Operators and Matrices
Volume 16, Number 4 (2022), 1219–1237
doi:10.7153/oam-2022-16-79

GENERALIZED CHOI–KRAUS DILATIONS OF LINEAR MAPS BETWEEN MATRIX ALGEBRAS

DEGUANG HAN, QIANFENG HU* AND RUI LIU

(*Communicated by Y.-T. Poon*)

Abstract. By the generalized Stinespring's dilation theorem, every linear map between two matrix algebras M_n and M_d has $*$-homomorphism dilation due to the fact that such a map is always completely bounded. In fact, since every such a map has a generalized Choi-Kraus representation $\varphi(X) = \sum_{k=1}^{L} A_k X B_k^*$, it automatically induces a $*$-homomorphism dilation by the representation matrix system $\{A_k, B_k\}$, which we call it the generalized Choi-Kraus dilation for a linear map, and the Choi-Kraus dilation when $A_k = B_k$ for a completely positive (CP) map. The purpose of this paper is to examine the connections between the generalized Choi-Kraus dilations with other well-established dilations including the universal dilation. We prove that any linearly minimal $*$-homomorphism dilation is equivalent to a linearly minimal generalized Choi-Kraus dilation, and present a necessary and sufficient condition for the equivalence of two linearly minimal generalized Choi-Kraus dilations. While all the linearly minimal Choi-Kraus dilations for a CP map are unitarily equivalent, the linearly minimal generalized Choi-Kraus dilations, even for a CP map, are not necessarily equivalent. In fact, a linear map admits only one equivalent class of linearly minimal generalized Choi-Kraus dilations if and only if its generalized Choi matrix has full rank.

1. Introduction

Dilation theory [1, 13, 14] is a paradigm for studying operators by exhibiting an operator as a compression of another operators which is well behaved in some sense, which boosts the study of classes of operators not only on Hilbert spaces but also Banach spaces [5]. While the finite dimensional approach to dilation theory or special representations of linear maps on vector spaces [6, 9] provides insights into the concrete property of maps, for example dilation dimension, positivity (including completely positivity [2], k-positivity [7, 16]), entanglement detection of quantum states [17], or preservation of separability of states [8] etc. Exploring the limits of the finite dimensional approach may enlighten us to develop the techniques for dilation theory of operators in infinite dimensional spaces. A typical example is quantum channel (completely positive and trace preserving map, CPTP) in infinite dimensional Hilbert spaces [19] and interpolations of some special quantum channels [10].

Mathematics subject classification (2020): 46G10, 46L07, 47A20, 47A15, 81P47.
Keywords and phrases: Linearly minimal $*$-homomorphism dilation system, generalized Choi-Kraus dilation, principle dilation, universal dilation, equivalent dilations.
This research is supported by National Science Foundation (Nos. DMS-2105038), the National Natural Science Foundation of China (Grants Nos. 11671214, 11971348 and 12071230), and the Young Academia Leaders Program of Nankai University.
* Corresponding author.

Recently for operator-valued measures [5, 11], Ramsey et al. [18] studied the measurable operator-valued functions with respect to an operator-valued measure and Han et al.[5] developed the theory about projection-valued dilations for operator-valued measures or more generally bounded homomorphism dilations for bounded linear maps on Banach algebras. Moreover, to better understand the underlying algebraic structure of the dilation theory, in [6], they also explored a pure algebraic version of the dilation theory for a linear system which is a triple $(\varphi, \mathscr{A}, V)$ such that φ is a linear map from a unital algebra $\mathscr{A}$ to the space of all linear maps from vector space V to V, i.e., $L(V)$. In this paper, we turn to a class of concrete maps: linear maps between matrix algebras.

One of the well-known dilation theorems is due to Stinespring [21] and asserts that a completely positive map ϕ from C^*-algebras $\mathscr{A}$ into $B(\mathscr{H})$ (the linear bounded operators on the Hilbert space $\mathscr{H}$) admits a $*$-homomorphisms π into $B(\mathscr{K})$ for some other Hilbert space $\mathscr{K}$, that is, there exists a bounded operator $V : \mathscr{H} \to \mathscr{K}$ such that for any $a \in \mathscr{A}$, $\phi(a) = V^*\pi(a)V$, which we call the Stinespring dilation of φ. Applying the above result to the completely bounded maps leads to Wittstock's decomposition theorem [22, 23, 14] : Let $\varphi : \mathscr{A} \to B(\mathscr{H})$ be a completely bounded map. Then there exists a Hilbert space $\mathscr{K}$ and bounded operators $V_1, V_2 : \mathscr{H} \to \mathscr{K}$ such that $\varphi(a) = V_1^*\pi(a)V_2$ for all $a \in \mathscr{A}$. This is also refereed to as the generalized Stinespring's dilation theorem [14, 15]. Due to Roger smith [20], any bounded linear map from a C^*-algebras into $M_n(\mathbb{C})$ is completely bounded. So all linear maps from the matrix algebra $M_n(\mathbb{C})$ to $M_d(\mathbb{C})$ are completely bounded hence every such a linear map has a $*$-homomorphism dilation.

