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# EXPLICIT SOLUTIONS OF MATRIX AND DYNAMICAL SCHRÖDINGER EQUATIONS AND OF KDV EQUATION IN TERMS OF SQUARE ROOTS OF THE GENERALISED MATRIX EIGENVALUES

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*Abstract.* In this paper, we consider matrix Schrödinger equation, dynamical Schrödinger equation and matrix KdV. We construct their explicit solutions using our GBDT version of Bäcklund–Darboux transformation and square roots of the generalised matrix eigenvalues. A separate section is dedicated to several examples including the case of strongly singular potentials.

## 1. Introduction

Schrödinger and KdV equations belong to the group of the most well-known and actively studied equations and their explicit solutions are of great interest. In particular, Bäcklund–Darboux transformations and related dressing procedures and commutation methods are fruitful approaches to the construction of explicit solutions of linear and integrable nonlinear equations (see, e.g., [2,3,7,8,12,15,16,17,18,20,21,23,24,25,27] and references therein). GBDT (generalized Bäcklund-Darboux transformation), which we use here, was first introduced in our paper [18], and a more general version of GBDT for first order systems rationally depending on the spectral parameter was treated in [19,23] (see also some references therein).

We construct GBDT and explicit solutions for the matrix Schrödinger equation

$$-y''(x,\lambda)+u(x)y(x,\lambda)=\lambda y(x,\lambda) \quad (u=u^*), \quad y':=\frac{d}{dx}y, \tag{1.1}$$

for the dynamical Schrödinger equation

$$\mathrm{i}\frac{\partial \psi}{\partial t}(x,t)=-\frac{\partial^2 \psi}{\partial x^2}(x,t)+u(x)\psi(x,t), \tag{1.2}$$

and for the matrix KdV equation

$$\frac{\partial u}{\partial t}-3u\frac{\partial u}{\partial x}-3\frac{\partial u}{\partial x}u+\frac{\partial^3 u}{\partial x^3}=0. \tag{1.3}$$

Here, $\mathrm{i}$ is the imaginary unit ($\mathrm{i}^2=-1$), $\lambda$ is the so called spectral parameter ($\lambda\in\mathbb{C}$, $\mathbb{C}$ stands for the complex plane), and $u(x)$ and $u(x,t)$ are $h\times h$ matrix functions.

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REMARK 1.1. In (1.1) and (1.2), we assume that $x$ belongs to some finite or infinite interval $\mathscr{I}$ $(x \in \mathscr{I})$, whereas $t$ in (1.2) belongs to the real axis $\mathbb{R}$. Without loss of generality, we assume also that $0 \in \mathscr{I}$ and speak later about parameter matrices $S(0)$ and $\Pi(0)$.

The main step in the construction of the explicit solutions of (1.1)–(1.3) via GBDT is the construction of the generalised eigenfunctions $\Pi(x)$ (or $\Pi(x,t)$). We consecutively construct $\Pi$ for our three systems using square roots of the generalised matrix eigenvalues. In this way, the results of the papers [5, 9, 14] are further developed and wider classes of $\Pi$ and solutions of (1.1)–(1.3) are obtained.

The next section is dedicated to the general construction of the solutions of (1.1)–(1.3). Interesting examples, including the case of strongly singular potentials, are treated in Section 3.

As usual, $\mathbb{N}$ is the set of positive integer numbers and $I_h$ is the $h \times h$ identity matrix.

## 2. Matrix Schrödinger and KdV equations

**1.** GBDT for Schrödinger equation (1.1) is determined by 3 parameter matrices. More precisely, we choose some *initial* system (1.1) (or, equivalently, the initial potential $u = u^*$ of Schrödinger equation (1.1)) and fix $n \in \mathbb{N}$. Then, we fix $n \times n$ matrices $A$ and $S(0)$, and an $n \times m$ $(m = 2h)$ matrix $\Pi(0)$ such that the following relations hold:

$$AS(0) - S(0)A^* = \Pi(0)j\Pi(0)^*, \quad S(0) = S(0)^*, \quad j := \begin{bmatrix} 0 & I_h \\ -I_h & 0 \end{bmatrix}. \tag{2.1}$$

Here, $j^* = j^{-1} = -j$. GBDT for Schrödinger equation is summed up, for instance, in [5, Sections 2,3]. In order to construct the potentials and solutions explicitly, we set here (similar to [9]) $u(x) \equiv 0$. That is, our initial system is trivial.

