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NUMERICAL RADII OF WEIGHTED SHIFT OPERATORS USING DETERMINANTAL POLYNOMIALS

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Abstract. In this paper, we introduce the expression of the determinantal polynomials for the weighted shift operators with weights

$$(w_1,\dots,w_{2n-1},b,a,b,a,b,\dots) \quad \text{and} \quad (w_1,\dots,w_{2n},a,b,a,b,\dots)$$

and using these we can find the numerical radii of the above operators. The purpose of this paper is to generalize the results of [14] and [4].

1. Introduction

Let $\mathscr{H}$ be a complex, separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and $B(\mathscr{H})$ denotes the set of all bounded linear operators on $\mathscr{H}$. For $T\in B(\mathscr{H})$, the *numerical range* of T is the subset of the complex plane $\mathbb{C}$ defined as

$$W(T)=\big\{\langle Tx,x\rangle:\ x\in\mathscr{H},\ \|x\|=1\big\},$$

and the *numerical radius* of T is defined as

$$w(T)=\sup\{|z|:z\in W(T)\}.$$

It is known that $W(T)$ is a nonempty, bounded and convex subset of $\mathbb{C}$ (see [9, 8]). Throughout the paper, $\mathrm{Re}(T)=\frac{T+T^*}{2}$ denotes the real part of the operator T.

Let T be a weighted shift matrix with nonnegative weights $(w_1,\dots,w_{n-1})$ represented as follows,

$$T=T(w_1,\dots,w_{n-1})=\begin{pmatrix} 0 & & & & \\ w_1 & 0 & & & \\ & w_2 & 0 & & \\ & & \ddots & \ddots & \\ & & & w_{n-1} & 0 \end{pmatrix}.$$

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The characteristic polynomial of $\mathrm{Re}(T)$ denoted as

$$p_n(t) = \det\big(tI_n - \mathrm{Re}(T(w_1, w_2, \ldots, w_{n-1}))\big)$$

has the recurrence

$$p_n(t) = t p_{n-1}(t) - \frac{1}{4} w_{n-1}^2 p_{n-2}(t).$$

From Lemma 1 of [13], we have

$$p_n(t) = t^n + \sum_{1 \leqslant k \leqslant n/2} \left(\frac{-1}{4}\right)^k S_k(w_1, \ldots, w_{n-1}) t^{n-2k},$$

where the circularly symmetric function is

$$S_k(w_1, w_2, \ldots, w_{n-1}) = \sum w_{i_1}^2 w_{i_2}^2 \cdots w_{i_k}^2,$$

the sum being taken over

$$1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n-1, \;\; i_2 - i_1 \geqslant 2, \;\; i_3 - i_2 \geqslant 2, \ldots, i_k - i_{k-1} \geqslant 2.$$

To avoid any confusion, the circularly symmetric function $S_k(w_1, w_2, \ldots, w_{n-1})$ is denoted by $S_k^{(n-1)}$.

Let T be a weighted shift operator with bounded weights $(w_1, w_2, \ldots)$ on the Hilbert space $\ell^2(\mathbb{N})$ represented by the infinite matrix as follows,

$$T = T(w_1, w_2, \ldots) = \begin{pmatrix} 0 & & & & \\ w_1 & 0 & & & \\ & w_2 & 0 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \ddots \\ & & & & \ddots \end{pmatrix}.$$

Weighted shift operators are special tridiagonal operators. Periodic tridiagonal operators and their generalization are studied in [2, 3].

Since a weighted shift operator T is unitarily equivalent to $e^{i\theta}T$ for any real θ, its numerical range is always open or closed circular disc centered at the origin (see [13]). In particular, $W\big(T(1, 1, \ldots)\big)$ is an open unit disc centered at the origin (see [10]).

Define a unitary operator

$$U = \mathrm{diag}(u_1, u_1 u_2, u_1 u_2 u_3, \ldots),$$

where $\{u_n : n = 1, 2, 3, \ldots\}$ is a sequence of complex numbers with $|u_n| = 1$, $n = 1, 2, 3, \ldots$. Then the operator UTU^* is a weighted shift operator with weights

$$(u_2 w_1, u_3 w_2, u_4 w_3, \ldots, u_{n+1} w_n, \ldots),$$

by choosing $u_1 = 1$, $u_{n+1} = \overline{w_n}/|w_n|$ if $w_n \neq 0$, and $u_{n+1} = 1$ if $w_n = 0$. Then

$$UTU^* = |T|,$$

where $|T|$ is the entrywise absolute value $(|a_{ij}|)$ of the operator $T = (a_{ij})$. So one can always assume that the weights of a weighted shift operator are nonnegative.

In 1976, Ridge [12] has given a description of numerical radius for a weighted shift operator T of period p. He has shown that, for a weighted shift operator T with weights of periodic sequence a, b, the numerical radius is $\frac{a+b}{2}$. The computations of numerical radii of weighted shift operators with various weights such as $(w_1, 1, 1, \ldots)$, $(1, w_2, 1, 1, \ldots)$, $(w_1, w_2, 1, 1, \ldots)$ and $(w_1, w_2, a, b, a, b, \ldots)$ have done in [1, 5, 15, 4].

In 1983, Stout [13] has provided an algorithm to get the numerical radius of a weighted shift operator $T(w_1, w_2, \ldots)$ with square summable weights by introducing the analytic function

$$\begin{aligned} F_T(z) &= \det\big(I - z\mathrm{Re}(T(w_1, w_2, \ldots))\big) \\ &= 1 + \sum_{k=1}^{\infty} \left(\frac{-1}{4}\right)^k c_k z^{2k}, \end{aligned}$$

where

$$c_k = \sum w_{i_1}^2 w_{i_2}^2 \cdots w_{i_k}^2,$$

the sum being taken over

$$1 \leqslant i_1 < i_2 < \cdots < i_k < \infty, \;\; i_2 - i_1 \geqslant 2, \;\; i_3 - i_2 \geqslant 2, \ldots, i_k - i_{k-1} \geqslant 2,$$

and the radius $w(T(w_1, w_2, \ldots)) = 1/\lambda$, where λ is the minimal positive root of $F_T(z) = 0$.

In 2015, Undrakh et al. [14] have determined the numerical radius of a weighted shift operator $T = T(w_1, \ldots, w_n, 1, 1, \ldots)$ in terms of the weighted shift matrix $T(w_1, \ldots, w_n)$. They have shown that if $w(T) > 1$, then $w(T) - \sqrt{w(T)^2 - 1}$ is the minimal positive root of the determinantal polynomial $F_n(z)$ defined by

$$F_n(z) = Q_{n-1}(z) - w_n^2 z^2 Q_{n-2}(z),$$

where the determinantal polynomial

$$Q_l(z) = \det\big((z^2 + 1)I_{l+1} - 2z\mathrm{Re}(T(w_1, \ldots, w_l))\big),$$

$l = 1, 2, \ldots$ with initials $Q_0(z) = z^2 + 1$, $Q_{-1}(z) = 1$. Recently, Chien et al. [6] have calculated the numerical radius of a weighted shift operator $T(w_1, w_2, \ldots)$ by introducing a q-analog expression of the determinantal polynomials $Q_n(z)$ and $F_n(z)$ of a weighted shift operator $T = T(w_1, w_2, \ldots)$ with weights (w_n) satisfying

$$\begin{aligned} &\lim_{n\to\infty} w_n = 1, \\ &\prod_{n=1}^{\infty} w_n = \beta \text{ for some } 0 < \beta < \infty, \end{aligned}$$

and

$$\sum_{n=1}^{\infty} |w_n^2 - 1| < \infty.$$

In this paper, we compute the expression of the determinantal polynomials for the weighted shift operators with weights $(w_1,\dots,w_{2n-1},b,a,b,a,b,\dots)$ and $(w_1,\dots,w_{2n},a,b,a,b,\dots)$. Further using these polynomials, we can find the numerical radii of the above operators. Thus we generalize the results given in [14, 4].

