$ is $|\psi(0)|^2e^{|\varphi(0)|^2}\neq 0$. This contradiction shows that $|a|<1$ if (2.2) holds. The proof is complete. $\square$

Assume $f\in H(\mathbb{C}^n)$, for any $\zeta\in\mathbb{C}^n$, we denote by $f_\zeta(u)=f(u\zeta)$ $(u\in\mathbb{C})$. Then f_ζ is called the slice function of f at ζ which is an entire function on $\mathbb{C}$. Given an $n\times n$ matrix A, denote by $\|A\|$ the operator norm of A. Now we are going to extend Proposition 2.2 to several variables as follows.

PROPOSITION 2.3. *Let $\psi\in H(\mathbb{C}^n)$ with $\psi(0)\neq 0$ and φ be an entire map on $\mathbb{C}^n$. If there exists a constant $M>0$ such that*

$$|\psi(z)|^2\frac{1+|z|^{2m}}{1+|\varphi(z)|^{2m}}e^{|\varphi(z)|^2-|z|^2}\leqslant M$$

for every $z\in\mathbb{C}^n$, then $\varphi(z)=Az+b$ for some $n\times n$ matrix A with $\|A\|\leqslant 1$ and $b\in\mathbb{C}^n$. And for $u\in\mathbb{C}$, $\psi_\zeta(u)=\psi(0)e^{-u\langle A\zeta,b\rangle}$ whenever $|A\zeta|=|\zeta|$ for some $\zeta\in\mathbb{C}^n$. In particular, if A is unitary, then $\psi(z)=\psi(0)e^{-\langle z,A^*b\rangle}, z\in\mathbb{C}^n$.

Furthermore, if $\lim\limits_{|z|\to\infty}|\psi(z)|^2\frac{1+|z|^{2m}}{1+|\varphi(z)|^{2m}}e^{|\varphi(z)|^2-|z|^2}=0$, then $\varphi(z)=Az+b$ for some matrix A with $\|A\|<1$ and $b\in\mathbb{C}^n$.

Proof. The proof is a modification of [18, Lemma 6], so we omit the details. $\square$

For simplicity, we will use the following notations.

$$m_z(\psi,\varphi)=|\psi(z)|\frac{1+|z|^m}{1+|\varphi(z)|^m}e^{\frac{|\varphi(z)|^2-|z|^2}{2}},\quad z\in\mathbb{C}^n$$

and

$$m(\psi,\varphi)=\sup\{m_z(\psi,\varphi):z\in\mathbb{C}^n\}.$$

Then we have the following necessary conditions for the boundedness of $W_{\psi,\varphi}$.

THEOREM 2.4. *Let $\psi\in H(\mathbb{C}^n)$ and φ be an entire map on $\mathbb{C}^n$. If $W_{\psi,\varphi}$ is bounded on $\mathscr{F}^{p,m}(\mathbb{C}^n)$, then $\psi\in\mathscr{F}^{p,m}(\mathbb{C}^n)$ and $m(\psi,\varphi)<\infty$. In this case, $\varphi(z)=Az+b$ for some $n\times n$ matrix A with $\|A\|\leqslant 1$ and $b\in\mathbb{C}^n$.*

Proof. It is clear that $\psi(z) = W_{\psi,\varphi}(1) \in \mathscr{F}^{p,m}(\mathbb{C}^n)$. For any $w \in \mathbb{C}^n$, by (1.1), we have

$$\|W_{\psi,\varphi}K_{m,w}\|_{p,m} \gtrsim |W_{\psi,\varphi}K_{m,w}(z)|\frac{1+|z|^m}{e^{\frac{1}{2}|z|^2}} = |\psi(z)||K_{m,w}(\varphi(z))|\frac{1+|z|^m}{e^{\frac{1}{2}|z|^2}}.$$

In particular, with $w = \varphi(z)$, it follows from (1.2) that $m(\psi,\varphi) \lesssim \|W_{\psi,\varphi}\|$. In this case, by proposition 2.3, we have $\varphi(z) = Az + b$ with $\|A\| \leqslant 1$ and $b \in \mathbb{C}^n$. $\square$

REMARK 2.5. If A is a zero matrix, i.e. $\varphi(z) \equiv b$, then $W_{\psi,\varphi}$ is an operator of finite rank. In this case, $W_{\psi,\varphi}$ is bounded on $\mathscr{F}^{p,m}(\mathbb{C}^n)$ if and only if $\psi \in \mathscr{F}^{p,m}(\mathbb{C}^n)$.

REMARK 2.6. If A is invertible, the condition in Theorem 2.4 is also sufficient.

As in [17], when A is not a zero matrix and not invertible, the method we use in this case is the so-called singular value decomposition of an $n \times n$ matrix (see [10, Theorem 2.6.3]).