In this paper we examine all the dilations of linear maps between two matrix algebras in connection with their generalized Choi-Kraus representations. In [2] Choi introduced the Choi-Kraus representations for completely positive maps on complex matrices, which has played an essential role in operator algebra [14] and has a wide range of applications in mathematical physics, such as quantum measurement theory [4], quantum information theory [12]. Following the same argument, every linear map between matrix algebras has a generalized Choi-Kraus representation that naturally induces a $*$-homomorphism dilation, and we will call it the generalized Choi-Kraus dilations. Moreover, the non-uniqueness of generalized Choi-Kraus representations leads to the diversity of those dilations. In this paper we establish some classification results among these dilations as well as some other known dilations. We show that any linearly minimal $*$-homomorphism dilation of a linear map between matrix algebras is equivalent to one of its linearly minimal generalized Choi-Kraus dilations. This allows us to focus only on generalized Choi-Kraus dilations. We derive a condition under which a generalized Choi-Kraus representation induces a principle dilation. By comparing with the universal dilation and the principle dilation, we classify generalized Choi-Kraus dilations in terms of the invariant subspaces of their associated matrix representation systems. For a completely positive map, all the linearly minimal Choi-Kraus dilations are unitarily equivalent. However, the generalized ones are not necessarily equivalent. We prove that all the linearly minimal generalized Choi-Kraus dilations of a linear map are equivalent if and only its generalized Choi matrix has the full rank. Moreover, the number of the inequivalent classes of linearly minimal dilations will only be 1, 2, ∞.

2. Preliminary

Let $\mathbb{C}$ be the complex number field. We use the following standard notations.

- Vectors $v \in \mathbb{C}^n$ will be treated as column vectors $v = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ and its adjoint gives a row vector $v^* := \left(\overline{\alpha}_1, \cdots, \overline{\alpha}_n\right)$, and $v(j) = \alpha_j$ is the j-th entry of the vector v. We use $\{e_i\}_{i=1}^n$ for the standard basis of $\mathbb{C}^n$, still considered as column vectors.
- $M_{n,m}(\mathbb{C})$ — the $n \times m$ complex matrices, abbreviated as $M_{n,m}$ $(M_n = M_{n,n}(\mathbb{C}))$. M_n^+ — the set of positive semidefinite matrices, and we write $X \geqslant 0$ if $X \in M_n^+$. $E_{i,j}$ — the standard matrix units. I_n — the identity matrix in M_n and $A(i,j)$ — the (i,j)-entry of a matrix A.
- $L(M_n, M_d)$ — the set of all linear maps from M_n to M_d and $CP(M_n, M_d)$ — the set of all completely positive maps from M_n to M_d.
- $\ker A$ — the kernel of the matrix A. $\mathrm{Im} A$ — the image space of A. $\mathrm{Rank}(A)$ — the rank of A. A^* — the adjoint of matrix A, A^T — the transpose of matrix A.
- $A \otimes B$ — tensor product of the matrices A and B. Suppose $A = (a_{i,j})$, for convenience, we set $A \otimes B$ in block form by
$$\begin{pmatrix} a_{1,1}B & \cdots & a_{1,n}B \\ \vdots & \ddots & \vdots \\ a_{n,1}B & \cdots & a_{n,n}B \end{pmatrix}$$
which is also referred as the Kronecker product of A and B.

Note that we actually follow the physicists' convention to take inner products to be linear in the second variable and conjugate linear in first. Besides, we can also write $|v\rangle := v$ and $\langle w| := w^*$, then $v^* w = \langle v \mid w \rangle$ and $vw^* =: |v\rangle\langle w|$.

DEFINITION 2.1. A linear map $\varphi : M_n \to M_d$ is positive if
$$\varphi(X) \geqslant 0, \quad \text{whenever } X \geqslant 0.$$
For any $\varphi \in L(M_n, M_d), k \in \mathbb{N}$, its k-th ampliation $\varphi_k : M_k(M_n) \to M_k(M_d)$ is defined by
$$\varphi_k\Big(\big(X_{i,j}\big)\Big) = \Big(\varphi(X_{i,j})\Big), \quad 1 \leqslant i, j \leqslant k.$$
φ is said to be a completely positive (CP, for short) map if its k-th ampliation is positive for all $k \in \mathbb{N}$.

To characterize completely positive linear maps on complex matrices, Choi [2] defines the Choi matrix $\big(\phi(E_{i,j})\big)$ of $\phi \in CP(M_n, M_d)$. Similarly, we define the generalized Choi matrix G_φ of $\varphi \in L(M_n, M_d)$ by
$$G_\varphi = (\varphi(E_{i,j})) \in M_n(M_d).$$

If we write G_φ as $\sum_{i=1}^m v_i w_i^*$, where for $1 \leqslant i \leqslant m$, $v_i = \begin{pmatrix} \alpha_1^i \\ \vdots \\ \alpha_n^i \end{pmatrix}$, $w_i = \begin{pmatrix} \beta_1^i \\ \vdots \\ \beta_n^i \end{pmatrix} \in \mathbb{C}^{nd}$, and $\alpha_j^i, \beta_j^i \in \mathbb{C}^d, 1 \leqslant j \leqslant n$. Let $A_i = \left[\alpha_1^i, \vdots, \cdots, \vdots, \alpha_n^i\right]_{d,n}$ and $B_i := \left[\beta_1^i, \vdots, \cdots, \vdots, \beta_n^i\right]_{d,n}$, then it can be verified that

$$\varphi(X) = \sum_{i=1}^m A_i X B_i^*, \tag{1}$$

which is called a *generalized Choi-Kraus representation* of φ [15], where $\{A_i, B_i^*\}_{i=1}^m$ is called the Kraus matrix pairs.