The matrix functions $\Pi(x)$ and $S(x)$ with fixed values $\Pi(0)$ and $S(0)$ are determined by the relations (see [5, (3.7)]):

$$\Pi'(x) = A\Pi(x) \begin{bmatrix} 0 & 0 \\ I_h & 0 \end{bmatrix} - \Pi(x) \begin{bmatrix} 0 & I_h \\ u(x) & 0 \end{bmatrix}, \tag{2.2}$$

$$S'(x) = \Pi(x) \begin{bmatrix} 0 & 0 \\ 0 & I_h \end{bmatrix} \Pi(x)^*. \tag{2.3}$$

Setting $u(x) \equiv 0$ and partitioning $\Pi$ into two $n \times h$ blocks

$$\Pi(x) = \begin{bmatrix} \Lambda_1(x) & \Lambda_2(x) \end{bmatrix}, \quad \Pi(0) = \begin{bmatrix} \vartheta_1 & \vartheta_2 \end{bmatrix}, \tag{2.4}$$

we rewrite the relations (2.2) and (2.3) as

$$\Lambda_1'(x) = A\Lambda_2(x), \quad \Lambda_2'(x) = -\Lambda_1(x); \tag{2.5}$$

$$S(x) = S(0) + \int_0^x \Lambda_2(\xi)\Lambda_2(\xi)^* d\xi. \tag{2.6}$$

The potential and solution of the GBDT-transformed Schrödinger equation are expressed in terms of $\Pi(x)$ and $S(x)$ [5,9], and it remains to calculate the matrix functions $\Pi(x)$ and $S(x)$ determined by (2.5), (2.6) and the given triple $\{A, S(0), \Pi(0)\}$.

LEMMA 2.1. *Let an $n\times n$ matrix $Q$ be a square root of $A$: $Q^2 = A$. Then, the matrix functions*

$$\Lambda_1(x) := -\mathrm{i}Q\big(\mathrm{e}^{\mathrm{i}xQ}f_1 - \mathrm{e}^{-\mathrm{i}xQ}f_2\big), \quad \Lambda_2(x) := \mathrm{e}^{\mathrm{i}xQ}f_1 + \mathrm{e}^{-\mathrm{i}xQ}f_2, \tag{2.7}$$

*where $f_1$ and $f_2$ are $n\times h$ matrices, satisfy (2.5). Correspondingly, the matrix function $\Pi(x) := \begin{bmatrix}\Lambda_1(x) & \Lambda_2(x)\end{bmatrix}$ satisfies (2.2) (where $u\equiv 0$). In the case*

$$-\mathrm{i}Q(f_1 - f_2) = \vartheta_1, \quad f_1 + f_2 = \vartheta_2, \tag{2.8}$$

*the matrix $\begin{bmatrix}\Lambda_1(x) & \Lambda_2(x)\end{bmatrix}$ takes the required value $\begin{bmatrix}\vartheta_1 & \vartheta_2\end{bmatrix}$ (determined by the triple $\{A, S(0), \Pi(0) = \begin{bmatrix}\vartheta_1 & \vartheta_2\end{bmatrix}\}$) at $x = 0$. Moreover, the integral part in (2.6) may (for each $Q$ and $\Pi(0)$) be explicitly calculated.*

*Proof.* Simple direct calculations show that $\Lambda_1$ and $\Lambda_2$ given by (2.7) satisfy (2.5) or, equivalently, that

$$\Pi(x) := \begin{bmatrix}\Lambda_1(x) & \Lambda_2(x)\end{bmatrix} \tag{2.9}$$

satisfies (2.2). Clearly, $\Pi(x)$ given by (2.7)–(2.9) takes the required value at $x = 0$. Finally, we note that the entries of $\Lambda_2(x)$ are sums of the terms of the form $p_k(x)\mathrm{e}^{\mathrm{i}xc_k}$, where $c_k \in \mathbb{C}$ and $p_k(x)$ are polynomials. Hence, the last statement in the lemma is valid. $\square$