LEMMA 2.7. *If A is an $n \times n$ matrix of rank r, then A can be written as $A = U_1\Lambda U_2$, where U_1 and U_2 are $n \times n$ unitary matrices and $\Lambda = diag\{\lambda_1,\cdots,\lambda_n\}$ is a diagonal matrix with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r > \lambda_{r+1} = \cdots = \lambda_n = 0$. The $\lambda_i (i = 1,\cdots,n)$ are the non-negative square roots of the eigenvalues of AA^*. Moreover, if we require that they are listed in decreasing order, then Λ is uniquely determined from A.*

We notice that if U is unitary, then the composition operator C_U ia an isometry on the Fock-Sobolev spaces for any $0 < p < \infty$ with $C_U^{-1} = C_{U^*}$. Let $\psi \in H(\mathbb{C}^n)$ and $\varphi(z) = Az + b$ is an entire map of $\mathbb{C}^n$ with $\|A\| \leqslant 1$. If $A = U_1\Lambda U_2$ is the singular decomposition of A, then $W_{\psi,\varphi} = C_{U_2}W_{\Psi,\Phi}C_{U_1}$, where $\Psi(z) = \psi(U_2^*z)$ and $\Phi(z) = \Lambda z + U_1^*b$. Thus, $W_{\psi,\varphi}$ is bounded(compact) if and only if $W_{\Psi,\Phi}$ is bounded(compact). We call (Ψ,Φ) is the normalization of (ψ,φ) respect to the decomposition $A = U_1\Lambda U_2$.

For each $z = (z_1,\cdots,z_n) \in \mathbb{C}^n$ and $0 \leqslant r \leqslant n$, as in [17], we denote by $z_{[r]} = (z_1,\cdots,z_r)$ if $r \neq 0$ and otherwise $z_{[r]} = \emptyset$, $z'_{[r]} = (z_{r+1},\cdots,z_n)$ if $r \neq n$ and otherwise $z'_{[r]} = \emptyset$. For each $n \times n$ matrix A, we denote by $A_{[r]}$ the first $r \times r$ sub-matrix of A.

For any $\psi \in H(\mathbb{C}^{n-r})$ and $0 < r \leqslant n$, let

$$F_{\psi,p}(z_{[r]}) = \left(\int_{\mathbb{C}^{n-r}} |\psi(z_{[r]},z'_{[r]})|^p(1+|(z_{[r]},z'_{[r]})|^{mp})e^{-\frac{p}{2}|z'_{[r]}|^2}dV_{n-r}(z'_{[r]})\right)^{\frac{1}{p}}.$$

Then we can characterize the boundedness and compactness for the operator $W_{\psi,\varphi}$ in terms of the following quantities:

$$M_{[z_{[r]}]}(\psi,\varphi) = \frac{F_{\psi,p}(z_{[r]})}{1+|\varphi(z)|^m}e^{\frac{|\varphi(z)|^2-|z_{[r]}|^2}{2}}, \quad z_{[r]} \in \mathbb{C}^r$$

and $M(\psi,\varphi) = \sup\{M_{z_{[r]}}(\psi,\varphi) : z_{[r]} \in \mathbb{C}^r\}$.

LEMMA 2.8. *Let* $0 < p < \infty$ *and* $0 < r \leqslant n$*, then we have*

$$F_{f,p}(z_{[r]}) \lesssim e^{\frac{1}{2}|z_{[r]}|^2} \|f\|_{p,m}$$

for all $f \in H(\mathbb{C}^n)$ *and all* $z_{[r]} \in \mathbb{C}^r$.

Proof. By the proof of [8, Proposition 2.1], we get

$$|f(z_{[r]}, z'_{[r]})|^p \lesssim e^{\frac{p}{2}|z_{[r]}|^2} \int_{|w_{[r]}-z_{[r]}|<1} |f(w_{[r]}, z'_{[r]})|^p e^{-\frac{p}{2}|w_{[r]}|^2} dV_r(w_{[r]}).$$

It follows that

$$\begin{aligned}
& e^{-\frac{p}{2}|z_{[r]}|^2} F^p_{f,p}(z_{[r]}) \\
\lesssim & \int_{\mathbb{C}^{n-r}} (1 + |(z_{[r]}, z'_{[r]})|^{mp}) e^{-\frac{p}{2}|z'_{[r]}|^2} dV_{n-r}(z'_{[r]}) \\
& \quad \times \int_{|w_{[r]}-z_{[r]}|<1} |f(w_{[r]}, z'_{[r]})|^p e^{-\frac{p}{2}|w_{[r]}|^2} dV_r(w_{[r]}) \\
\lesssim & \int_{\mathbb{C}^{n-r}} e^{-\frac{p}{2}|z'_{[r]}|^2} dV_{n-r}(z'_{[r]}) \\
& \quad \int_{|w_{[r]}-z_{[r]}|<1} |f(w_{[r]}, z'_{[r]})|^p e^{-\frac{p}{2}|w_{[r]}|^2} \left(1 + [(1 + |w_{[r]}|)^2 + |z'_{[r]}|^2]\right)^{\frac{mp}{2}} dV_r(w_{[r]}) \\
\lesssim & \int_{\mathbb{C}^n} |f(z)|^p (1 + |z|^{mp}) e^{-\frac{p}{2}|z|^2} dV_n(z) \\
\simeq & \|f\|^p_{p,m}.
\end{aligned}$$

The proof is complete. □

THEOREM 2.9. *Let* $0 < p < \infty$*,* m *is a non-negative integer,* $\psi \in \mathscr{F}^{p,m}(\mathbb{C}^n)$ *and* $\varphi(z) = Az + b$ *is an entire map of* $\mathbb{C}^n$*, where* A *is an* $n \times n$ *matrix of rank* r *with* $\|A\| \leqslant 1$ *and* $0 < r \leqslant n$*.* (Ψ, Φ) *is the normalization of* (ψ, φ) *with respect to the singular value decomposition* $A = U_1 \Lambda U_2$*. Then*

(i) $W_{\psi,\varphi}$ *is bounded on* $\mathscr{F}^{p,m}(\mathbb{C}^n)$ *if and only if* $M(\Psi, \Phi) < \infty$.