2. Weighted shift operators

In this section, we consider the weighted shift operators

$$T = T(w_1,w_2,\dots,w_{2n-1},b,a,b,a,b,\dots) \text{ and } T_1 = T_1(w_1,w_2,\dots,w_{2n},a,b,a,b,\dots)$$

with positive weights

$$(w_1,w_2,\dots,w_{2n-1},b,a,b,a,b,\dots) \text{ and } (w_1,w_2,\dots,w_{2n},a,b,a,b,\dots),$$

respectively acting on a complex separable Hilbert space $\ell^2(\mathbb{N})$ identified with the Hardy space H^2. Then $T = S+K$ and $T_1 = S+K_1$ where $S = S(a,b,a,b,\dots)$ and K, K_1 are compact operators. Hence, $W_e(T) = W_e(S) = W_e(T_1)$, where the *essential numerical range* $W_e(T)$ of an operator $T \in B(\mathscr{H})$ is defined as

$$W_e(T) = \bigcap \{\overline{W(T+K)} : K \text{ compact on } \mathscr{H}\}.$$

Properties of essential numerical range can be found in [7, 11].

PROPOSITION 2.1. *Let T be a weighted shift operator with positive weights $(w_1, w_2,\dots)$ satisfying the conditions*

$$\lim_{n\to\infty} w_{2n+1} = a \text{ and } \lim_{n\to\infty} w_{2n} = b \text{ where } a,b > 0.$$

Then $w(T) > \frac{a+b}{2}$ if and only if $\operatorname{Re}(T)$ has an eigenvalue greater than $\frac{a+b}{2}$.

Proof. If $w(T) > \frac{a+b}{2}$ then from Lemma 2.1 of [4] we can prove that $w(T)$ is an eigenvalue of $\operatorname{Re}(T)$. Conversely, if α is an eigenvalue of $\operatorname{Re}(T)$ greater than $\frac{a+b}{2}$ then $w(T) = w(\operatorname{Re}(T)) > \frac{a+b}{2}$. $\square$

LEMMA 2.2. *Let T be a weighted shift operator with weights $(w_1,w_2,\dots,w_{2n-1}, b,a,b,a,b,\dots)$ where $w_1,w_2,\dots,w_{2n-1},a,b > 0$. If there exists a non zero $f \in H^2$ such that $\operatorname{Re}(T)f = \alpha f$ then*

$$f'(0) = \frac{2\alpha}{w_1} f(0) \tag{1}$$

$$\frac{f^{(k)}(0)}{k!} = \frac{2\alpha}{w_k}\frac{f^{(k-1)}(0)}{(k-1)!} - \frac{w_{k-1}}{w_k}\frac{f^{(k-2)}(0)}{(k-2)!}, \text{ for } k = 2,\dots,2n-1 \tag{2}$$

$$\frac{f^{(2n)}(0)}{(2n)!} = \frac{2\alpha}{b}\frac{f^{(2n-1)}(0)}{(2n-1)!} - \frac{w_{2n-1}}{b}\frac{f^{(2n-2)}(0)}{(2n-2)!} \tag{3}$$

$$b\frac{f^{(2k-1)}(0)}{(2k-1)!} + a\frac{f^{(2k+1)}(0)}{(2k+1)!} = 2\alpha\frac{f^{(2k)}(0)}{(2k)!}, \text{ for } k = n, n+1, \dots \tag{4}$$

$$a\frac{f^{(2k)}(0)}{(2k)!} + b\frac{f^{(2k+2)}(0)}{(2k+2)!} = 2\alpha\frac{f^{(2k+1)}(0)}{(2k+1)!}, \text{ for } k = n, n+1, \dots \tag{5}$$

Here we denote $f^{(0)} = f(0) \neq 0$.

Proof. The weighted shift operator T on the Hardy space H^2 satisfies

$$\begin{aligned} Tf(z) =& f(0)w_1 z + f'(0)w_2 z^2 + \cdots + \frac{f^{(2n-2)}(0)}{(2n-2)!}w_{2n-1}z^{2n-1} + \frac{f^{(2n-1)}(0)}{(2n-1)!}bz^{2n} \\ &+ \frac{f^{(2n)}(0)}{(2n)!}az^{2n+1} + \cdots \end{aligned}$$

$$\begin{aligned} \text{and } T^* f(z) =& f'(0)w_1 + \frac{f''(0)}{2!}w_2 z + \cdots + \frac{f^{(2n-1)}(0)}{(2n-1)!}w_{2n-1}z^{2n-2} + \frac{f^{(2n)}(0)}{(2n)!}bz^{2n-1} \\ &+ \frac{f^{(2n+1)}(0)}{(2n+1)!}az^{2n} + \cdots \end{aligned}$$

where $f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \cdots \in H^2$. Let $\mathrm{Re}(T)f = \alpha f$ holds for some non-zero $f \in H^2$. Then we have

$$\begin{aligned} &\Big(f(0)w_1 z + f'(0)w_2 z^2 + \cdots + \frac{f^{(2n-2)}(0)}{(2n-2)!}w_{2n-1}z^{2n-1} + \frac{f^{(2n-1)}(0)}{(2n-1)!}bz^{2n} \\ &+ \frac{f^{(2n)}(0)}{(2n)!}az^{2n+1} + \cdots\Big) + \Big(f'(0)w_1 + \frac{f''(0)}{2!}w_2 z + \cdots + \frac{f^{(2n-1)}(0)}{(2n-1)!}w_{2n-1}z^{2n-2} \\ &+ \frac{f^{(2n)}(0)}{(2n)!}bz^{2n-1} + \frac{f^{(2n+1)}(0)}{(2n+1)!}az^{2n} + \cdots\Big) = 2\alpha f(z). \end{aligned} \tag{6}$$

Compare coordinates-wise Eq. (6) we get the required result. □

LEMMA 2.3. *Let* T *be a weighted shift operator with weights* $(w_1, w_2, \dots, w_{2n-1}, b, a, b, a, b, \dots)$ *where* $w_1, w_2, \dots, w_{2n-1}, a, b > 0$. *If there exists a non zero* $f \in H^2$ *such that* $\mathrm{Re}(T)f = \alpha f$ *then*

$$\begin{aligned} &bf(0) - \sum_{i=1}^{n}(w_{2i-1} - a)\left\{\frac{f^{(2i-2)}(0)}{(2i-2)!}z^{2i} + \frac{f^{(2i-1)}(0)}{(2i-1)!}z^{2i-1}\right\} \\ &- \sum_{i=1}^{n-1}(w_{2i} - b)\left\{\frac{f^{(2i-1)}(0)}{(2i-1)!}z^{2i+1} + \frac{f^{(2i)}(0)}{(2i)!}z^{2i}\right\} \\ &= \left(\frac{a+b}{2}\right)\left(z^2 - \frac{4\alpha}{a+b}z + 1\right)f(z) + \left(\frac{a-b}{2}\right)(z^2 - 1)f(-z). \end{aligned} \tag{7}$$

Proof. Let there exists nonzero f such that $\mathrm{Re}(T)f = \alpha f$. Then from Eq. (6) of Lemma 2.2 we have

$$\begin{aligned}
2\alpha f(z) = &\ \big(af(0)z + bf'(0)z^2 + a\frac{f''(0)}{2!}z^3 + b\frac{f'''(0)}{3!}z^4 + \cdots\big)\\
&+\big(af'(0) + b\frac{f''(0)}{2!}z + a\frac{f'''(0)}{3!}z^2 + b\frac{f^{iv}(0)}{4!}z^3 + \cdots\big)\\
&+\Big\{(w_1-a)f(0)z + (w_2-b)f'(0)z^2 + \cdots + (w_{2n-2}-b)\frac{f^{(2n-3)}(0)}{(2n-3)!}z^{2n-2}\\
&+(w_{2n-1}-a)\frac{f^{(2n-2)}(0)}{(2n-2)!}z^{2n-1}\Big\} + \Big\{(w_1-a)f'(0) + (w_2-b)\frac{f''(0)}{2!}z\\
&+\cdots + (w_{2n-2}-b)\frac{f^{(2n-2)}(0)}{(2n-2)!}z^{2n-3} + (w_{2n-1}-a)\frac{f^{(2n-1)}(0)}{(2n-1)!}z^{2n-2}\Big\}.
\end{aligned}$$