REMARK 2.2. If $\det A \neq 0$ square roots $Q$ of $A$ always exist (see, e.g., [6, Chapter VIII, §6] with further details and references in [22, Section 2]). Clearly, $Q$ is invertible in this case. Therefore, the matrices

$$f_1 := \big(\vartheta_2 + \mathrm{i}Q^{-1}\vartheta_1\big)/2, \quad f_2 := \big(\vartheta_2 - \mathrm{i}Q^{-1}\vartheta_1\big)/2 \tag{2.10}$$

are well-defined. It is immediate that $f_1$ and $f_2$ given by (2.10) satisfy (2.8).

If the solutions $Z_k$ of the matrix equations

$$\mathrm{i}(QZ_1 - Z_1Q^*) = f_1f_1^*, \quad \mathrm{i}(QZ_2 + Z_2Q^*) = f_1f_2^*, \quad -\mathrm{i}(QZ_3 - Z_3Q^*) = f_2f_2^* \tag{2.11}$$

exist, formula (2.6) and the second equality in (2.7) yield the following representation of $S(x)$:

$$S(x) = \mathrm{e}^{\mathrm{i}xQ}Z_1\mathrm{e}^{-\mathrm{i}xQ^*} + \mathrm{e}^{\mathrm{i}xQ}Z_2\mathrm{e}^{\mathrm{i}xQ^*} + \mathrm{e}^{-\mathrm{i}xQ}Z_2^*\mathrm{e}^{-\mathrm{i}xQ^*} + \mathrm{e}^{-\mathrm{i}xQ}Z_3\mathrm{e}^{\mathrm{i}xQ^*}. \tag{2.12}$$

In view of (2.5), the matrix functions $\Lambda_k(x)$ also admit an essentially less convenient than (2.7) representation

$$\begin{bmatrix}\Lambda_1(x)\\ \Lambda_2(x)\end{bmatrix} = \mathrm{e}^{x\mathscr{A}}\begin{bmatrix}\vartheta_1 & \vartheta_2\end{bmatrix}, \quad \mathscr{A} = \begin{bmatrix}0 & A\\ -I_n & 0\end{bmatrix} \tag{2.13}$$

(see [5, (3.21)]). In the case $S(0)=I_n$, explicit (although somewhat inconvenient) expressions for $S(x)$ are presented in [9] in terms of the matrix exponent $\mathrm{e}^{\mathrm{i}xA_\gamma}$, where $A_\gamma$ is a $4n\times 4n$ matrix.

Assume that $y(x,\lambda)$ satisfies the trivial Schrödinger equation (1.1) (with $u\equiv 0$) and put $Y_0(x,\lambda):=\begin{bmatrix} y(x,\lambda)\\ y'(x,\lambda)\end{bmatrix}\in\mathbb{C}^m$ $(m=2h)$. Using [5, Proposition 3.5] (for the case $u\equiv 0$) and Lemma 2.1 above, we obtain the next theorem.

THEOREM 2.3. *Let a triple $\{A,S(0),\Pi(0)\}$ satisfy (2.1) and assume that $Q^2=A$. Let the matrix functions $\Pi(x)=\begin{bmatrix}\Lambda_1(x) & \Lambda_2(x)\end{bmatrix}$ and $S(x)$ be explicitly defined by formulas (2.6) and (2.7)–(2.9). Define (in the points of invertibility of $S(x)$) the GBDT-transformed $h\times h$ potential $\widetilde{u}(x)$ by the relations*

$$\widetilde{u}(x)=2\big(X_{12}(x)+X_{21}(x)+X_{22}(x)^2\big),\quad X_{ik}(x):=\Lambda_i(x)^*S(x)^{-1}\Lambda_k(x). \tag{2.14}$$