Conversely, if $\varphi(X) = \sum_{i=1}^m A_i X B_i^*$ for some $A_i := \left[\alpha_1^i, \vdots, \cdots, \vdots, \alpha_n^i\right]_{d,n}$ and $B_i := \left[\beta_1^i, \vdots, \cdots, \vdots, \beta_n^i\right]_{d,n}$, and set $v_i = \begin{pmatrix} \alpha_1^i \\ \vdots \\ \alpha_n^i \end{pmatrix}$, $w_i = \begin{pmatrix} \beta_1^i \\ \vdots \\ \beta_n^i \end{pmatrix} \in \mathbb{C}^{nd}$, then $G_\varphi = \sum_{i=1}^m v_i w_i^*$. In particular, if we define

$$A_\varphi = \sum_{i=1}^m v_i e_i^*, \quad B_\varphi = \sum_{i=1}^m w_i e_i^*, \tag{2}$$

then

$$G_\varphi = A_\varphi B_\varphi^*. \tag{3}$$

Similar to Choi rank $\mathrm{Cr}(\phi)$ of $\phi \in CP(M_n, M_d)$ to $\varphi \in L(M_n, M_d)$, the generalized Choi rank of φ is given by

$$\mathrm{Gr}(\varphi) := \min\left\{k : \varphi(X) = \sum_{i=1}^k A_i X B_i^*\right\}$$

and we have

$$\mathrm{Gr}(\varphi) = \mathrm{Rank}(G_\varphi). \tag{4}$$

Of course, these concepts coincide when the linear maps are completely positive maps.

Now we recall the dilation theory on general linear systems in [6]. A linear system is a triple $(\varphi, \mathscr{A}, V)$ such that φ is a linear map from a unital algebra $\mathscr{A}$ to $L(V)$, where V is a vector space and $L(V)$ denotes the space of all linear maps from V to V.

DEFINITION 2.2. A homomorphism dilation system of a linear system $(\varphi, \mathscr{A}, V)$ is a unital homomorphism π from $\mathscr{A}$ to $L(W)$ for some vector space W such that there exist an injective linear map $T : V \to W$ and a surjective linear map $S : W \to V$ such that for all $a \in \mathscr{A}$, the following diagram commutes:

$$\begin{tikzcd} W \arrow[r, "\pi(a)"] & W \arrow[d, "S"] \\ V \arrow[u, "T"] \arrow[r, "\varphi(a)"] & V \end{tikzcd}.$$

That is

$$\varphi(a) = S\pi(a)T, \quad \forall\, a \in \mathscr{A}.$$

We will use (π, S, T, W) to denote this homomorphism dilation system (also called dilation for short). For our convenience, we call S as the synthesis operator, T as the analysis operator, W as the dilation space.

For a given linear system, there are already two special ways to construct dilations: the canonical dilation and the universal dilation. While the canonical dilation was initially introduced in [5, Section 4.1] for dilation for operator-valued measure and finally named together with the universal dilation as the main tools to investigate the structural properties of dilations for linear systems [6]. Here we list the universal dilation for later use.

The universal dilation: Let $W_u = \mathscr{A} \otimes V$. For $\sum_i c_i a_i \otimes x_i \in W_u$ where $\{c_i\} \subset \mathbb{C}, \{a_i\} \subset \mathscr{A}, \{x_i\} \subset V$, define

$$\begin{aligned}
&\pi_u : \mathscr{A} \to L(W_u), \quad && \pi_u(b)\Big(\sum_i c_i a_i \otimes x_i\Big) = \sum_i c_i (b a_i) \otimes x_i,\ \forall\, b \in \mathscr{A}.\\
&T_u : V \to W_u, && T_u x = 1 \otimes x,\ \forall\, x \in V.\\
&S_u : W_u \to V, && S_u\Big(\sum_i c_i a_i \otimes x_i\Big) = \sum_i c_i \varphi(a_i) x_i.
\end{aligned}$$

It can be checked that all the maps are well-defined and π_u is a homomorphism and

$$S_u \pi_u(a) T_u x = \varphi(a)x,\ \forall a \in \mathscr{A}, x \in V.$$

Thus, (π_u, S_u, T_u, W_u) is a homomorphism dilation system and we call it the universal dilation of $(\varphi, \mathscr{A}, V)$.