For $z = 0$ the Eq. (7) holds trivially. Now for $z \neq 0$ the above equation implies

$$\begin{aligned}
2\alpha f(z) = &\ \Big(\frac{a+b}{2}\Big)\Big\{zf(z) + \frac{f(z)-f(0)}{z}\Big\} + \Big(\frac{a-b}{2}\Big)\Big\{zf(-z) - \frac{f(-z)-f(0)}{z}\Big\}\\
&+\Big\{(w_1-a)f(0)z + (w_2-b)f'(0)z^2 + \cdots + (w_{2n-2}-b)\frac{f^{(2n-3)}(0)}{(2n-3)!}z^{2n-2}\\
&+(w_{2n-1}-a)\frac{f^{(2n-2)}(0)}{(2n-2)!}z^{2n-1}\Big\} + \Big\{(w_1-a)f'(0) + (w_2-b)\frac{f''(0)}{2!}z\\
&+\cdots + (w_{2n-2}-b)\frac{f^{(2n-2)}(0)}{(2n-2)!}z^{2n-3} + (w_{2n-1}-a)\frac{f^{(2n-1)}(0)}{(2n-1)!}z^{2n-2}\Big\}.
\end{aligned}$$

After simplifying the above equation we get our required result. □

LEMMA 2.4. *Let T be a weighted shift operator with weights $(w_1, w_2, \ldots, w_{2n-1}, b, a, b, a, b, \ldots)$ where $w_1, w_2, \ldots, w_{2n-1}, a, b > 0$. If there exists a non zero $f \in H^2$ such that* $\mathrm{Re}(T)f = \alpha f$ *then*

$$\begin{aligned}
&(ab + 2\alpha bz + b^2z^2)f(0) - \sum_{i=1}^{n}(w_{2i-1}-a)\Big\{b\frac{f^{(2i-1)}(0)}{(2i-1)!}z^{2i-1} + a\frac{f^{(2i-2)}(0)}{(2i-2)!}z^{2i}\\
&+2\alpha\frac{f^{(2i-1)}(0)}{(2i-1)!}z^{2i} + a\frac{f^{(2i-1)}(0)}{(2i-1)!}z^{2i+1} + 2\alpha\frac{f^{(2i-2)}(0)}{(2i-2)!}z^{2i+1} + b\frac{f^{(2i-2)}(0)}{(2i-2)!}z^{2i+2}\Big\}\\
&-\sum_{i=1}^{n-1}(w_{2i}-b)\Big\{a\frac{f^{(2i)}(0)}{(2i)!}z^{2i} + b\frac{f^{(2i-1)}(0)}{(2i-1)!}z^{2i+1} + 2\alpha\frac{f^{(2i)}(0)}{(2i)!}z^{2i+1} + b\frac{f^{(2i)}(0)}{(2i)!}z^{2i+2}\\
&+2\alpha\frac{f^{(2i-1)}(0)}{(2i-1)!}z^{2i+2} + a\frac{f^{(2i-1)}(0)}{(2i-1)!}z^{2i+3}\Big\}\\
&= \big(abz^4 + (a^2+b^2-4\alpha^2)z^2 + ab\big)f(z). \qquad (8)
\end{aligned}$$

Proof. If we put $-z$ in place of z in Eq. (7) of Lemma 2.3 then we have

$$\begin{aligned} & bf(0)-\sum_{i=1}^{n}(w_{2i-1}-a)\left\{\frac{f^{(2i-2)}(0)}{(2i-2)!}z^{2i}-\frac{f^{(2i-1)}(0)}{(2i-1)!}z^{2i-1}\right\} \\ & -\sum_{i=1}^{n-1}(w_{2i}-b)\left\{\frac{f^{(2i)}(0)}{(2i)!}z^{2i}-\frac{f^{(2i-1)}(0)}{(2i-1)!}z^{2i+1}\right\} \\ = & \left(\frac{a+b}{2}\right)\left(z^2+\frac{4\alpha}{a+b}z+1\right)f(-z)+\left(\frac{a-b}{2}\right)(z^2-1)f(z). \end{aligned}$$

After solving (7) and above equation simultaneously we get,

$$\begin{aligned} & \left(\frac{a+b}{2}\right)\left(z^2+\frac{4\alpha}{a+b}z+1\right) \\ & \times\left\{bf(0)-\sum_{i=1}^{n}(w_{2i-1}-a)\left(\frac{f^{(2i-2)}(0)}{(2i-2)!}z^{2i}+\frac{f^{(2i-1)}(0)}{(2i-1)!}z^{2i-1}\right)\right. \\ & \left.-\sum_{i=1}^{n-1}(w_{2i}-b)\left(\frac{f^{(2i-1)}(0)}{(2i-1)!}z^{2i+1}+\frac{f^{(2i)}(0)}{(2i)!}z^{2i}\right)\right\} \\ & -\left(\frac{a-b}{2}\right)(z^2-1)\left\{bf(0)-\sum_{i=1}^{n}(w_{2i-1}-a)\left(\frac{f^{(2i-2)}(0)}{(2i-2)!}z^{2i}-\frac{f^{(2i-1)}(0)}{(2i-1)!}z^{2i-1}\right)\right. \\ & \left.-\sum_{i=1}^{n-1}(w_{2i}-b)\left(\frac{f^{(2i)}(0)}{(2i)!}z^{2i}-\frac{f^{(2i-1)}(0)}{(2i-1)!}z^{2i+1}\right)\right\} \\ = & \left(abz^4+(a^2+b^2-4\alpha^2)z^2+ab\right)f(z). \end{aligned}$$

After simplification we get the required result. $\square$

Now, we define the L.H.S of (8) by

$$\begin{aligned} & H_{2n-1}(z) \\ = & (ab+2\alpha bz+b^2z^2)f(0)-\sum_{i=1}^{n}(w_{2i-1}-a)\left\{b\frac{f^{(2i-1)}(0)}{(2i-1)!}z^{2i-1}+a\frac{f^{(2i-2)}(0)}{(2i-2)!}z^{2i}\right. \\ & \left.+2\alpha\frac{f^{(2i-1)}(0)}{(2i-1)!}z^{2i}+a\frac{f^{(2i-1)}(0)}{(2i-1)!}z^{2i+1}+2\alpha\frac{f^{(2i-2)}(0)}{(2i-2)!}z^{2i+1}+b\frac{f^{(2i-2)}(0)}{(2i-2)!}z^{2i+2}\right\} \\ & -\sum_{i=1}^{n-1}(w_{2i}-b)\left\{a\frac{f^{(2i)}(0)}{(2i)!}z^{2i}+b\frac{f^{(2i-1)}(0)}{(2i-1)!}z^{2i+1}+2\alpha\frac{f^{(2i)}(0)}{(2i)!}z^{2i+1}\right. \\ & \left.+b\frac{f^{(2i)}(0)}{(2i)!}z^{2i+2}+2\alpha\frac{f^{(2i-1)}(0)}{(2i-1)!}z^{2i+2}+a\frac{f^{(2i-1)}(0)}{(2i-1)!}z^{2i+3}\right\} \end{aligned} \tag{9}$$

and let,

$$F_{2n-1}(z)=\frac{w_1w_2\cdots w_{2n-1}}{f(0)}H_{2n-1}(z). \tag{10}$$

For $\alpha > \frac{a+b}{2}$ we have $\beta = \frac{4\alpha^2-a^2-b^2}{2ab} > 1$.