(ii) $W_{\psi,\varphi}$ *is compact on* $\mathscr{F}^{p,m}(\mathbb{C}^n)$ *if and only if* $\lim_{|z_{[r]}| \to \infty} M_{z_{[r]}}(\Psi, \Phi) = 0$.

Proof. If $W_{\psi,\varphi}$ is bounded, then so is $W_{\Psi,\Phi}$. By Lemma 2.8, we have

$$\|W_{\Psi,\Phi} K_{m,w}\|^m_{p,m} \gtrsim F^p_{\Psi K_{m,w} \circ \Phi, p}(z_{[r]}) e^{-\frac{p}{2}|z_{[r]}|^2}$$

for each $w \in \mathbb{C}^n$. Since $K_{m,w}(\Phi(z))$ is only dependent on $z_{[r]}$, we have

$$F_{\Psi K_{m,w} \circ \Phi, p}(z_{[r]}) = |K_{m,w}(\Phi(z))| F_{\Psi,p}(z_{[r]}).$$

In particular, taking $w=\Phi(z)$ yields that

$$\frac{F_{\Psi,p}(z_{[r]})}{1+|\Phi(z)|^m}e^{\frac{|\Phi(z)|^2-|z_{[r]}|^2}{2}}\lesssim \|W_{\Psi,\Phi}\|\leqslant \|W_{\psi,\varphi}\|$$

for all $z_{[r]}\in\mathbb{C}^n$.

Conversely, denote by $\hat{b}=U_1^*b$. Then for any $f\in\mathscr{F}^{p,m}(\mathbb{C}^n)$, we have

$$\begin{aligned}
&\|W_{\Psi,\Phi}f\|_{p,m}^p\\
\lesssim&\int_{\mathbb{C}^n}|\Psi(z)|^p|f\circ\Phi(z)|^p|z|^{mp}e^{-\frac{p}{2}|z|^2}dV_n(z)\\
\simeq&\int_{\mathbb{C}^r}|f\circ\Phi(z)|^pF_{\Psi,p}^p(z_{[r]})e^{-\frac{p}{2}|z_{[r]}|^2}dV_r(z_{[r]})\\
\lesssim& M(\Psi,\Phi)^p\int_{\mathbb{C}^r}|f\circ\Phi(z)|^p(1+|\Phi(z)|^m)^pe^{-\frac{p}{2}|\Phi(z)|^2}dV_r(z_{[r]})\\
\simeq& M(\Psi,\Phi)^p|det\Lambda_{[r]}^{-1}|^2\\
&\times\int_{\mathbb{C}^r}|f(w_{[r]},\hat{b}'_{[r]})|^p(1+|(w_{[r]},\hat{b}'_{[r]})|^m)^pe^{-\frac{p}{2}(|(w_{[r]}|^2+|\hat{b}'_{[r]})|^2)}dV_r(w_{[r]})
\end{aligned}$$

Then it follows from Lemma 2.8 that $\|W_{\Psi,\Phi}f\|_{p,m}^p\lesssim M(\Psi,\Phi)^p|det\Lambda_{[r]}^{-1}|^2\|f\|_{p,m}^p$, which implies that $W_{\Psi,\Phi}$ is bounded, then so is $W_{\psi,\varphi}$. Furthermore, $\|W_{\psi,\varphi}\|\simeq M(\Psi,\Phi)$.

Now we assume that $W_{\psi,\varphi}$ is compact, then so is $W_{\Psi,\Phi}$. For each $w\in\mathbb{C}^n$, consider the function $k_{m,w}(z)=\frac{1+|w|^m}{e^{|w|^2/2}}K_{m,w}(z)$, $z\in\mathbb{C}^n$. Then $\|k_{m,w}\|_{p,m}\simeq 1$ and converges to zero as $|w|\to\infty$ uniformly on compact subsets of $\mathbb{C}^n$. It follows that $k_{m,w}$ converges to zero weakly in $\mathscr{F}^{p,m}(\mathbb{C}^n)$. Thus

$$\lim_{|w|\to\infty}\|W_{\Psi,\Phi}k_{m,w}\|_{p,m}=0. \tag{2.3}$$

By the proof of the necessity for the boundedness, we have

$$M_{z_{[r]}}(\Psi,\Phi)\lesssim\|W_{\Psi,\Phi}k_{m,w}(\Phi(z))\|_{p,m}.$$

This, together with (2.3), implies that $\lim_{|z_{[r]}|\to\infty}M_{z_{[r]}}(\Psi,\Phi)=0$.

Conversely, let $\{f_k\}$ be an arbitrary bounded sequence in $\mathscr{F}^{p,m}(\mathbb{C}^n)$, which converges to zero as $k\to\infty$ uniformly on compact subsets of $\mathbb{C}^n$. Then for any $R>0$, we have

$$\begin{aligned}
\|W_{\Psi,\Phi}f_k\|_{p,m}^p&\lesssim\int_{\mathbb{C}^n}|\Psi(z)|^p|f_k\circ\Phi(z)|^p|z|^{mp}e^{-\frac{p}{2}|z|^2}dV_n(z)\\
&\lesssim\int_{\mathbb{C}^r}|f_k\circ\Phi(z)|^pF_{\Psi,p}^p(z_{[r]})e^{-\frac{p}{2}|z_{[r]}|^2}dV_r(z_{[r]})\\
&\lesssim\left(\int_{|z_{[r]}|\leqslant R}+\int_{|z_{[r]}|>R}\right)|f_k\circ\Phi(z)|^pF_{\Psi,p}^p(z_{[r]})e^{-\frac{p}{2}|z_{[r]}|^2}dV_r(z_{[r]})\\
&:=I_1(R)+I_2(R),
\end{aligned}$$