While for $\varphi \in L(M_n, M_d)$, suppose its generalized Choi-Kraus representation is $\varphi(X) = \sum_{i=1}^m A_i X B_i^*$ where $A_i, B_i \in M_{d,n}, 1 \leqslant i \leqslant m$. Define $\pi : M_n \to M_{mn}$, and $A, B : \mathbb{C}^{mn} \to \mathbb{C}^d$ by

$$A = (A_1, \cdots, A_m), \quad \pi_1(X) = \begin{pmatrix} X & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X \end{pmatrix} = I_m \otimes X, \quad B = (B_1, \cdots, B_m).$$

It is easy to verify that π is a unital $*$-homomorphism and for all $X \in M_n, x \in \mathbb{C}^d$,

$$A\pi(X)B^* x = \sum_{i=1}^m A_i X B_i^* x = \varphi(X)x.$$

If $\varphi \in CP(M_n, M_d)$ with the Choi-Kraus representation $\varphi(X) = \sum_{i=1}^m A_i X A_i^*$. Define $\pi : M_n \to M_{mn}$, and $A : \mathbb{C}^{mn} \to \mathbb{C}^d$ by

$$A = (A_1, \cdots, A_m), \quad \pi(X) = \begin{pmatrix} X & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X \end{pmatrix} = I_m \otimes X.$$

Then π is also a unital $*$-homomorphism and $\varphi(X) = A\pi(X)A^*$ for all $X \in M_n$.

DEFINITION 2.3. For $\varphi \in L(M_n, M_d)$, the above constructed dilation $(\pi, A, B^*, \mathbb{C}^{mn})$ is called the generalized Choi-Kraus dilation induced by $\varphi(X) = \sum_{i=1}^m A_i X B_i^*$. For $\varphi \in CP(M_n, M_d)$, $(\pi, A, A^*, \mathbb{C}^{mn})$ is called the Choi-Kraus dilation of φ induced by $\varphi(X) = \sum_{i=1}^m A_i X A_i^*$.

REMARK 1. (i) A CP map can have a generalized Choi-Kraus representation $\varphi(X) = \sum_{i=1}^m A_i X B_i^*$, where A_i's are not necessarily the same as B_i's. In this case the induced generalized Choi-Kraus dilation could be quite different (inequivalent) from the Choi-Kraus dilation induced by $\varphi(X) = \sum_{i=1}^m C_i X C_i^*$ (see Proposition 3.8 or Example 3.12).

(ii) There is a simple question concerning the generalized Choi-Kraus dilation $(\pi, A, B^*, \mathbb{C}^{mn})$ for φ is whether $\mathrm{span}\{\pi(X)B^*x, X \in M_n, x \in \mathbb{C}^d\}$ is $\mathbb{C}^{mn}$. It is in general not true (see Corollary 3.5). While, given a homomorphism dilation system (π, S, T, W) of a linear system $(\varphi, \mathscr{A}, V)$, we choose the dilation space as the subspace

$$\pi(\mathscr{A})TV = \mathrm{span}\{\pi(a)Tx : a \in \mathscr{A}, x \in X\}$$

so that the restriction of π to $\pi(\mathscr{A})TV$ still defines a homomorphism.

Besides, if $\ker S$ contains a nonzero π-invariant subspace K, then we define $\widetilde{W} = W/K$, and let $\widetilde{S} : \widetilde{W} \to V$, $\widetilde{T} : V \to \widetilde{W}$ and $\widetilde{\pi} : \mathscr{A} \to L(\widetilde{W})$ be the induced linear maps. Then we have

$$\widetilde{S}\widetilde{\pi}(a)\widetilde{T}(x) = \varphi(a)x, \forall a \in \mathscr{A}, x \in V.$$

Thus $(\widetilde{\pi}, \widetilde{S}, \widetilde{T}, \widetilde{W})$ is a homomorphism dilation of $(\varphi, \mathscr{A}, V)$ and we call it the reduced homomorphism dilation of (π, S, T, W) with respect to K.

Thus, we give the following definition.

DEFINITION 2.4. A homomorphism dilation system (π, S, T, W) of a linear system $(\varphi, \mathscr{A}, V)$ is called *reducible* if $\ker S$ contains a nonzero π-invariant subspace, otherwise it is called *irreducible*. The space $\pi(\mathscr{A})TV$ is called the linearly minimal dilation space whose dimension is called the dilation dimension. And we call the dilation is *linearly minimal* if $W = \pi(\mathscr{A})TV$. It is called a *principle dilation* if it is both linearly minimal and irreducible.

Clearly, given a dilation (π, S, T, W), we replace W with the subspace $\pi(\mathscr{A})TV$, then $(\pi, S, T, \pi(\mathscr{A})TV)$ is also a dilation of $(\varphi, \mathscr{A}, V)$, which is a linearly minimal one. In what follows we mainly focus on linearly minimal dilations. In particular, the universal dilation is a linearly minimal dilation with the largest dilation dimension $\dim(\mathscr{A})\dim(V)$.

Han et al. [6] established several results to better understand the structural properties of all linearly minimal dilations, we list some here for later reference.

DEFINITION 2.5. For a linear system $(\varphi, \mathscr{A}, V)$, we say two linearly minimal homomorphism dilation systems (π_1, S_1, T_1, W_1), (π_2, S_2, T_2, W_2) are equivalent if there exists a bijective linear map $R : W_1 \to W_2$ such that

$$RT_1 = T_2, \quad S_2R = S_1, \quad \pi_1(a) = R^{-1}\pi_2(a)R, \ \forall a \in \mathscr{A}.$$

Moreover, if R is a unitary, we call two dilations are unitarily equivalent.