Since $q=\sqrt{\beta-\sqrt{\beta^2-1}}$ is a root of $abz^4+(a^2+b^2-4\alpha^2)z^2+ab=0$, therefore from the relation of the sequence $\frac{f^{(k)}(0)}{k!}$ in Eq. (1) and Eq. (2) for $k=1,2,\ldots,2n-1$, we have

$$\begin{aligned} w_1\cdots w_k\frac{f^{(k)}(0)}{k!}q^k &= f(0)q^k\Big((2\alpha)^k-(2\alpha)^{k-2}S_1^{(k-1)}+(2\alpha)^{k-4}S_2^{(k-1)}-\cdots\Big)\\ &= f(0)q^k\Big((2\alpha)^k+\sum_{1\leqslant l\leqslant\lfloor\frac{k}{2}\rfloor}(-1)^l(2\alpha)^{k-2l}S_l^{(k-1)}\Big)\\ &= f(0)\Big(p^k+\sum_{1\leqslant l\leqslant\lfloor\frac{k}{2}\rfloor}(-1)^l q^{2l}p^{k-2l}S_l^{(k-1)}\Big), \end{aligned} \tag{11}$$

where $p=2\alpha q=\sqrt{abq^4+(a^2+b^2)q^2+ab}$. From [14], we have

$$\det(xI_{n+1}-2y\mathrm{Re}(T(w_1,\ldots,w_n)))=\sum_{0\leqslant l\leqslant\lfloor\frac{n+1}{2}\rfloor}(-1)^l S_l^{(n)}x^{n+1-2l}y^{2l}.$$

Here if we put $x=\sqrt{abz^4+(a^2+b^2)z^2+ab}$ and $y=z$ we have

$$\begin{aligned} Q_n(z) &= \det\Big(\sqrt{abz^4+(a^2+b^2)z^2+ab}I_{n+1}-2z\mathrm{Re}(T(w_1,\ldots,w_n))\Big)\\ &= \sum_{0\leqslant l\leqslant\lfloor\frac{n+1}{2}\rfloor}(-1)^l S_l^{(n)}(abz^4+(a^2+b^2)z^2+ab)^{\frac{n+1-2l}{2}}z^{2l} \end{aligned}$$

for $n=1,2,\ldots$ and $Q_0(z)=\sqrt{abz^4+(a^2+b^2)z^2+ab}$, $Q_{-1}(z)=1$.

Now from Eq. (11), for $k=1,2,\ldots,2n-1$ we have

$$w_1\cdots w_k\frac{f^{(k)}(0)}{k!}q^k=f(0)Q_{k-1}(q). \tag{12}$$

LEMMA 2.5. *Let $q=\sqrt{\beta-\sqrt{\beta^2-1}}$ and define $T_{2m+3}(p,q)$ as follows*

$$\begin{aligned} T_{2m+3}(p,q) &= (w_1\cdots w_{2m+3})(b+(a-w_1)q^2)\\ &\quad -(w_2\cdots w_{2m+3})(b+aq^2)\big\{(w_1-a)+(w_2-b)q^2\big\}\\ &\quad -w_{2m+3}(b+aq^2)\frac{Q_{2m+2}(q)}{p}-w_{2m+3}\big\{(w_{2m+2}-b)-aq^2\big\}Q_{2m+1}(q)\\ &\quad -\sum_{i=1}^{m}\big\{(w_{2i}-b)+(w_{2i+1}-a)q^2\big\}(w_{2i+1}\cdots w_{2m+3})Q_{2i-1}(q)\\ &\quad -\sum_{i=1}^{m}(b+aq^2)\big\{(w_{2i+1}-a)+(w_{2i+2}-b)q^2\big\}(w_{2i+2}\cdots w_{2m+3})\frac{Q_{2i}(q)}{p}. \end{aligned}$$

Then for all $m\geqslant 1$, $T_{2m+3}(p,q)=0$ under the relation $p=\sqrt{abq^4+(a^2+b^2)q^2+ab}$.

Proof. We will prove the above result by using induction on m under the relation $p=\sqrt{abq^4+(a^2+b^2)q^2+ab}$. For $m=1$ it is easy to check that $T_5(p,q)=0$. Suppose $T_{2m+3}(p,q)=0$. Now,

$$\begin{aligned}
&T_{2m+5}(p,q)\\
&=(w_2\cdots w_{2m+3}w_{2m+4}w_{2m+5})\big(p^2-w_1^2q^2-w_2q^2(aq^2+b)\big)\\
&\quad-(aq^2+b)w_{2m+5}\frac{Q_{2m+4}(q)}{p}\\
&\quad-w_{2m+5}\big\{(w_{2m+4}-b)-aq^2\big\}Q_{2m+3}(q)\\
&\quad-\sum_{i=1}^{m+1}\big\{(w_{2i}-b)+(w_{2i+1}-a)q^2\big\}(w_{2i+1}\cdots w_{2m+4}w_{2m+5})Q_{2i-1}(q)\\
&\quad-\sum_{i=1}^{m+1}(b+aq^2)\big\{(w_{2i+1}-a)+(w_{2i+2}-b)q^2\big\}(w_{2i+2}\cdots w_{2m+4}w_{2m+5})\frac{Q_{2i}(q)}{p}\\
&=w_{2m+4}w_{2m+5}T_{2m+3}(p,q)+w_{2m+3}w_{2m+4}w_{2m+5}(aq^2+b)\frac{Q_{2m+2}(q)}{p}\\
&\quad-(aq^2+b)w_{2m+5}\frac{Q_{2m+4}(q)}{p}+\big\{(w_{2m+2}-b)-aq^2\big\}w_{2m+3}w_{2m+4}w_{2m+5}Q_{2m+1}(q)\\
&\quad-\big\{(w_{2m+4}-b)-aq^2\big\}w_{2m+5}Q_{2m+3}(q)\\
&\quad-w_{2m+3}w_{2m+4}w_{2m+5}\big\{(w_{2m+2}-b)+(w_{2m+3}-a)q^2\big\}Q_{2m+1}(q)\\
&\quad-(aq^2+b)w_{2m+4}w_{2m+5}\big\{(w_{2m+3}-a)+(w_{2m+4}-b)q^2\big\}\frac{Q_{2m+2}(q)}{p}\\
&=p^2w_{2m+4}w_{2m+5}\frac{Q_{2m+2}(q)}{p}-w_{2m+4}^2w_{2m+5}(aq^2+b)q^2\frac{Q_{2m+2}(q)}{p}\\
&\quad-(aq^2+b)w_{2m+5}\frac{Q_{2m+4}(q)}{p}-\big\{(w_{2m+4}-b)-aq^2\big\}w_{2m+5}Q_{2m+3}(q)\\
&\quad-w_{2m+3}^2w_{2m+4}w_{2m+5}q^2Q_{2m+1}(q).
\end{aligned}$$

Thus, we get,

$$\begin{aligned}
T_{2m+5}(p,q)=\;&pw_{2m+4}w_{2m+5}Q_{2m+2}(q)-w_{2m+4}^2w_{2m+5}(aq^2+b)q^2\frac{Q_{2m+2}(q)}{p}\\
&-w_{2m+5}(aq^2+b)\Big(\frac{Q_{2m+4}(q)}{p}-Q_{2m+3}(q)\Big)-w_{2m+4}w_{2m+5}Q_{2m+3}(q)\\
&-w_{2m+3}^2w_{2m+4}w_{2m+5}q^2Q_{2m+1}(q).
\end{aligned}\tag{13}$$

Also, we know that $S_l^{(2m+4)}-S_l^{(2m+3)}=w_{2m+4}^2S_{l-1}^{(2m+2)}$ for $1\leqslant l\leqslant m+2$ (see [13]). Therefore,

$$\begin{aligned}
\frac{Q_{2m+4}(q)}{p}-Q_{2m+3}(q)&=\sum_{1\leqslant l\leqslant m+2}(-1)^lp^{2m+4-2l}q^{2l}\big(S_l^{(2m+4)}-S_l^{(2m+3)}\big)\\
&=\sum_{1\leqslant l\leqslant m+2}(-1)^lp^{2m+4-2l}q^{2l}w_{2m+4}^2S_{l-1}^{(2m+2)}
\end{aligned}$$

$$= w_{2m+4}^2 \sum_{0\leqslant l\leqslant m+1} (-1)^{l+1} p^{2m-2l+2} q^{2l+2} S_l^{(2m+2)}$$
$$= -w_{2m+4}^2 q^2 \frac{Q_{2m+2}(q)}{p}.$$