where

$$\begin{aligned} I_1(R) &= \int_{|z_{[r]}|\leqslant R} |f_k\circ\Phi(z)|^p F^p_{\Psi,p}(z_{[r]}) e^{-\frac{p}{2}|z_{[r]}|^2} dV_r(z_{[r]}) \\ &\lesssim \|\Psi\|^p_{p,m} \max_{|z_{[r]}|\leqslant R} |f_k\circ\Phi(z)|^p. \end{aligned}$$

And by the proof of the sufficiency for the boundedness, we have

$$\begin{aligned} I_2(R) &= \int_{|z_{[r]}|> R} |f_k\circ\Phi(z)|^p F^p_{\Psi,p}(z_{[r]}) e^{-\frac{p}{2}|z_{[r]}|^2} dV_r(z_{[r]}) \\ &\lesssim |det\Lambda^{-1}_{[r]}|^2 \|f_k\|^p_{p,m} \left(\sup_{|z_{[r]}|>R} M_{z_{[r]}}(\Psi,\Phi)\right)^p. \end{aligned}$$

Letting $R\to\infty$ and then $k\to\infty$ yields that $\lim_{k\to\infty}\|W_{\Psi,\Phi}f_k\|=0$, which shows that $W_{\Psi,\Phi}$ is compact by [8, Lemma 3.2], then so is $W_{\psi,\varphi}$. The proof is complete. □

3. Unitary weighted composition operators

In this section, we investigate unitary weighted composition operators on $\mathscr{F}^{2,m}(\mathbb{C}^n)$. When $m=0$, it was studied by Zhao in [19], which showed that $W_{\psi,\varphi}$ is unitary on $\mathscr{F}^2(\mathbb{C}^n)$ if and only if there exist an unitary matrix A, a vector $b\in\mathbb{C}^n$ and a constant α with $|\alpha|=1$ such that $\varphi(z)=Az+b$ and $\psi(z)=\alpha e^{\langle z,A^{-1}b\rangle-\frac{|b|^2}{2}}$. However, we will show that no nontrivial unitary weighted composition operators exist on $\mathscr{F}^{2,m}(\mathbb{C}^n)$ when $m\geqslant 1$, which corresponds with the result on the Dirichlet space over the unit disk (see [12]).

Firstly, by modifying the proof of [18, Theorem 8], we can characterize the invertible bounded weighted composition operators on $\mathscr{F}^{2,m}(\mathbb{C}^n)$, and we omit the details of the proof.

PROPOSITION 3.1. *Let $\psi\in H(\mathbb{C}^n)$ and φ be an entire map on $\mathbb{C}^n$. Then the operator $W_{\psi,\varphi}$ is invertible on $\mathscr{F}^{2,m}(\mathbb{C}^n)$ if and only if the following two conditions holds:*

(i) $\varphi(z)=Az+b$, where A is an invertible $n\times n$ matrix with $\|A\|=\|A^{-1}\|=1$ and $b\in\mathbb{C}^n$.

(ii) there exist positive constants M_1 and M_2, such that $M_1\leqslant m_z(\psi,\varphi)\leqslant M_2$ for all $z\in\mathbb{C}^n$.

For any $b\in\mathbb{C}^n$, let $\varphi_b(z)=z-b$. By Proposition 2.3 and Theorem 2.4, if W_{ψ,φ_b} is bounded on $\mathscr{F}^{2,m}(\mathbb{C}^n)$, then $\psi(z)=\psi(0)e^{\langle z,b\rangle}$. In this case,

$$W^*_{\psi,\varphi_b}K_{m,b}(z)=\overline{\psi(b)}K_{m,\varphi_b(b)}(z)=\overline{\psi(0)}e^{|b|^2} \tag{3.1}$$

for every $z\in\mathbb{C}^n$. We will prove $b=0$ if W_{ψ,φ_b} is unitary on $\mathscr{F}^{2,m}(\mathbb{C}^n)$ when $m\geqslant 1$.

LEMMA 3.2. *Suppose m is a positive integer, then for any $b\in\mathbb{C}^n$,*

$$\int_{\mathbb{C}^n}|z+b|^{2m}e^{-|z|^2}dV_n(z)\geqslant\int_{\mathbb{C}^n}|z|^{2m}e^{-|z|^2}dV_n(z).$$

Equality holds if and only if $b=0$.

Proof. Without loss of generality, by taking a proper unitary transform if necessary, we may assume that $b=|b|e_1$, where $e_1=(1,0,\cdots,0)$. Then we have

$$\begin{aligned}\int_{\mathbb{C}^n}|z+b|^{2m}e^{-|z|^2}dV_n(z)&=\int_{\mathbb{C}^n}(|z|^2+|b|^2+2Re|b|z_1)^m e^{-|z|^2}dV_n(z)\\&=\int_{\mathbb{C}^n}|z|^{2m}e^{-|z|^2}dV_n(z)+I(b),\end{aligned}$$

where

$$I(b)=\sum_{\substack{k_1\neq m\\ k_1+k_2+k_3=m}}C_{k_1,k_2,k_3}\int_{\mathbb{C}^n}|z|^{2k_1}|b|^{2k_2}(2|b|Rez_1)^{k_3}e^{-|z|^2}dV_n(z)$$

and $C_{k_1,k_2,k_3}>0$.