THEOREM 2.6. *All the principle homomorphism dilation systems for a linear system are equivalent.*

Any linearly minimal dilation is equivalent to a reduced dilation of its universal dilation. To be more precise, we have

THEOREM 2.7. *Let* (π_1, S_1, T_1, W_1) *be a linearly minimal homomorphism dilation system and* (π_u, S_u, T_u, W_u) *be the universal dilation system for a linear system* $(\varphi, \mathscr{A}, V)$*. Then* (π_1, S_1, T_1, W_1) *is equivalent to a reduced homomorphism dilation system of* (π_u, S_u, T_u, W_u) *with respect to*

$$K = \Big\{ w = \sum_i c_i a_i \otimes x_i \in W_u : \sum_i c_i \pi_1(a_i) T_1 x_i = 0 \Big\},$$

which is called the reduced subspace associated with (π_1, S_1, T_1, W_1)*.*

Moreover, two linearly minimal homomorphism dilation systems are equivalent if and only if the corresponding reduced subspaces are the same.

The above theorem indicates that the reduced subspace determines an equivalent class of linearly minimal dilations, thus we can give a classification of all linearly minimal dilations based on the reduced subspaces.

Note that the above results are about homomorphism dilation systems of linear systems, when we consider the $*$-homomorphism dilations of the linear maps on matrix algebras, those theorems remain valid in our setting.

3. Generalized Choi-Kraus dilations

The first result tells us that it is enough to work with linearly minimal generalized Choi-Kraus dilations.

THEOREM 3.1. *For* $\varphi \in L(M_n, M_d)$*, any linearly minimal* $*$*-homomorphism dilation of* φ *is equivalent to a linearly minimal generalized Choi-Kraus dilation.*

Proof. Suppose (π, S, T, W_1) is a linearly minimal $*$-homomorphism. Then the reduced subspace associated with it is

$$K = \Big\{ w = \sum_i c_i a_i \otimes x_i \in W_u : \sum_i c_i \pi(a_i) T x_i = 0 \Big\}.$$

By Theorem 2.7, it remains to show that there is a linearly minimal generalized Choi-Kraus dilation such that the corresponding reduced subspace is K as well.

We first recall the universal dilation of $\varphi \in L(M_n, M_d)$, as a special linear system $(\varphi, M_n, \mathbb{C}^d)$, the dilation space is $W_u = M_n \otimes \mathbb{C}^d$, in the light of the one-to-one correspondence between W_u and $\mathbb{C}^{n^2 d}$. That is, here we choose $\mathbb{C}^{n^2 d}$ as the dilation space of the universal dilation. Then the maps $T_u : \mathbb{C}^d \to W_u, \pi_u : M_n \to L(W_u)$ are realized concretely as $T_u : \mathbb{C}^d \to \mathbb{C}^{n^2 d}$, $\pi_u : M_n \to M_{n^2 d}$ and formulated by

$$T_u = \begin{pmatrix} E_{1,1} \\ \vdots \\ E_{1,d} \\ E_{2,1} \\ \vdots \\ \vdots \\ E_{n,d} \end{pmatrix}, \quad \pi_u(X) = \begin{pmatrix} X & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X \end{pmatrix} = I_{nd} \otimes X,\ \forall\, X \in M_n. \tag{5}$$

Write $G_\varphi = \left(\varphi(E_{i,j})\right) = \sum_{i=1}^{nd} v_i e_i^*$, where $v_i = \begin{pmatrix} \alpha_1^i \\ \vdots \\ \alpha_n^i \end{pmatrix} \in \mathbb{C}^{nd}$ and $\alpha_j^i \in \mathbb{C}^d, 1 \leqslant j \leqslant n$ and $\{e_i\}_{i=1}^{nd}$ is the standard basis of $\mathbb{C}^{nd}$. Define $V_i = \left(\alpha_1^i, \cdots, \alpha_n^i\right) \in M_{d,n}$. Then $S_u = (V_1, \cdots, V_{nd})$.

Represent $K \subset W_u = \mathbb{C}^{n^2 d} = \mathbb{C}^{nd} \otimes \mathbb{C}^n$, for arbitrary $w \in K$ as $\sum_{i=1}^n w_i \otimes e_i$, where $\{w_i\}_{1=i}^n \subset \mathbb{C}^{nd}$ and $\{e_i\}_{1=i}^n$ is standard basis of $\mathbb{C}^n$. Since K is π_u-invariant subspace in $\ker S_u$, that is, for all $X \in M_n$, $\pi_u(X)w \in K \subset \ker S_u$, it follows that

$$\pi_u(E_{i,j})w \in K \subset \ker S_u, 1 \leqslant i, j \leqslant n.$$

On the one hand, fix j, for $1 \leqslant i \leqslant n$, as

$$\pi_u(E_{i,j})w = (I_{nd} \otimes E_{i,j})w = w_j \otimes e_i,$$

then deriving the condition $\{w_j \otimes e_i\}_{i=1}^n \subset \ker S_u$ implies that $w_j \in \ker G_\varphi$. On the other hand, for each i, for $1 \leqslant j \leqslant n$, $\pi(E_{i,j})w \in K$, that is, $\{w_j \otimes e_i\}_{j=1}^n \subset K$. Thus we conclude form those properties that $K = M \otimes \mathbb{C}^n$, where M is a subspace of $\ker G_\varphi$.