From Eq. (13) we have

$$T_{2m+5}(p,q) = w_{2m+4}w_{2m+5}\big(pQ_{2m+2}(q) - Q_{2m+3}(q)\big) \\ -w_{2m+3}^2 w_{2m+4}w_{2m+5} q^2 Q_{2m+1}(q). \tag{14}$$

Now we have $pQ_0(q) - Q_1(q) = q^2 w_1^2 Q_{-1}(q)$ and for $m \geqslant 2$,

$$pQ_{2m-2}(q) = \sum_{0\leqslant l\leqslant m-1} (-1)^l q^{2l} p^{2m-2l} S_l^{(2m-2)}.$$

So,

$$\begin{aligned}
& pQ_{2m-2}(q) - Q_{2m-1}(q) \\
&= \sum_{1\leqslant l\leqslant m-1} (-1)^l q^{2l} p^{2m-2l} \big(S_l^{(2m-2)} - S_l^{(2m-1)}\big) + (-1)^{m+1} q^{2m} S_m^{(2m-1)} \\
&= w_{2m-1}^2 \sum_{0\leqslant l\leqslant m-2} (-1)^l q^{2l+2} p^{2m-2l-2} S_l^{(2m-3)} + (-1)^{m+1} q^{2m} S_m^{(2m-1)} \\
&= w_{2m-1}^2 q^2 \Big(\sum_{0\leqslant l\leqslant m-1} (-1)^l q^{2l} p^{2m-2l-2} S_l^{(2m-3)} - (-1)^{m-1} q^{2m-2} S_{m-1}^{(2m-3)} \Big) \\
&\quad + (-1)^{m+1} q^{2m} S_m^{(2m-1)} \\
&= w_{2m-1}^2 q^2 Q_{2m-3}(q) + (-1)^{m+1} q^{2m} \big(S_m^{(2m-1)} - w_{2m-1}^2 S_{m-1}^{(2m-3)}\big) \\
&= w_{2m-1}^2 q^2 Q_{2m-3}(q).
\end{aligned}$$

Thus for $m \geqslant 1$ we have

$$pQ_{2m-2}(q) - Q_{2m-1}(q) = w_{2m-1}^2 q^2 Q_{2m-3}(q). \tag{15}$$

Now from Eq. (14) and replacing m by $m+2$ in Eq. (15), we have $T_{2m+5}(p,q) = 0$. Hence by induction on m we get our result. □

THEOREM 2.6. *Let $n \geqslant 1$ and $T = T(w_1, w_2, \ldots, w_{2n-1}, b, a, b, a, b, \ldots)$ be a weighted shift operator where $w_1, w_2, \ldots, w_{2n-1}, a, b > 0$. Let $f(z)$ be a nonzero formal power series:*

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!} z^2 + \cdots + \frac{f^{(n)}(0)}{n!} z^n + \cdots.$$

Suppose $\alpha > \frac{a+b}{2}$ and $\beta = \frac{4\alpha^2 - a^2 - b^2}{2ab}$ then the following holds:

1. *If α is an eigenvalue of* $\mathrm{Re}(T)$*, then the value* $q=\sqrt{\beta-\sqrt{\beta^2-1}}$ *is a zero of the polynomial* $G_{2n-1}(z)$ *where*

$$G_{2n-1}(z)=(a^2z^2+ab)\frac{Q_{2n-2}(z)}{\sqrt{abz^4+(a^2+b^2)z^2+ab}}-w_{2n-1}^2z^2Q_{2n-3}(z).$$

2. *If the coefficients of* $f(z)$ *satisfy the conditions* (1)–(5) *and* $q=\sqrt{\beta-\sqrt{\beta^2-1}}$ *is a zero of the polynomial* $G_{2n-1}(z)$*, then* $f\in H^2$ *and hence* α *is an eigenvalue of* $\mathrm{Re}(T)$.

Proof. Since $q=\sqrt{\beta-\sqrt{\beta^2-1}}$ is a root of $z^4-2\beta z^2+1=0$, then with the help of Eq. (8) and Eq. (10), it follows that $F_{2n-1}(q)=H_{2n-1}(q)=0$.

1. For $n=1$, the weighted shift operator $T(w_1,b,a,b,a,b,\ldots)$, $\mathrm{Re}(T)f=\alpha f$ where $f\in H^2$ implies

$$\begin{aligned}&bf(0)-(w_1-a)f'(0)z-(w_1-a)f(0)z^2\\&=\left(\frac{a+b}{2}\right)\left(z^2-\frac{4\alpha}{a+b}z+1\right)f(z)+\left(\frac{a-b}{2}\right)(z^2-1)f(-z).\end{aligned}\tag{16}$$

Put $(-z)$ in Eq. (16) we get

$$\begin{aligned}&bf(0)+(w_1-a)f'(0)z-(w_1-a)f(0)z^2\\&=\left(\frac{a+b}{2}\right)\left(z^2+\frac{4\alpha}{a+b}z+1\right)f(-z)+\left(\frac{a-b}{2}\right)(z^2-1)f(z).\end{aligned}\tag{17}$$

Simplifying Eq. (16) and (17) we get

$$\begin{aligned}&\big(abz^4+(a^2+b^2-4\alpha^2)z^2+ab\big)f(z)\\&=(ab+2\alpha bz+b^2z^2)f(0)-(w_1-a)(af'(0)z^3+bf(0)z^4+bf'(0)z\\&\quad+af(0)z^2+2\alpha f'(0)z^2+2\alpha f(0)z^3).\end{aligned}$$

Putting $z=q$, we get

$$\begin{aligned}&(w_1-a)(af'(0)q^3+bf(0)q^4+bf'(0)q+af(0)q^2+2\alpha f'(0)q^2+2\alpha f(0)q^3)\\&-(ab+2\alpha bq+b^2q^2)f(0)=0.\end{aligned}$$

Simplifying the above equation by using Eq. (1) and applying $p=2\alpha q$, we have

$$(p+a+bq^2)(ab+a^2q^2-w_1^2q^2)=0.$$

Since $p+a+bq^2\neq 0$, therefore, q is a root of

$$G_1(z)=ab+a^2z^2-w_1^2z^2=(a^2z^2+ab)\frac{Q_0(z)}{\sqrt{abz^4+(a^2+b^2)z^2+ab}}-w_1^2z^2Q_{-1}(z).$$

Now consider the case of $n=2$. Putting $z=q$ in Eq. (9) and after some simplifications by using Eq. (12) we get

$$\begin{aligned}
0 &= (p+a+bq^2)\Big\{w_1w_2w_3(b+(a-w_1)q^2)\\
&\quad -w_3\big((w_2-b)+(w_3-a)q^2\big)(p^2-q^2w_1^2)\\
&\quad -(b+aq^2)\big((w_3-a)(p^2-q^2(w_1^2+w_2^2))+w_2w_3((w_1-a)+(w_2-b)q^2)\big)\Big\}\\
&= (p+a+bq^2)\Big\{(a^2q^2+ab)(p^2-q^2(w_1^2+w_2^2))-w_3^2q^2(p^2-q^2w_1^2)\Big\}.
\end{aligned}$$

Since $p+a+bq^2\neq 0$, therefore

$$(a^2q^2+ab)(p^2-q^2(w_1^2+w_2^2))-w_3^2q^2(p^2-q^2w_1^2)=0.$$

So, q is a root of

$$\begin{aligned}
G_3(z) &= (a^2z^2+ab)(abz^4+(a^2+b^2)z^2+ab-z^2(w_1^2+w_2^2))\\
&\quad -w_3^2z^2(abz^4+(a^2+b^2)z^2+ab-z^2w_1^2)\\
&= (a^2z^2+ab)\frac{Q_2(z)}{\sqrt{abz^4+(a^2+b^2)z^2+ab}}-w_3^2z^2Q_1(z).
\end{aligned}$$

Now our aim is to compute the polynomial for $n>2$.