*Then, the function*

$$\widetilde{y}(x,\lambda)=[I_h\quad 0]w_A(x,\lambda)Y_0(x,\lambda), \tag{2.15}$$

$$w_A(x,\lambda):=I_m-j\Pi(x)^*S(x)^{-1}(A-\lambda I_n)^{-1}\Pi(x). \tag{2.16}$$

*satisfies the transformed matrix Schrödinger equation*

$$-\widetilde{y}''(x,\lambda)+\widetilde{u}(x)\widetilde{y}(x,\lambda)=\lambda\widetilde{y}(x,\lambda). \tag{2.17}$$

REMARK 2.4. It is easy to see that the vector functions $Y_0$ have the form

$$Y_0(x,\lambda)=W_0(x,\lambda)f_0,\quad W_0(x,\lambda):=\begin{bmatrix}\mathrm{e}^{\mathrm{i}x\sqrt{\lambda}}I_h & \mathrm{e}^{-\mathrm{i}x\sqrt{\lambda}}I_h\\ \mathrm{i}\sqrt{\lambda}\,\mathrm{e}^{\mathrm{i}x\sqrt{\lambda}}I_h & -\mathrm{i}\sqrt{\lambda}\,\mathrm{e}^{-\mathrm{i}x\sqrt{\lambda}}I_h\end{bmatrix}, \tag{2.18}$$

where $f_0\in\mathbb{C}^m$ are arbitrary constant vectors.

It is also immediate from (2.14) that $\widetilde{u}=\widetilde{u}^*$.

**2.** The same $\Pi(x)$ and $S(x)$ provide explicit solutions of the dynamical Schrödinger systems

$$\mathrm{i}\frac{\partial}{\partial t}\widetilde{\psi}(x,t)=\big(\widetilde{H}\widetilde{\psi}\big)(x,t),\quad \widetilde{H}:=-\frac{\partial^2}{\partial x^2}+\widetilde{u}(x). \tag{2.19}$$

Using again Lemma 2.1, we reformulate [5, Theorem 3.1] (for the case $u\equiv 0$).

THEOREM 2.5. *Let a triple $\{A,S(0),\Pi(0)\}$ satisfy (2.1) and assume that $Q^2=A$. Let the matrix functions $\Pi(x)=\begin{bmatrix}\Lambda_1(x) & \Lambda_2(x)\end{bmatrix}$ and $S(x)$ be explicitly defined by formulas (2.6) and (2.7)–(2.9). Define (in the points of invertibility of $S(x)$) the GBDT-transformed $h\times h$ potential $\widetilde{u}(x)$ by the relations (2.14).*

*Then, in the points of invertibility of $S(x)$, the $m\times n$ matrix function*

$$\widetilde{\psi}(x,t)=\begin{bmatrix}0 & I_h\end{bmatrix}\Pi(x)^*S(x)^{-1}\mathrm{e}^{-\mathrm{i}tA} \tag{2.20}$$

*satisfies the transformed dynamical Schrödinger system (2.19).*

**3.** In order to construct explicit solutions of the matrix KdV

$$\frac{\partial \widetilde{u}}{\partial t} - 3\widetilde{u}\frac{\partial \widetilde{u}}{\partial x} - 3\frac{\partial \widetilde{u}}{\partial x}\widetilde{u} + \frac{\partial^3 \widetilde{u}}{\partial x^3} = 0, \tag{2.21}$$

we add the variable $t$ in our matrix functions and consider

$$\Pi(x,t) = \begin{bmatrix}\Lambda_1(x,t) & \Lambda_2(x,t)\end{bmatrix} \quad \text{and} \quad S(x,t) \quad (x \in \mathscr{I}_1,\, t \in \mathscr{I}_2),$$

where $\mathscr{I}_1$ and $\mathscr{I}_2$ are intervals containing $0$. Instead of the matrix identity (2.1), we require

$$AS(0,0) - S(0,0)A^* = \Pi(0,0)j\Pi(0,0)^* \quad \big(S(0,0) = S(0,0)^*\big). \tag{2.22}$$