We construct a matrix D such that $\ker D = M$, a slight of variation of Douglas' Factorization Theorem in [3], there exists C such that $CD = G_\varphi$, as we can write $C = \sum_{i=1}^m v_i e_i^*, D = \sum_{i=1}^m e_i w_i^*$ for some m, where $\{e_i\}_{i=1}^m$ is the standard basis of $\mathbb{C}^m$ and $\{v_i, w_i\}_{i=1}^m \subset \mathbb{C}^{nd}$. Besides, we can rearrange vectors $\{v_i, w_i\}_{i=1}^m$ into the Kraus matrix pairs $\{A_i, B_i\}_{i=1}^m \subset M_{d,n}$ such that $\varphi(X) = \sum_{i=1}^m A_i X B_i^*$. Then we can show (by the following Corollary 3.6 which does not depend on this theorem) that the reduced subspace associated with the linearly minimal dilation (π, A, B^*, W_2) induced by $\varphi(X) = \sum_{i=1}^m A_i X B_i^*$ is K as well. Therefore, (π, S, T, W_1) is equivalent to the linearly minimal generalized Choi-Kraus dilation (π, A, B^*, W_2). □

PROPOSITION 3.2. *Let $\varphi \in L(M_n, M_d)$ with $\mathrm{Gr}(\varphi) = r$. Then the generalized Choi-Kraus dilation $(\pi, A, B^*, \mathbb{C}^{nr})$ of φ induced by its generalized Choi-Kraus representation $\varphi(X) = \sum_{i=1}^r A_i X B_i^*$ is a principle dilation.*

Proof. By definition, the generalized Choi-Kraus dilation $(\pi, A, B^*, \mathbb{C}^{nr})$ of φ is given by

$$A = (A_1, \cdots, A_r), \quad B = (B_1, \cdots, B_r), \quad \pi(X) = \begin{pmatrix} X & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X \end{pmatrix} = I_r \otimes X, \forall X \in M_n.$$

We first show it is irreducible, namely, $\ker A$ does not contain any nontrivial π-invariant subspaces. Suppose K is π-invariant subspace in $\ker A$, meaning that if $v \in K$, then $\pi(X)v \in K \subset \ker A$ for all $X \in M_n$. Note that $K \subset W \subset \mathbb{C}^{nr}$, we set

$$v = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_r \end{pmatrix} \in \mathbb{C}^{nr} \quad \text{where} \quad \alpha_i = \begin{pmatrix} \alpha_i(1) \\ \alpha_i(2) \\ \vdots \\ \alpha_i(n) \end{pmatrix} \in \mathbb{C}^n.$$

We first choose $E_{1,1}$, then $\pi_1(E_{1,1})v \in K \subset \ker A$ and thus

$$A\pi_1(E_{1,1})v = (A_1, \cdots, A_r) \begin{pmatrix} E_{1,1}\alpha_1 \\ \vdots \\ E_{1,1}\alpha_r \end{pmatrix} = \mathbf{0} \text{ (zero vector)}.$$

That is,

$$\alpha_1(1) \begin{pmatrix} A_1(1,1) \\ \vdots \\ A_1(d,1) \end{pmatrix} + \alpha_2(1) \begin{pmatrix} A_2(1,1) \\ \vdots \\ A_2(d,1) \end{pmatrix} + \cdots + \alpha_r(1) \begin{pmatrix} A_r(1,1) \\ \vdots \\ A_r(d,1) \end{pmatrix} = \mathbf{0} \text{ (zero vector)}.$$

Continuing this way, namely, we take $\{E_{i,1}\}_{i=2}^n$ and arrange the conditions $\{\pi_1(E_{i,1})v\}_{i=1}^n \subset \ker A$ into

$$\alpha_1(1)A_1 + \alpha_2(1)A_2 + \cdots + \alpha_r(1)A_r = \mathbf{0} \text{ (zero matrix)}.$$

As $\mathrm{Gr}(\varphi) = r$ and $\varphi(X) = \sum_{i=1}^r A_i X B_i^*$, then we claim $\{A_i\}_{i=1}^r$ is linearly independent, otherwise, we can recombine the Kraus matrix pairs $\{A_i, B_i^*\}_{i=1}^r$ into $\{C_i, D_i^*\}_{i=1}^k$ where $k < r$, which contradicts the minimality of the generalized Choi rank. Consequently,

$$\alpha_1(1) = \alpha_2(1) = \cdots = \alpha_r(1) = 0.$$

For each $2 \leqslant j \leqslant n$, we repeat the above argument, that is, we take $E_{1,j}, E_{2,j}, \cdots, E_{n,j}$ such that $\pi(E_{i,j})v \in \ker A$ for all $1 \leqslant i \leqslant n$. Then it follows that

$$\alpha_1(j)A_1 + \alpha_2(j)A_2 + \cdots + \alpha_r(j)A_r = \mathbf{0} \text{ (zero matrix)}.$$

Similarly, by the linear independence of $\{A_i\}_{i=1}^r$, we get $\alpha_1(j) = \alpha_2(j) = \cdots = \alpha_r(j) = 0$. To summarise,

$$\alpha_i(j) = 0, \ 1 \leqslant i \leqslant r, \ 1 \leqslant j \leqslant n.$$

that is, $v=0$, meaning $\{0\}$ is the only π-invariant subspace in $\ker A$.