From Eq. (9) we have

$$\begin{aligned}
&H_{2n-1}(z)\\
&= (ab+2\alpha bz+b^2z^2)f(0)-(w_1-a)\big(bz^4+2\alpha z^3+az^2\big)f(0)\\
&\quad -(w_{2n-1}-a)(az^{2n+1}+2\alpha z^{2n}+bz^{2n-1})\frac{f^{(2n-1)}(0)}{(2n-1)!}\\
&\quad -\sum_{i=1}^{n-1}\Big\{(w_{2i}-b)(b+2\alpha z+az^2)z^{2i+1}+(w_{2i-1}-a)(az^2+2\alpha z+b)z^{2i-1}\Big\}\\
&\quad \times\frac{f^{(2i-1)}(0)}{(2i-1)!}-\sum_{i=1}^{n-1}\Big\{(w_{2i+1}-a)(bz^2+2\alpha z+a)z^{2i+2}\\
&\quad +(w_{2i}-b)(bz^2+2\alpha z+a)z^{2i}\Big\}\frac{f^{(2i)}(0)}{(2i)!}.
\end{aligned}$$

Hence we get,

$$\begin{aligned}
&H_{2n-1}(z)\\
&= 2\alpha\Big\{(bz+(a-w_1)z^3)f(0)-\sum_{i=1}^{n-1}\{(w_{2i-1}-a)z^{2i}+(w_{2i}-b)z^{2i+2}\}\frac{f^{(2i-1)}(0)}{(2i-1)!}\\
&\quad -(w_{2n-1}-a)\frac{f^{(2n-1)}(0)}{(2n-1)!}z^{2n}-\sum_{i=1}^{n-1}\{(w_{2i}-b)z^{2i+1}+(w_{2i+1}-a)z^{2i+3}\}\frac{f^{(2i)}(0)}{(2i)!}\Big\}
\end{aligned}$$

$$+\{ab+(b^2+a^2-aw_1)z^2+b(a-w_1)z^4\}f(0)-\sum_{i=1}^{n-1}\{b(w_{2i-1}-a)z^{2i-1}$$
$$+(aw_{2i-1}+bw_{2i}-a^2-b^2)z^{2i+1}+a(w_{2i}-b)z^{2i+3}\}\frac{f^{(2i-1)}(0)}{(2i-1)!}$$
$$-(bz^{2n-1}+az^{2n+1})(w_{2n-1}-a)\frac{f^{(2n-1)}(0)}{(2n-1)!}-\sum_{i=1}^{n-1}\{a(w_{2i}-b)z^{2i}$$
$$+(aw_{2i+1}+bw_{2i}-a^2-b^2)z^{2i+2}+b(w_{2i+1}-a)z^{2i+4}\}\frac{f^{(2i)}(0)}{(2i)!}. \tag{18}$$

Putting $z=q$ in the above equation and by using Eq. (10) and Eq. (12) we get,

$$\begin{aligned}
0 &= (w_1\cdots w_{2n-1})(b+(a-w_1)q^2)p+(w_1\cdots w_{2n-1})\\
&\quad\times\{ab+(a^2+b^2-aw_1)q^2+b(a-w_1)q^4\}\\
&\quad-(w_{2n-1}-a)pQ_{2n-2}(q)-(b+aq^2)(w_{2n-1}-a)Q_{2n-2}(q)\\
&\quad-\sum_{i=1}^{n-1}\{(w_{2i-1}-a)+(w_{2i}-b)q^2\}(w_{2i}\cdots w_{2n-1})pQ_{2i-2}(q)-\sum_{i=1}^{n-1}\{b(w_{2i-1}-a)\\
&\quad+(aw_{2i-1}+bw_{2i}-a^2-b^2)q^2+a(w_{2i}-b)q^4\}(w_{2i}\cdots w_{2n-1})Q_{2i-2}(q)\\
&\quad-\sum_{i=1}^{n-1}\{(w_{2i}-b)+(w_{2i+1}-a)q^2\}(w_{2i+1}\cdots w_{2n-1})pQ_{2i-1}(q)-\sum_{i=1}^{n-1}\{a(w_{2i}-b)\\
&\quad+(aw_{2i+1}+bw_{2i}-a^2-b^2)q^2+b(w_{2i+1}-a)q^4\}(w_{2i+1}\cdots w_{2n-1})Q_{2i-1}(q)\\
&= p\Big\{(w_1\cdots w_{2n-1})(b+(a-w_1)q^2)-(b+aq^2)(w_{2n-1}-a)\frac{Q_{2n-2}(q)}{p}\\
&\quad-\{b(w_1-a)+(aw_1+bw_2-a^2-b^2)q^2+a(w_2-b)q^4\}(w_2\cdots w_{2n-1})\\
&\quad-\sum_{i=1}^{n-1}\{(w_{2i}-b)+(w_{2i+1}-a)q^2\}(w_{2i+1}\cdots w_{2n-1})Q_{2i-1}(q)-\sum_{i=1}^{n-2}\{b(w_{2i+1}-a)\\
&\quad+(aw_{2i+1}+bw_{2i+2}-a^2-b^2)q^2+a(w_{2i+2}-b)q^4\}(w_{2i+2}\cdots w_{2n-1})\frac{Q_{2i}(q)}{p}\Big\}\\
&\quad+(w_1\cdots w_{2n-1})\{ab+(a^2+b^2-aw_1)q^2+b(a-w_1)q^4\}\\
&\quad-(w_{2n-1}-a)pQ_{2n-2}(q)\\
&\quad-\sum_{i=1}^{n-1}\{(w_{2i-1}-a)+(w_{2i}-b)q^2\}(w_{2i}\cdots w_{2n-1})pQ_{2i-2}(q)-\sum_{i=1}^{n-1}\{a(w_{2i}-b)\\
&\quad+(aw_{2i+1}+bw_{2i}-a^2-b^2)q^2+b(w_{2i+1}-a)q^4\}(w_{2i+1}\cdots w_{2n-1})Q_{2i-1}(q)\\
&= p\Big\{(w_1\cdots w_{2n-1})(b+(a-w_1)q^2)-(b+aq^2)(w_{2n-1}-a)\frac{Q_{2n-2}(q)}{p}\\
&\quad-(w_2\cdots w_{2n-1})(b+aq^2)\{(w_1-a)+(w_2-b)q^2\}-\sum_{i=1}^{n-1}\{(w_{2i}-b)
\end{aligned}$$

$$+(w_{2i+1}-a)q^2\big\}(w_{2i+1}\cdots w_{2n-1})Q_{2i-1}(q)-\sum_{i=1}^{n-2}(b+aq^2)\big\{(w_{2i+1}-a)$$
$$+(w_{2i+2}-b)q^2\big\}(w_{2i+2}\cdots w_{2n-1})\frac{Q_{2i}(q)}{p}\Big\}$$
$$+(a+bq^2)(w_1\cdots w_{2n-1})(b+(a-w_1)q^2)$$
$$-(a+bq^2)(b+aq^2)(w_{2n-1}-a)\frac{Q_{2n-2}(q)}{p}$$
$$-(a+bq^2)(w_2\cdots w_{2n-1})(b+aq^2)\big\{(w_1-a)+(w_2-b)q^2\big\}$$
$$-(a+bq^2)\sum_{i=1}^{n-1}\big\{(w_{2i}-b)+(w_{2i+1}-a)q^2\big\}(w_{2i+1}\cdots w_{2n-1})Q_{2i-1}(q)$$
$$-(a+bq^2)\sum_{i=1}^{n-2}(b+aq^2)\big\{(w_{2i+1}-a)+(w_{2i+2}-b)q^2\big\}$$
$$\times(w_{2i+2}\cdots w_{2n-1})\frac{Q_{2i}(q)}{p},$$

$$0=(p+a+bq^2)\Big\{(w_1\cdots w_{2n-1})(b+(a-w_1)q^2)$$
$$-(b+aq^2)(w_{2n-1}-a)\frac{Q_{2n-2}(q)}{p}$$
$$-(w_2\cdots w_{2n-1})(b+aq^2)\big\{(w_1-a)+(w_2-b)q^2\big\}$$
$$-\sum_{i=1}^{n-1}\big\{(w_{2i}-b)+(w_{2i+1}-a)q^2\big\}(w_{2i+1}\cdots w_{2n-1})Q_{2i-1}(q)-\sum_{i=1}^{n-2}(b+aq^2)$$
$$+\big\{(w_{2i+1}-a)(w_{2i+2}-b)q^2\big\}(w_{2i+2}\cdots w_{2n-1})\frac{Q_{2i}(q)}{p}\Big\}.$$