We partition $\Pi$ into the $n \times h$ blocks

$$\Pi(x,t) = \begin{bmatrix}\Lambda_1(x,t) & \Lambda_2(x,t)\end{bmatrix}, \quad \Pi(0,0) = \begin{bmatrix}\vartheta_1 & \vartheta_2\end{bmatrix}. \tag{2.23}$$

Equations (2.5) take the form

$$\frac{\partial}{\partial x}\Lambda_1(x,t) = A\Lambda_2(x,t), \quad \frac{\partial}{\partial x}\Lambda_2(x,t) = -\Lambda_1(x,t), \tag{2.24}$$

and another pair of PDEs is added (see [9, p. 372]):

$$\frac{\partial}{\partial t}\Lambda_1(x,t) = 4A^2\Lambda_2(x,t), \quad \frac{\partial}{\partial t}\Lambda_2(x,t) = -4A\Lambda_1(x,t). \tag{2.25}$$

Finally, $S(x,t)$ is determined by the relations (see [9, (5.6) and (5.9)]):

$$\frac{\partial}{\partial x}S = \Lambda_2\Lambda_2^*, \quad \frac{\partial S}{\partial t} = 4(A\Lambda_2\Lambda_2^* + \Lambda_2\Lambda_2^*A^* + \Lambda_1\Lambda_1^*). \tag{2.26}$$

Similar to Lemma 2.1, we derive the following lemma.

LEMMA 2.6. *Let an $n \times n$ matrix $Q$ be a square root of $A$: $Q^2 = A$. Then, the matrix functions*

$$\Lambda_1(x,t) := -\mathrm{i}Q\big(\mathrm{e}^{\mathrm{i}(xQ+4tQ^3)}f_1 - \mathrm{e}^{-\mathrm{i}(xQ+4tQ^3)}f_2\big), \tag{2.27}$$

$$\Lambda_2(x,t) := \mathrm{e}^{\mathrm{i}(xQ+4tQ^3)}f_1 + \mathrm{e}^{-\mathrm{i}(xQ+4tQ^3)}f_2, \tag{2.28}$$

*where $f_1$ and $f_2$ are $n \times h$ matrices, satisfy* (2.24) *and* (2.25). *If* (2.8) *holds, we have $\begin{bmatrix}\Lambda_1(0,0) & \Lambda_2(0,0)\end{bmatrix} = \Pi(0,0)$. Moreover, $S(x,t)$ may be explicitly calculated $($for each $Q$ and $\Pi(0,0))$ using* (2.26).

REMARK 2.7. If $\det A \neq 0$, the square root $Q$ always exists and the matrices $f_1, f_2$ satisfying (2.8) are given by (2.10) $($similar to the case of Remark 2.2$)$.

Equalities (2.24) and the first equality in (2.26) yield

$$\frac{\partial}{\partial x}(AS - SA^*) = \frac{\partial}{\partial x}(\Lambda_1\Lambda_2^* - \Lambda_2\Lambda_1^*). \tag{2.29}$$

Equalities (2.25) and the second equality in (2.26) yield

$$\frac{\partial}{\partial t}(AS - SA^*) = 4\big(A^2\Lambda_2\Lambda_2^* - \Lambda_2\Lambda_2^*(A^*)^2 + A\Lambda_1\Lambda_1^* - \Lambda_1\Lambda_1^*A^*\big),$$
$$\frac{\partial}{\partial t}(\Lambda_1\Lambda_2^* - \Lambda_2\Lambda_1^*) = 4\big(A^2\Lambda_2\Lambda_2^* - \Lambda_2\Lambda_2^*(A^*)^2 + A\Lambda_1\Lambda_1^* - \Lambda_1\Lambda_1^*A^*\big),$$

that is,

$$\frac{\partial}{\partial t}(AS - SA^*) = \frac{\partial}{\partial t}(\Lambda_1\Lambda_2^* - \Lambda_2\Lambda_1^*). \tag{2.30}$$

From (2.22), (2.29) and (2.30) we derive

$$AS(x,t) - S(x,t)A^* = \Pi(x,t)j\Pi(x,t)^*. \tag{2.31}$$

According to the proof of [9, Theorem 0.5], relations (2.24)–(2.26) and (2.31) imply that the matrix function

$$\widetilde{u}(x,t) = 2\big(X_{12}(x,t) + X_{21}(x,t) + X_{22}(x,t)^2\big), \tag{2.32}$$

where

$$X_{ik}(x,t) := \Lambda_i(x,t)^*S(x,t)^{-1}\Lambda_k(x,t), \tag{2.33}$$

satisfies KdV (2.21). Using also Lemma 2.6, we obtain the following theorem.