Next, suppose W is the linearly minimal dilation space, that is, $W=\operatorname{span}\{\pi(X)B^*x : X\in M_n, x\in\mathbb{C}^d\}$. Note that

$$\sum_k c_k\pi(X_k)B^*x_k=\begin{pmatrix}\sum_k c_kX_kB_1^*x_k\\ \vdots\\ \sum_k c_kX_kB_m^*x_k\end{pmatrix}$$

$$=\begin{pmatrix} B_1^*(1,1)I_n & \cdots & B_1^*(1,d)I_n & B_1^*(2,1)I_n & \cdots\cdots & B_1^*(n,d)I_n\\ \vdots & \cdots & \vdots & \vdots & \cdots\cdots & \vdots\\ B_m^*(1,1)I_n & \cdots & B_m^*(1,d)I_n & B_m^*(2,1)I_n & \cdots\cdots & B_m^*(n,d)I_n\end{pmatrix}\begin{pmatrix}\sum_k c_kX_kE_{1,1}x_k\\ \vdots\\ \sum_k c_kX_kE_{1,d}x_k\\ \sum_k c_kX_kE_{2,1}x_k\\ \vdots\\ \vdots\\ \sum_k c_kX_kE_{n,d}x_k\end{pmatrix}. \tag{6}$$

Using the notations defined in Equation (2) and Equation (5), we have,

$$\sum_k c_k\pi(X_k)B^*x_k=(B_\varphi^*\otimes I_n)\Big(\sum_k c_k\pi_u(X_k)T_ux_k\Big)=(B_\varphi^*\otimes I_n)\big(\sum_k c_kX_k\otimes x_k\big).$$

Hence

$$W=(B_\varphi^*\otimes I_n)(W_u)=(B_\varphi^*\otimes I_n)(\mathbb{C}^{n^2d}). \tag{7}$$

That is, the linearly minimal dilation space of dilation $(\pi,A,B^*,\mathbb{C}^{nr})$ is $\operatorname{Im}(B_\varphi^*\otimes I_n)$.

A similar argument gives that $\{B_i\}_{i=1}^r$ is linearly independent, and thus $\operatorname{Rank}(B_\varphi^*)=r$. Therefore $\dim W=\operatorname{Rank}(B_\varphi^*\otimes I_n)=nr$. Moreover, since $W\subset\mathbb{C}^{nr}$, it follows that $W=\mathbb{C}^{nr}$. Therefore we have proved that $(\pi,A,B,\mathbb{C}^{nr})$ is irreducible and linearly minimal hence is a principle dilation. $\square$

REMARK 2. With the notations as above, if $\varphi(X)=\sum_{i=1}^r V_iXW_i^*$ is another generalized Choi-Kraus representation of φ, then, by Theorem 2.6, $(\pi,V,W^*,\mathbb{C}^{nr})$ induced by $\varphi(X)=\sum_{i=1}^r V_iXW_i^*$ is equivalent to $(\pi,A,B^*,\mathbb{C}^{nr})$. Moreover, the reduced subspace associated with $(\pi,A,B^*,\mathbb{C}^{nr})$ is

$$K=\left\{\sum_k c_kX_k\otimes x_k\in W_u : \sum_k c_k\pi(X_k)B^*x_k=0\right\}.$$

By the above argument, it follows that $K=\ker(B_\varphi^*\otimes I_n)$. A direct application of the Sylvester Theorem in matrix theory [24], which states that for $P\in M_{m,n}, Q\in M_{n,l}$, then

$$\operatorname{Rank}(PQ)=\operatorname{Rank}(Q)-\dim(\operatorname{Im}Q\cap\ker P),$$

it gives that if $\operatorname{Rank}(PQ)=\operatorname{Rank}(Q)$, then $\ker PQ=\ker Q$. As $\operatorname{Rank}(B_\varphi^*)=r=\operatorname{Gr}(\varphi)=\operatorname{Rank}(G_\varphi)$, we have the reduced subspace

$$K=\ker(B_\varphi^*\otimes I_n)=\ker(A_\varphi\otimes I_n)(B_\varphi^*\otimes I_n)=\ker(G_\varphi\otimes I_n).$$

Meanwhile, by the fact that in the finite dimensional case, if a linearly minimal homomorphism dilation system is equivalent to a principle dilation, then it's also a principle dilation by Corollary 3.2 in [6]. Thus by Theorem 2.6 and 2.7, a generalized Choi-Kraus dilation of $\varphi \in L(M_n, M_d)$ is a principle dilation if and only if the reduced subspace is $\ker(G_\varphi \otimes I_n)$.

From Theorem 2.7, we know that any linearly minimal dilation is equivalent to a reduced dilation of the universal dilation. Meanwhile any linearly minimal dilation can be reduced to a principle dilation with respect to a maximal invariant subspace by [6]. Next, we quantify the relation between the generalized Choi-Kraus dilation with the universal dilation defined in Equation (5) and the principle dilation shown in Proposition 3.2.