Since $p+a+bq^2\neq 0$, therefore, we have

$$0=(w_1\cdots w_{2n-1})(b+(a-w_1)q^2)-(b+aq^2)(w_{2n-1}-a)\frac{Q_{2n-2}(q)}{p}$$
$$-(w_2\cdots w_{2n-1})(b+aq^2)\big\{(w_1-a)+(w_2-b)q^2\big\}-\sum_{i=1}^{n-1}\big\{(w_{2i}-b)$$
$$+(w_{2i+1}-a)q^2\big\}(w_{2i+1}\cdots w_{2n-1})Q_{2i-1}(q)-\sum_{i=1}^{n-2}(b+aq^2)\big\{(w_{2i+1}-a)$$
$$+(w_{2i+2}-b)q^2\big\}(w_{2i+2}\cdots w_{2n-1})\frac{Q_{2i}(q)}{p}.$$

Now from above equation we have,

$$\Big(\frac{a^2q^2+ab}{p}\Big)Q_{2n-2}(q)-w_{2n-1}(b+aq^2)\frac{Q_{2n-2}(q)}{p}$$
$$+(w_1\cdots w_{2n-1})(b+(a-w_1)q^2)$$

$$
\begin{aligned}
&-(w_2\cdots w_{2n-1})(b+aq^2)\{(w_1-a)+(w_2-b)q^2\}\\
&-w_{2n-1}\{(w_{2n-2}-b)+(w_{2n-1}-a)q^2\}Q_{2n-3}(q)\\
&-\sum_{i=1}^{n-2}\{(w_{2i}-b)+(w_{2i+1}-a)q^2\}(w_{2i+1}\cdots w_{2n-1})Q_{2i-1}(q)\\
&-\sum_{i=1}^{n-2}(b+aq^2)\{(w_{2i+1}-a)+(w_{2i+2}-b)q^2\}(w_{2i+2}\cdots w_{2n-1})\frac{Q_{2i}(q)}{p}=0.
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&\left\{\Big(\frac{a^2q^2+ab}{p}\Big)Q_{2n-2}(q)-w_{2n-1}^2q^2Q_{2n-3}(q)\right\}+(w_1\cdots w_{2n-1})(b+(a-w_1)q^2)\\
&-(w_2\cdots w_{2n-1})(b+aq^2)\{(w_1-a)+(w_2-b)q^2\}-w_{2n-1}(b+aq^2)\frac{Q_{2n-2}(q)}{p}\\
&-w_{2n-1}\{(w_{2n-2}-b)-aq^2\}Q_{2n-3}(q)\\
&-\sum_{i=1}^{n-2}\{(w_{2i}-b)+(w_{2i+1}-a)q^2\}(w_{2i+1}\cdots w_{2n-1})Q_{2i-1}(q)\\
&-\sum_{i=1}^{n-2}(b+aq^2)\{(w_{2i+1}-a)+(w_{2i+2}-b)q^2\}(w_{2i+2}\cdots w_{2n-1})\frac{Q_{2i}(q)}{p}=0.
\end{aligned}
$$

i.e.,

$$
\Big(\frac{a^2q^2+ab}{p}\Big)Q_{2n-2}(q)-w_{2n-1}^2q^2Q_{2n-3}(q)+T_{2n-1}(p,q)=0.
$$

From Lemma 2.5 we have $T_{2n-1}(p,q)=0$ for $n>2$. Hence q is a root of

$$
G_{2n-1}(z)=(a^2z^2+ab)\frac{Q_{2n-2}(z)}{\sqrt{abz^4+(a^2+b^2)z^2+ab}}-w_{2n-1}^2z^2Q_{2n-3}(z).
$$

2. Let $G_{2n-1}(q)=0$. Then we have

$$
\Big(\frac{a^2q^2+ab}{p}\Big)Q_{2n-2}(q)-w_{2n-1}^2q^2Q_{2n-3}(q)=0.
$$

i.e.,

$$
\Big(\frac{abq^4+(a^2+b^2)q^2+ab-abq^4-b^2q^2}{p}\Big)Q_{2n-2}(q)-w_{2n-1}^2q^2Q_{2n-3}(q)=0.
$$

i.e.,

$$
\big(pQ_{2n-2}(q)-w_{2n-1}^2q^2Q_{2n-3}(q)\big)-\Big(\frac{abq^4+b^2q^2}{p}\Big)Q_{2n-2}(q)=0. \tag{19}
$$

Using Eq. (12) and (15) in the above equation we have

$$\begin{aligned}
0 &= 2\alpha q Q_{2n-2}(q) - w_{2n-1}^2 q^2 Q_{2n-3}(q) - \Big(\frac{abq^4+b^2q^2}{p}\Big)Q_{2n-2}(q)\\
&= \frac{w_1\cdots w_{2n-1}}{f(0)}q^{2n}\Big\{2\alpha\frac{f^{(2n-1)}(0)}{(2n-1)!} - w_{2n-1}\frac{f^{(2n-2)}(0)}{(2n-2)!}\Big\}\\
&\quad -\Big(\frac{abq^4+b^2q^2}{p}\Big)Q_{2n-2}(q)\\
&= b\frac{w_1\cdots w_{2n-1}}{f(0)}q^{2n}\frac{f^{(2n)}(0)}{(2n)!} - \frac{w_1\cdots w_{2n-1}}{f(0)}\Big(\frac{abq^2+b^2}{p}\Big)\frac{f^{(2n-1)}(0)}{(2n-1)!}q^{2n+1}\\
&= bw_1\cdots w_{2n-1}\frac{q^{2n}}{f(0)}\Big\{\frac{f^{(2n)}(0)}{(2n)!} - \Big(\frac{aq^2+b}{p}\Big)q\frac{f^{(2n-1)}(0)}{(2n-1)!}\Big\}.
\end{aligned}$$

So we have

$$\frac{f^{(2n)}(0)}{(2n)!} - \Big(\frac{a(\beta-\sqrt{\beta^2-1})+b}{2\alpha}\Big)\frac{f^{(2n-1)}(0)}{(2n-1)!} = 0. \tag{20}$$

Now from Eq. (4) we have

$$b\frac{f^{(2n-1)}(0)}{(2n-1)!} + a\frac{f^{(2n+1)}(0)}{(2n+1)!} - (a(\beta-\sqrt{\beta^2-1})+b)\frac{f^{(2n-1)}(0)}{(2n-1)!} = 0.$$

i.e.,

$$\frac{f^{(2n+1)}(0)}{(2n+1)!} - (\beta-\sqrt{\beta^2-1})\frac{f^{(2n-1)}(0)}{(2n-1)!} = 0. \tag{21}$$