We integrate in polar coordinates to get

$$\begin{aligned}&\int_{\mathbb{C}^n}|z|^{2k_1}|b|^{2k_2}(2|b|Rez_1)^{k_3}e^{-|z|^2}dV_n(z)\\&=2^{k_3+1}n|b|^{2k_2+k_3}\int_0^\infty r^{2n+2k_1+k_3-1}e^{-r^2}dr\int_{\mathbb{S}_n}(Re\zeta_1)^{k_3}d\sigma_n(\zeta).\end{aligned}$$

Here, $\mathbb{S}_n=\{z\in\mathbb{C}^n:|z|=1\}$ and $d\sigma_n$ is the normalized area measure on $\mathbb{S}_n$. If $n=1$, then we have

$$\int_{\mathbb{S}_1}(Re\zeta)^{k_3}d\sigma_1(\zeta)=\frac{1}{2\pi}\int_0^{2\pi}cos^{k_3}\theta d\theta.$$

It is easy to verify that $\int_0^{2\pi}cos^{k_3}\theta d\theta\geqslant 0$ when k_3 is even and $\int_0^{2\pi}cos^{k_3}\theta d\theta=0$ when k_3 is odd. If $n\geqslant 2$, then by [22, Lemma 1.9], we have

$$\begin{aligned}\int_{\mathbb{S}^n}(Re\zeta_1)^{k_3}d\sigma(\zeta)&=(n-1)\int_{\mathbb{D}}(1-|z|^2)^{n-2}(Rez)^{k_3}dA(z)\\&=(n-1)\int_0^{2\pi}cos^{k_3}\theta d\theta\int_0^1 r^{k_3-1}(1-r^2)^{n-2}dr\geqslant 0.\end{aligned}$$

Here, $\mathbb{D}$ denotes the unit disk of the complex plane $\mathbb{C}$ and dA denotes the normalized area measure on $\mathbb{C}$. Consequently, we get $I(b)\geqslant 0$ and equality holds if and only if $b=0$. □

PROPOSITION 3.3. *Let m be a positive integer and $\psi\in H(\mathbb{C}^n)$. Then W_{ψ,φ_b} is unitary on $\mathscr{F}^{2,m}(\mathbb{C}^n)$ if and only if $b=0$ and ψ is a constant function of unimodule.*

Proof. The sufficiency is trivial. Now we assume W_{ψ,φ_b} is unitary on $\mathbb{C}^n$, then we have

$$K_{m,w}(z) = W_{\psi,\varphi_b}W^*_{\psi,\varphi_b}K_{m,w}(z) = \psi(z)\overline{\psi(w)}K_{m,\varphi_b(w)}(\varphi_b(z))$$

for all $z,\ w \in \mathbb{C}^n$. Taking $z = w = 0$, we get

$$|\psi(0)|^2\|K_{m,b}\|^2_{2,m} = |\psi(0)|^2K_{m,b}(b) = 1. \tag{3.2}$$

Because every unitary operator is an isometry, we have $\|W^*_{\psi,\varphi_b}K_{m,b}\|_{2,m} = \|K_{m,b}\|_{2,m}$. This, together with (3.1) and (3.2), implies that $|\psi(0)|^2e^{|b|^2} = 1$. Notice that $\psi(z) = \psi(0)e^{\langle z,b\rangle}$, then by a change of variables, we obtain

$$\|W_{\psi,\varphi_b}f\|^2_{2,m} = \omega(n,2,m)\int_{\mathbb{C}^n}|f(z)|^2|z+b|^{2m}e^{-|z|^2}dV_n(z)$$

for every $f \in \mathscr{F}^{2,m}(\mathbb{C}^n)$. In particular, taking $f = 1$, then Lemma 3.2 tells us that $b = 0$. And then ψ is a constant function of unimodule. $\square$

THEOREM 3.4. *Let m be a positive integer, $\psi \in H(\mathbb{C}^n)$ and φ be an entire map on $\mathbb{C}^n$. Then $W_{\psi,\varphi}$ is unitary on $\mathscr{F}^{2,m}(\mathbb{C}^n)$ if and only if $\varphi(z) = Az$ for some unitary $n\times n$ matrix A and ψ is a constant function of unimodule.*

Proof. If $W_{\psi,\varphi}$ is unitary, then by Proposition 3.1, we have $\varphi(z) = Az + b$ for some invertible $n\times n$ matrix with $\|A\| = \|A^{-1}\| = 1$ and $b \in \mathbb{C}^n$. Assume $A = U_1\Lambda U_2$ is the singular value decomposition of A, then Λ is invertible and $\|\Lambda\| = \|\Lambda^{-1}\| = 1$, which shows that Λ must be an identity matrix. Denote by (Ψ,Φ) the normalization of (ψ,φ), where $\Psi(z) = \psi(U_2^*z)$ and $\Phi(z) = z + U_1^*b$. It follows from Proposition 3.3 that $U_1^*b = 0$ and Ψ is a constant function of unimodule. Therefore, $b = 0$ and ψ is a constant function of unimodule. Furthermore, we have $W_{\psi,\varphi}W^*_{\psi,\varphi}K_{m,w}(z) = K_{m,w}(z)$ for all $z,\ w \in \mathbb{C}^n$. By taking $w = z$, we obtain $|\varphi(z)| = |z|$ for every $z \in \mathbb{C}^n$, which implies that A is a unitary matrix.