THEOREM 2.8. *Let a triple $\{A, S(0,0), \Pi(0,0)\}$ satisfy (2.22) and assume that $Q^2 = A$. Let the matrix functions $\Lambda_1(x,t)$, $\Lambda_2(x,t)$ and $S(x,t)$ be explicitly defined by formulas (2.26)–(2.28) and (2.8). Define $($in the points of invertibility of $S(x,t))$ the GBDT-transformed $h \times h$ potential $\widetilde{u}(x,t)$ by the relations (2.32) and (2.33). Then, $\widetilde{u}$ satisfies KdV equation (2.21).*

## 3. Examples

Let us consider several useful examples of the potentials $\widetilde{u}$ of Schrödinger equations (2.17) and (2.19) generated $($via relations (2.6), (2.7) and (2.14)$)$ by some special triples $\{A, S(0), \Pi(0) = \begin{bmatrix} \vartheta_1 & \vartheta_2 \end{bmatrix}\}$ satisfying (2.1). The corresponding explicit solutions $\widetilde{y}$ and $\widetilde{\psi}$ follow (in terms of $\Lambda_1(x)$, $\Lambda_2(x)$ and $S(x)$) from Theorems 2.3 and 2.5, respectively.

REMARK 3.1. Rational potentials $($rational extensions$)$ are of interest in applications $($see, e.g., [10,11] and references therein$)$. If $Q$ $($or, equivalently, $A)$ is nilpotent, it follows from (2.6), (2.7) and (2.14) that the entries of the potential $\widetilde{u}(x)$ are rational functions.

The simplest example is the case $A=0$.

EXAMPLE 3.2. [9, p. 371]. Assume that $A=Q=0$ and $\vartheta_1=0$. Then, (2.1) holds for any $S(0)=S(0)^*$ and any $\vartheta_2$. The equalities (2.8) are valid in the case $f_1+f_2=\vartheta_2$. Thus, by virtue of (2.6), (2.7) and (2.14) we have

$$\begin{aligned} &\Lambda_1(x)=0, \quad \Lambda_2(x)=\vartheta_2, \quad S(x)=S(0)+x\vartheta_2\vartheta_2^*, \\ &\widetilde{u}(x)=2\big(\vartheta_2^*\big(S(0)+x\vartheta_2\vartheta_2^*\big)^{-1}\vartheta_2\big)^2 \end{aligned} \tag{3.1}$$

REMARK 3.3. The expression (3.1) for $\widetilde{u}$ may be simplified $($especially for the scalar case $h=1)$ in an easy way, see (3.5) below.

Indeed, assume that $S(0)$ is invertible and rewrite (3.1) as

$$\widetilde{u}(x)=2\big(\vartheta_2^*\big(I_n+x\theta\vartheta_2^*\big)^{-1}\theta\big)^2, \quad \theta:=S(0)^{-1}\vartheta_2. \tag{3.2}$$

Then, using geometric progressions, we rewrite the resolvent $\big(I_n+x\theta\vartheta_2^*\big)^{-1}$ in the form

$$\begin{aligned} \big(I_n+x\theta\vartheta_2^*\big)^{-1}&=I_n-x\theta\left(\sum_{k=1}^{\infty}(-x\vartheta_2^*\theta)^{k-1}\right)\vartheta_2^* \\ &=I_n-x\theta(I_h+x\vartheta_2^*\theta)^{-1}\vartheta_2^*. \end{aligned} \tag{3.3}$$