Also using Eq. (4) and (5) we have

$$\frac{f^{(2n-1+2r+4)}(0)}{(2n-1+2r+4)!} - 2\beta\frac{f^{(2n-1+2r+2)}(0)}{(2n-1+2r+2)!} + \frac{f^{(2n-1+2r)}(0)}{(2n-1+2r)!} = 0, \tag{22}$$

for $r = 0,1,2,\ldots$. So the characteristic equation of the difference equation (22) is

$$s^4 - 2\beta s^2 + 1 = 0.$$

Thus,

$$\frac{f^{(2n-1+2r)}(0)}{(2n-1+2r)!} = \mu_1(\beta+\sqrt{\beta^2-1})^r + \mu_2(\beta-\sqrt{\beta^2-1})^r, \text{ for } r = 0,1,2,\ldots$$

where μ_1, μ_2 are constants. Now from Eq. (21), we get $\mu_1 = 0$. Therefore,

$$\frac{f^{(2n-1+2r)}(0)}{(2n-1+2r)!} = \mu_2 q^{2r}, \text{ for } r = 0,1,2,\ldots.$$

Now from Eq. (4) we have

$$\frac{f^{(2n+2r)}(0)}{(2n+2r)!} = \mu_2 \frac{(b+aq^2)q}{\sqrt{abq^4+(a^2+b^2)q^2+ab}} q^{2r}, \text{ for } r = 0,1,2,\ldots.$$

Here the function f is analytic on the open disc $\{z \in \mathbb{C} : |z| < \frac{1}{q^2}\}$. Now from the last two equations we have $f \in H^2$. □

REMARK 2.7. In Theorem 2.6 if we put $w_{2n+1} = a$ in the following operator

$$T = T(w_1,\ldots,w_{2n},w_{2n+1},b,a,b,a,b,\ldots)$$

then we have $T = T_1 = T_1(w_1,\ldots,w_{2n},a,b,a,b,\ldots)$. So, in this case, $G_{2n+1}(z) = G_{2n}(z)$, where

$$G_{2n}(z) = (a^2z^2+ab)\frac{Q_{2n}(z)}{\sqrt{abz^4+(a^2+b^2)z^2+ab}} - a^2z^2Q_{2n-1}(z).$$

Now if we put $z = q$ in above equation then we have, $G_{2n+1}(q) = G_{2n}(q) = 0$. Therefore,

$$\begin{aligned}
0 &= (a^2q^2+ab)\sum_{0\leqslant l\leqslant n}(-1)^l q^{2l}p^{2n-2l}S_l^{(2n)} - a^2q^2\sum_{0\leqslant l\leqslant n}(-1)^l q^{2l}p^{(2n-2l)}S_l^{(2n-1)} \\
&= (a^2q^2+ab)\big(p^{2n}+\sum_{1\leqslant l\leqslant n}(-1)^l q^{2l}p^{2n-2l}S_l^{(2n)}\big) \\
&\quad -a^2q^2\big(p^{2n}+\sum_{1\leqslant l\leqslant n}(-1)^l q^{2l}p^{(2n-2l)}S_l^{(2n-1)}\big) \\
&= (a^2q^2+ab)\big(p^{2n}+\sum_{1\leqslant l\leqslant n}(-1)^l q^{2l}p^{2n-2l}S_l^{(2n-1)} \\
&\quad +w_{2n}^2\sum_{1\leqslant l\leqslant n}(-1)^l q^{2l}p^{2n-2l}S_{l-1}^{(2n-2)}\big) \\
&\quad -a^2q^2\big(p^{2n}+\sum_{1\leqslant l\leqslant n}(-1)^l q^{2l}p^{(2n-2l)}S_l^{(2n-1)}\big) \\
&= abQ_{2n-1}(q)+w_{2n}^2(a^2q^2+ab)\sum_{0\leqslant l\leqslant n-1}(-1)^{l+1}q^{2l+2}p^{2n-2l-2}S_l^{(2n-2)} \\
&= abQ_{2n-1}(q)-w_{2n}^2\frac{(a^2q^2+ab)q^2}{p}Q_{2n-2}(q).
\end{aligned}$$

Therefore, q is a root of

$$G_{2n}(z) = abQ_{2n-1}(z)-w_{2n}^2(a^2z^2+ab)z^2\frac{Q_{2n-2}(z)}{\sqrt{abz^4+(a^2+b^2)z^2+ab}}.$$

Particular cases

1. Replacing $w_2 = 1$ and $a = b = 1$ in Theorem 2.6, we get the determinantal polynomial of the operator $T(w_1, 1, 1, 1, \ldots)$ as
$$G_1(z) = z^2 + 1 - w_1^2 z^2.$$
This polynomial is also obtained in [14]. The zeros of $G_1(z)$ determine the numerical radius $w(T(w_1, 1, 1, 1, \ldots))$ (cf. [1]).

2. Replacing $w_1 = 1$ and $a = b = 1$ in Remark 2.7, we get the determinantal polynomial of the operator $T(1, w_2, 1, 1, \ldots)$ as
$$G_2(z) = (z^2+1)^2 - z^2 - w_2^2 z^2 (z^2+1).$$
This polynomial is also obtained in [14]. The zeros of $G_1(z)$ determine the numerical radius $w(T(1, w_2, 1, 1, \ldots))$ (cf. [5]).

3. Replacing $a = b = 1$, our operators reduces to the weighted shift operator with weights $(w_1, w_2, \ldots, w_n, 1, 1, \ldots)$. From Theorem 2.6 and Remark 2.7 we get the determinantal polynomial $G_n(z) = Q_{n-1}(z) - w_n^2 z^2 Q_{n-2}(z)$ which is obtained in [14].

4. For $n = 2$, our operator reduces to weighted shift operator T with weights $(w_1, w_2, a, b, a, b, \ldots)$. If $bw_1^2 + (a+b)w_2^2 > (a+b)^2 b$, then $\alpha = \|\mathrm{Re}(T)\| > \frac{a+b}{2}$ is an eigenvalue of $\mathrm{Re}(T)$. By Remark 2.7 we have,
$$G_2(z) = a\big\{a(b^2 - w_2^2)z^4 + b(a^2 + b^2 - w_1^2 - w_2^2)z^2 + ab^2\big\}.$$
The minimal positive root less than 1 of $G_2(z) = 0$ is
$$q = \sqrt{\frac{2ab}{(w_1^2 + w_2^2 - a^2 - b^2) + \sqrt{(a^2 + b^2 - w_1^2 - w_2^2)^2 - 4a^2(b^2 - w_2^2)}}}$$
and using $4w(T)^2 = ab\left(q^2 + \frac{1}{q^2}\right) + a^2 + b^2$, we get the numerical radius of T. This formula is obtained in [4].

EXAMPLE 2.8. Consider the weighted shift operator $T = T(3, 4, 5, 2, 1, 2, 1, \ldots)$. Then by Theorem 2.6 we have
$$G_3(z) = -48z^6 + 84z^4 - 88z^2 + 4.$$
The eigenvalue greater than 1.5 of the self-adjoint operator $\mathrm{Re}(T)$ lie in the set
$$\left\{\frac{1}{2}\sqrt{2\left(z^2 + \frac{1}{z^2}\right) + 5} \; : 0 < z < 1, \; -48z^6 + 84z^4 - 88z^2 + 4 = 0\right\}.$$
The only element of this set is 3.4334 (approx.) for $z = 0.218070005$ and therefore the approximate value of the numerical radius of $T(3, 4, 5, 2, 1, 2, 1, \ldots)$ is 3.4334.

EXAMPLE 2.9. Consider the weighted shift operator $T = T(4,5,6,7,1,2,1,2,\ldots)$. Then by the Remark 2.7 we have

$$G_4(z) = -90z^8 + 1300z^6 + 3878z^4 - 464z^2 + 8.$$

The eigenvalues greater than 1.5 of the self-adjoint operator $\mathrm{Re}(T)$ lie in the set

$$\left\{ \frac{1}{2}\sqrt{2(z^2+\frac{1}{z^2})+5} \ : 0 < z < 1,\ -90z^8 + 1300z^6 + 3878z^4 - 464z^2 + 8 = 0 \right\}.$$

The elements of this set are approximately 5.0153 and 2.5623 for $z = 0.144662$ and 0.308082 respectively. Since the eigenvalue $5.0153 > 2.5623$ therefore the approximate value of the numerical radius of $T = T(4,5,6,7,1,2,1,2,\ldots)$ is 5.0153.

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