The sufficiency is trivial. We complete the proof. $\square$

4. Self-adjoint and $\mathscr{J}$-symmetry

A bounded operator T on a separate complex Hilbert space $\mathscr{H}$ is called complex symmetric if there exists a conjugation $\mathscr{C}$, such that $T = \mathscr{C}T^*\mathscr{C}$. In this case, T is also called $\mathscr{C}$-symmetric. Here, a *conjugation* is a conjugate linear, isometric involution on $\mathscr{H}$. Precisely, $\mathscr{C}$ is called a conjugation on $\mathscr{H}$ if it satisfies the following conditions:

(i) $\mathscr{C}(\lambda f + \mu g) = \overline{\lambda}\mathscr{C}f + \overline{\mu}\mathscr{C}g$ for all $f,\ g \in \mathscr{H}$ and $\lambda,\ \mu \in \mathbb{C}$;

(ii) $\langle \mathscr{C}f, \mathscr{C}g\rangle = \langle g, f\rangle$ for all $f,\ g \in \mathscr{H}$;

(iii) $\mathscr{C}^2 = I$ is the identity map on $\mathscr{H}$.

For example, for any $f \in H(\mathbb{C}^n)$ and $z \in \mathbb{C}^n$, let $(\mathscr{J} f)(z) = \overline{f(\overline{z})}$, then $\mathscr{J}$ is a conjugation on $\mathscr{F}^{2,m}(\mathbb{C}^n)$. In [6] and [7], Hai and Khoi studied complex symmetric weighted composition operators on the classical Fock space. In [9], Hu, Yang and Zhou proved that $W_{\psi,\varphi}$ is $\mathscr{J}$-symmetric on the Dirichlet space in the unit ball if and only if $W_{\psi,\varphi}$ is normal. In this section, we firstly characterize self-adjoint weighted composition operators on $\mathscr{F}^{2,m}(\mathbb{C}^n)$ with $m \geqslant 1$ and as an application, we show that there exist no nontrivial $\mathscr{J}$-symmetric weighted composition operators on $\mathscr{F}^{2,m}(\mathbb{C}^n)$ with $m \geqslant 1$, which corresponds with the result on the Dirichlet space in the unit ball.

For each $w \in \mathbb{C}^n$ and each multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, we denote by $K_{m,w}^{[\alpha]}$ the reproducing kernel for the partial derivative of mixed order α at w, that is

$$(\partial^\alpha f)(w) = \langle f, K_{m,w}^{[\alpha]} \rangle_m, \quad f \in \mathscr{F}^{2,m}(\mathbb{C}^n).$$

It can be shown that $K_{m,w}^{[\alpha]}(z) = \frac{\partial^{|\alpha|} K_{m,w}(z)}{\partial \overline{w_1}^{\alpha_1} \cdots \partial \overline{w_n}^{\alpha_n}}$.

LEMMA 4.1. *Let m be a nonnegative integer, $\psi \in H(\mathbb{C}^n)$ and $\varphi(z) = Az$ for some $n \times n$ matrix with $\|A\| \leqslant 1$. If $W_{\psi,\varphi}$ is self-adjoint on $\mathscr{F}^{2,m}(\mathbb{C}^n)$, then A is self-adjoint and $\psi \equiv c$ for some $c \in \mathbb{R}$.*

Proof. If $W_{\psi,\varphi}$ is self-adjoint on $\mathscr{F}^{2,m}(\mathbb{C}^n)$, then we have

$$\overline{\psi(w)} K_{m,Aw}(z) = (W_{\psi,\varphi}^* K_{m,w})(z) = (W_{\psi,\varphi} K_{m,w})(z) = \psi(z) K_{m,w}(Az) \tag{4.1}$$

for any $z, w \in \mathbb{C}^n$. In particular, with $w = 0$, we have $\psi(z) \equiv \overline{\psi(0)}$. And then by taking $z = 0$, we get $\psi(z) \equiv \psi(0) \in \mathbb{R}$. Therefore, according to (4.1), we have

$$K_{m,Aw}(z) = K_{m,w}(Az)$$

for any $z, w \in \mathbb{C}$. It follows that $\langle z, Aw \rangle = \langle Az, w \rangle$ for any $z, w \in \mathbb{C}^n$. Thus A is self-adjoint. The proof is complete. □

LEMMA 4.2. *Let m be a nonnegative integer, $\psi \in H(\mathbb{C}^n)$, $\varphi(z) = Az + b$ for some nonzero matrix A with $\|A\| \leqslant 1$ and $b \in \mathbb{C}^n$. Suppose $A_j = 0$ for some $j \in \{1, 2, \ldots, n\}$, here A_j denotes the j-th column of A. If $W_{\psi,\varphi}$ is self-adjoint on $\mathscr{F}^{2,m}(\mathbb{C}^n)$, then $b = 0$ and A is self-adjoint.*

Proof. Suppose $W_{\psi,\varphi}$ is self-adjoint on $\mathscr{F}^{2,m}(\mathbb{C}^n)$, then for any $z, w \in \mathbb{C}^n$, we have

$$\psi(z) K_{m,w}(\varphi(z)) = (W_{\psi,\varphi} K_{m,w})(z) = (W_{\psi,\varphi}^* K_{m,w})(z) = \overline{\psi(w)} K_{m,\varphi(w)}(z). \tag{4.2}$$