The equality (3.3) holds for small $x$ and so (in view of the analyticity) for all points of invertibility. Taking into account (3.3), we obtain

$$\begin{aligned} \vartheta_2^*\big(I_n+x\theta\vartheta_2^*\big)^{-1}\theta&=\vartheta_2^*\theta+(I_h+x\vartheta_2^*\theta)^{-1}\vartheta_2^*\theta-(I_h+x\vartheta_2^*\theta)(I_h+x\vartheta_2^*\theta)^{-1}\vartheta_2^*\theta \\ &=(I_h+x\vartheta_2^*\theta)^{-1}\vartheta_2^*\theta. \end{aligned} \tag{3.4}$$

Relations (3.2) and (3.4) yield

$$\widetilde{u}(x)=2\big((I_h+x\vartheta_2^*\theta)^{-1}\vartheta_2^*\theta\big)^2. \tag{3.5}$$

Our next example deals with a slightly more complicated subcase of the case $A=0$.

EXAMPLE 3.4. Let $h=1$, $n=2$, $A=0$ and

$$Q=\begin{bmatrix}0&1\\0&0\end{bmatrix}, \quad \vartheta_1=\begin{bmatrix}b\\0\end{bmatrix}, \quad \vartheta_2=\begin{bmatrix}c\\0\end{bmatrix}, \quad S(0)=\begin{bmatrix}0&0\\0&d\end{bmatrix} \quad (d\neq 0), \tag{3.6}$$

where $b,c,d\in\mathbb{R}$. Clearly, we have $Q^2=A=0$.

In the case of the given above $\vartheta_1$ and $\vartheta_2$, we also have $\Pi(0)j\Pi(0)^*=0$. Thus, (2.1) holds for any $S(0)=S(0)^*$ and we choose $S(0)$ as in (3.6). For $f_i=\begin{bmatrix} f_{i1} \\ f_{i2}\end{bmatrix}$ $(i=1,2)$ relations (2.8) are equivalent to

$$f_{11}+f_{21}=c, \quad f_{12}=-f_{22}=\mathrm{i}b/2. \tag{3.7}$$

Further we assume that (3.7) (and so (2.8)) is valid and use Theorems 2.3 and 2.5. Since $Q^2=0$, the series representations of $\mathrm{e}^{\pm \mathrm{i}xQ}$ and relations (2.7) and (2.8) imply that

$$\Lambda_1(x)=-\mathrm{i}Q(f_1-f_2)=\vartheta_1, \quad \Lambda_2(x)=f_1+f_2+\mathrm{i}Qx(f_1-f_2)=\vartheta_2-x\vartheta_1. \tag{3.8}$$

Using (2.6) and taking into account (3.6) and (3.8), we obtain

$$S(x)=\begin{bmatrix}\gamma(x) & 0 \\ 0 & d\end{bmatrix}, \quad \gamma(x):=(b^2/3)x^3-bcx^2+c^2x. \tag{3.9}$$

Finally, relations (2.14), (3.8) and (3.9) yield

$$\begin{aligned}\widetilde{u}(x)&=\frac{4b}{\gamma(x)}(c-bx)+\frac{2}{\gamma(x)^2}(c-bx)^4 \\ &=\frac{2(bx-c)}{\gamma(x)^2}\big((b^3/3)x^3-b^2cx^2+bc^2x-c^3\big).\end{aligned} \tag{3.10}$$

REMARK 3.5. The case of the singularities and strong singularities of $\widetilde{u}(x)$, for instance,

$$\widetilde{u}(x)\sim \ell(\ell+1)/x^2 \quad \text{for} \quad x\to 0, \tag{3.11}$$

is of special interest (see, e.g., [4, 11, 13, 14] and the references therein). Formula (3.10) (for the Example 3.4 above) shows that

$$\widetilde{u}(x)=\frac{2}{x^2}\big(1+O(x)\big) \quad \text{for} \quad c\neq 0,\ x\to 0; \tag{3.12}$$

$$\widetilde{u}(x)=\frac{6}{x^2} \quad \text{for} \quad c=0, \quad b\neq 0. \tag{3.13}$$

Thus, we have the case $\ell=1$ for $c\neq 0$ and $\ell=2$ for $c=0$.