In particular, with $w = 0$, we get

$$\psi(z) = \overline{\psi(0)} K_{m,\varphi(0)}(z). \tag{4.3}$$

And the by taking $z=0$, we get $\psi(0)\in\mathbb{R}$. Therefore, according to (4.2) and (4.3), we obtain

$$K_{m,\varphi(0)}(z)K_{m,w}(\varphi(z)) = K_{m,w}(\varphi(0))K_{m,\varphi(w)}(z).$$

Let $w=w^{(j)}=(0,\dots,w_j,\dots,0)$, $w_j\neq 0$, then $\varphi(w^{(j)})=\varphi(0)=b$. If $b\neq 0$, then

$$K_{m,w^{(j)}}(\varphi(0)) = K_{m,w^{(j)}}(\varphi(z))$$

for all $z\in\mathbb{C}^n$, which is impossible since $A\neq 0$. This contradiction shows that $b=0$. Then A is self-adjoint by Lemma 4.1. $\square$

We note that under the condition of Lemma 4.2, we have $b=0$ whenever A is not invertible.

THEOREM 4.3. *Let m be a positive integer, $\psi\in H(\mathbb{C}^n)$ and φ be an entire map on $\mathbb{C}^n$. Then $W_{\psi,\varphi}$ is self-adjoint on $\mathscr{F}^{2,m}(\mathbb{C}^n)$ if and only if there exist a self-adjoint matrix A with $\|A\|\leqslant 1$ and a constant $c\in\mathbb{R}$ such that $\varphi(z)=Az$ and $\psi\equiv c$.*

Proof. We begin to prove the sufficiency. If $\varphi(z)=Az$ and $\psi\equiv c$, where A is self-adjoint and $c\in\mathbb{R}$. Then we have

$$W^*_{\psi,\varphi}K_{m,w}(w) = \overline{\psi(w)}K_{m,Aw}(z) = cK_{m,w}(Az) = W_{\psi,\varphi}K_{m,w}(z)$$

for any $z,w\in\mathbb{C}^n$. Since the space spanned by the reproducing kernel functions is dense in $\mathscr{F}^{2,m}(\mathbb{C}^n)$, thus $W^*_{\psi,\varphi}f=W_{\psi,\varphi}f$ for any $f\in\mathscr{F}^{2,m}(\mathbb{C}^n)$, which shows that $W_{\psi,\varphi}$ is self-adjoint.

Now we prove the necessity. By Theorem 2.4, we know that $\varphi(z)=Az+b$ for some matrix A with $\|A\|\leqslant 1$ and $b\in\mathbb{C}^n$. Then by Lemma 4.1, it is enough to prove $b=0$. Through a simple calculation, we have

$$W^*_{\psi,\varphi}K^{[e_j]}_{m,w}(z) = \overline{(\partial_j\psi)(w)}K_{m,\varphi(w)}(z) + \overline{\psi(w)}\sum_{k=1}^{n}\overline{(\partial_j\varphi_k)(w)}K^{[e_k]}_{m,\varphi(w)}(z)$$

for all $z,\ w\in\mathbb{C}^n$ and $j=1,2,\dots,n$. Here e_j is the multi-index that has 1 in the jth spot and 0 everywhere else and φ_k is the kth coordinate function of φ. If $W_{\psi,\varphi}$ is self-adjoint, then

$$\psi(z)K^{[e_j]}_{m,w}(\varphi(z)) = \overline{(\partial_j\psi)(w)}K_{m,\varphi(w)}(z) + \overline{\psi(w)}\sum_{k=1}^{n}\overline{(\partial_j\varphi_k)(w)}K^{[e_k]}_{m,\varphi(w)}(z)$$

for all $z,\ w\in\mathbb{C}^n$ and $j=1,2,\dots,n$.

In particular, by taking $w=0$, we get

$$\frac{n}{n+m}\psi(z)\varphi_j(z) = \overline{(\partial_j\psi)(0)}K_{m,\varphi(0)}(z) + \overline{\psi(0)}\sum_{k=1}^{n}\overline{(\partial_j\varphi_k)(0)}K^{[e_k]}_{m,\varphi(0)}(z). \tag{4.4}$$

Then according to (4.3) and (4.4), we have

$$\frac{n}{n+m}\varphi_j(z) = \frac{\overline{(\partial_j\psi)(0)}}{\overline{\psi(0)}} + \sum_{k=1}^{n} \frac{\overline{(\partial_j\varphi_k)(0)} K^{[e_k]}_{m,\varphi(0)}(z)}{K_{m,\varphi(0)}(z)}, \quad j = 1,2,\dots,n. \tag{4.5}$$

If there is some $j \in \{1,2,\dots,n\}$, such that $\left(\partial_j\varphi_{k(j)}\right)(0) = 0$ for all $k = 1,2,\dots,n$, then $b = 0$ by Lemma 4.2. So we assume that for any $j \in \{1,2,\dots,n\}$, there is some $k(j)$, such that $(\partial_j\varphi_k(j))(0) \neq 0$, and $k(j)$ runs over $1,2,\dots,n$ when j runs over $1,2,\dots,n$. Otherwise some row of A must be 0, which then implies $b = 0$ by Lemma 4.2. Taking $z = z^{(k(j))} = (0,\dots,z_{k(j)},\dots,0)$ in (4.5), we obtain

$$\frac{n}{n+m}\varphi_j(z^{(k(j))}) = \frac{\overline{(\partial_j\psi)(0)}}{\overline{\psi(0)}} + \overline{(\partial_j\varphi_{k(j)})(0)} \frac{K^{[e_{k(j)}]}_{m,\varphi(0)}(z^{k(j)})}{K_{m,\varphi(0)}(z^{k(j)})}.$$

Since φ_j is linear with respect to $z_{k(j)}$ and $m \geqslant 1$, we must have $\varphi_{k(j)}(0) = 0$. Therefore, $b = 0$. The proof is complete. $\square$

THEOREM 4.4. *Let m be a positive integer, $\psi \in H(\mathbb{C}^n)$ and φ be an entire map on $\mathbb{C}^n$. Then $W_{\psi,\varphi}$ is $\mathscr{J}$-symmetric on $\mathscr{F}^{2,m}(\mathbb{C}^n)$ if and only if there exist a symmetric matrix A with $\|A\| \leqslant 1$ and a constant $c \in \mathbb{C}$ such that $\varphi(z) = Az$ and $\psi \equiv c$.*

Proof. The proof for the sufficiency is similar to Theorem 4.3. So we only need to prove the necessity. If $W_{\psi,\varphi}$ is $\mathscr{J}$-symmetric, then

$$W_{\psi,\varphi}\mathscr{J}K_{m,w}(z) = \mathscr{J}W^*_{\psi,\varphi}K_{m,w}(z)$$

and

$$W_{\psi,\varphi}\mathscr{J}K^{[e_j]}_{m,w}(z) = \mathscr{J}W^*_{\psi,\varphi}K^{[e_j]}_{m,w}(z)$$

for all $z,\ w \in \mathbb{C}^n$ and $j = 1,\cdots,n$. It follows that

$$\psi(z)K_{m,\overline{w}}(\varphi(z)) = \psi(w)K_{m,\overline{\varphi(w)}}(z) \tag{4.6}$$

and

$$\psi(z)K^{[e_j]}_m(\varphi(z),\overline{w}) = (\partial_j\psi)(w)K_{m,\overline{\varphi(w)}}(z) + \psi(w)\sum_{k=1}^{n}(\partial_j\varphi_k)(w)K^{[e_k]}_{m,\overline{\varphi(w)}}(z)$$

for all $z,\ w \in \mathbb{C}^n$ and $j = 1,\cdots,n$. According to the argument of Theorem 4.3, we get $\varphi(z) = Az$ for some matrix A with $\|A\| \leqslant 1$ and $\psi \equiv c$ for some $c \in \mathbb{C}$. Take these into (4.6), we have $\langle Az,\overline{w}\rangle = \langle z,\overline{Aw}\rangle$ for all $z,\ w \in \mathbb{C}^n$, which implies A is symmetric. The proof is complete. $\square$

LEMMA 4.5. *If U is a unitary symmetric matrix, then $\mathscr{J}C_{Uz}$ if a conjugation on $\mathscr{F}^{2,m}(\mathbb{C}^n)$.*

Proof. If U is a symmetric unitary matrix, then by Theorem 3.4 and Theorem 4.4, we have C_{Uz} if unitary and $\mathscr{J}$-symmetric on $\mathscr{F}^{2,m}(C^n)$. And the result follows from a direct calculation. $\square$

THEOREM 4.6. *Let m be a positive integer, $\psi \in H(\mathbb{C}^n)$ and φ be an entire map on $\mathbb{C}^n$. Suppose U is a symmetric unitary matrix. Then $W_{\psi,\varphi}$ is complex symmetric with respect to the conjugation $\mathscr{J} C_{Uz}$ if and only if there exists a symmetric matrix A with $\|A\| \leqslant 1$, which commutes with U such that $\varphi(z) = AUz$ and $\psi \equiv c$ for some $c \in \mathbb{C}$.*

Proof. The sufficiency follows from a simple calculation. Now we assume $W_{\psi,\varphi}$ is $\mathscr{J} C_{Uz}$-symmetric, then we have

$$\mathscr{J} C_{Uz} W_{\psi,\varphi} = W_{\psi,\varphi}^* \mathscr{J} C_{Uz} = (C_{uz} W_{\psi,\varphi})^* \mathscr{J}.$$

By Theorem 4.4, we get $\varphi \circ U(z) = Az$ for some symmetric matrix A with $\|A\| \leqslant 1$ and $\psi \circ U \equiv c$ for some $c \in \mathbb{C}$. It follows that $\varphi(z) = AUz$ and $\psi \equiv c$. Let $\widetilde{\varphi} = \varphi \circ U$, then $C_{\widetilde{\varphi}}$ is $\mathscr{J}$-symmetric, which implies AU is symmetric and then $AU = UA$. $\square$

5. Further remarks

In [11], Le characterized normal weighted composition operators on the classical Fock space $\mathscr{F}^2(\mathbb{C})$. Then Zhao, in [20], extends Le's results to several variables. In [12], the authors proved no nontrivial normal weighted composition operators exist on the Dirichlet space on the unit disc. Therefore, we have the following conjecture:

$W_{\psi,\varphi}$ is normal on $\mathscr{F}^{2,m}(\mathbb{C}^n)$ with $m \geqslant 1$ if and only if $\varphi(z) = Az$ for some normal matrix A with $\|A\| \leqslant 1$ and $\psi \equiv c$ for some $c \in \mathbb{C}$.

6. Declarations

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