Another example for the case $\ell=1$ and an example for the case $\ell=3$ have been treated in [13] and [14], respectively. (In both cases, we had $S(0)=0$ but $\det A\neq 0$.)

EXAMPLE 3.6. Let us simplify formulas for fundamental solutions in our example (3.13) where $\ell=2$. Since $h=1$ and $c=0$, relations (3.6)–(3.9) and (2.16) yield $A=0$, $\vartheta_2=0$ and

$$\Pi(x)=\begin{bmatrix} b & -bx \\ 0 & 0\end{bmatrix}, \quad S(x)=\begin{bmatrix}(b^2/3)x^3 & 0 \\ 0 & d\end{bmatrix}, \tag{3.14}$$

$$\begin{bmatrix}1 & 0\end{bmatrix} w_A(x,\lambda)=\begin{bmatrix}1-\frac{3}{\lambda x^2} & \frac{3}{\lambda x}\end{bmatrix}. \tag{3.15}$$

Hence, Theorem 2.3 and formulas (2.15) and (2.18) imply that

$$\phi(x,\lambda) := \begin{bmatrix}1 & 0\end{bmatrix} w_A(x,\lambda)W_0(x,\lambda)\begin{bmatrix}1\\0\end{bmatrix} = \mathrm{e}^{\mathrm{i}x\sqrt{\lambda}}\left(1+\frac{3\mathrm{i}}{\sqrt{\lambda}\,x}-\frac{3}{\lambda x^2}\right) \tag{3.16}$$

satisfies Schrödinger equation with the potential $\widetilde{u}(x) = 6/x^2$ :

$$-\widetilde{y}''(x,\lambda)+\frac{6}{x^2}\widetilde{y}(x,\lambda) = \lambda\widetilde{y}(x,\lambda). \tag{3.17}$$

In view of (2.15) and (2.18), another solution $\chi(x,\lambda)$ of this equation is obtained by the substitution of $-\sqrt{\lambda}$ instead of $\sqrt{\lambda}$ on the right-hand side of (3.16), that is,

$$\chi(x,\lambda) = \mathrm{e}^{-\mathrm{i}x\sqrt{\lambda}}\left(1-\frac{3\mathrm{i}}{\sqrt{\lambda}\,x}-\frac{3}{\lambda x^2}\right). \tag{3.18}$$

Clearly, the branch of $\sqrt{\lambda}$ in (3.16) and (3.18) may be chosen in an arbitrary way (the same branch for both formulas).

Now, it is easy to construct a nonsingular at $x=0$ solution $\mathscr{Y}$ of the Schrödinger equation (3.17) as a linear combination of $\phi$ and $\chi$.

Namely, we take $\mathscr{Y}(x,\lambda) := \phi(x,\lambda)-\chi(x,\lambda)$. In view of (3.16) and (3.18), it is easily verified that

$$\begin{aligned}\mathscr{Y}(x,\lambda) =&(1+\mathrm{i}x\sqrt{\lambda})\left(1+\frac{3\mathrm{i}}{\sqrt{\lambda}\,x}-\frac{3}{\lambda x^2}\right)\\ &-(1-\mathrm{i}x\sqrt{\lambda})\left(1-\frac{3\mathrm{i}}{\sqrt{\lambda}\,x}-\frac{3}{\lambda x^2}\right)+O(1) = \frac{6\mathrm{i}}{\sqrt{\lambda}\,x}-\frac{6\mathrm{i}}{\sqrt{\lambda}\,x}+O(1)\end{aligned}$$

for $x\to 0$. That is, $\mathscr{Y}(x,\lambda)$ is nonsingular at $x=0$.

Using double commutation method, S. Albeverio, R. Hryniv, and Ya. Mykytyuk [1] studied the change of $\ell$ in the term $(\ell/x)\sigma_1$ in a radial Dirac system when an eigenvalue is removed or inserted. (See also a related paper [26].) The change of $\ell$ in the case of GBDT for radial Dirac systems was studied in [23, pp. 237–239] (see also the references therein) and the analog of this result for the Schrödinger equations would be of interest.

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