THEOREM 3.3. *Let $\varphi \in L(M_n, M_d)$ with $Gr(\varphi) = r$. Let $(\pi_u, S_u, T_u, \mathbb{C}^{n^2 d})$ be the universal dilation, $(\pi_1, A, B^*, \mathbb{C}^{nr})$ be the principle dilation induced by $\varphi(X) = \sum_{i=1}^r A_i X B_i^*$ and $(\pi_2, E, F^*, \operatorname{Im}(F_\varphi^* \otimes I_n))$ be a linearly minimal generalized Choi-Kraus dilation induced by $\varphi(X) = \sum_{i=1}^m E_i X F_i^*$. Then*

(1) $(\pi_2, E, F^*, \operatorname{Im}(F_\varphi^* \otimes I_n))$ is equivalent to the reduced dilation of $(\pi_u, S_u, T_u, \mathbb{C}^{n^2 d})$ with respect to $K_1 = \ker(F_\varphi^* \otimes I_n)$.

(2) $(\pi_1, A, B^*, \mathbb{C}^{nr})$ is equivalent to the reduced dilation of $(\pi_2, E, F^*, \operatorname{Im}(F_\varphi^* \otimes I_n))$ with respect to $K_2 = \ker(E_\varphi \otimes I) \bigcap \operatorname{Im}(F_\varphi^* \otimes I_n)$.

Proof. Similar to the Equation (6), we have

$$\sum_k c_k \pi_2(X_k) F^* x_k = (F_\varphi^* \otimes I_n)\Big(\sum_k c_k \pi_u(X_k) T_u x_k\Big) = (F_\varphi^* \otimes I_n)\Big(\sum_k c_k X_k \otimes x_k\Big).$$

Then the reduced subspace $K_1 = \{\sum_k c_k X_k \otimes x_k \in W_u : \sum_k c_k \pi_2(X_k) F^* x_k = 0\}$ can be simplified as $\ker(F_\varphi^* \otimes I_n)$. Thus if we define the following induced map

$$\widetilde{F_\varphi^* \otimes I_n} : \mathbb{C}^{n^2 d}/K_1 \to \operatorname{Im}(F_\varphi^* \otimes I_n), \quad \Big[\sum_k c_k X_k \otimes x_k\Big] \mapsto \sum_k c_k \pi_2(X_k) F^* x_k,$$

then it is easy to verify that $\widetilde{F_\varphi^* \otimes I_n}$ is a well-defined bijection from the quotient space $\mathbb{C}^{n^2 d}/K_1$ into $\operatorname{Im}(F_\varphi^* \otimes I_n)$ and

$$\widetilde{F_\varphi^* \otimes I_n}\widetilde{T}_u = F, \quad \pi_2(X)\widetilde{F_\varphi^* \otimes I_n} = \widetilde{F_\varphi^* \otimes I_n}\widetilde{\pi_u(X)}, \forall X \in M_n,$$

where $\widetilde{S}_u, \widetilde{T}_u$ and $\widetilde{\pi_u(X)}$ are corresponding induced maps. Then it follows from $E\widetilde{F_\varphi^* \otimes I_n} = \widetilde{S}_u$ that $(\pi_2, E, F^*, \operatorname{Im}(F_\varphi^* \otimes I_n))$ is equivalent to the reduced dilation of $(\pi_u, S_u, T_u, \mathbb{C}^{n^2 d})$ with respect to $K_1 = \ker(F_\varphi^* \otimes I_n)$.

Next, if we take $w \in \ker(E_\varphi \otimes I) \bigcap \mathrm{Im}(F_\varphi^* \otimes I_n)$ represented by $w = \sum_k c_k \pi(X_k) F^* x_k$, then

$$Ew = E\Big(\sum_k c_k \pi(X_k) F^* x_k\Big) = \Big(E_1, \cdots, E_m\Big) \begin{pmatrix} \sum_k c_k X_k F_1^* x_k \\ \vdots \\ \vdots \\ \sum_k c_k X_k F_m^* x_k \end{pmatrix}$$

$$= \Big(E_{1,1}, \cdots, E_{d,1}, E_{2,1}, \cdots, \cdots E_{d,n}\Big) \begin{pmatrix} E_1(1,1)I_n & \cdots & E_m(1,1)I_n \\ \vdots & \vdots & \vdots \\ E_1(d,1)I_n & \cdots & E_m(d,1)I_n \\ E_1(1,2)I_n & \cdots & E_m(1,2)I_n \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ E_1(d,n)I_n & \cdots & E_m(d,n)I_n \end{pmatrix} \begin{pmatrix} \sum_k c_k X_k F_1^* x_k \\ \vdots \\ \vdots \\ \sum_k c_k X_k F_m^* x_k \end{pmatrix}$$

$$= \Big(E_{1,1}, \cdots, E_{d,1}, E_{1,2}, \cdots, \cdots, E_{d,n}\Big)(E_\varphi \otimes I_n) w = 0$$

and

$$(E_\varphi \otimes I_n)\pi_2(X) w = (E_\varphi \otimes I_n)(I_m \otimes X) w = (I_{nd} \otimes X)(E_\varphi \otimes I_n) w.$$

Thus $K_2 = \ker(E_\varphi \otimes I) \bigcap \mathrm{Im}(F_\varphi^* \otimes I_n)$ is a π-invariant subspace in $\ker E$. By Equation (2) and the minimality of number of the Kraus matrix pairs $\{A_i, B_i^*\}$, an easy computation shows that $\ker B_\varphi^* = \ker G_\varphi$. Meanwhile $G_\varphi = E_\varphi F_\varphi^*$. Thus $\ker F_\varphi^* \subset \ker G_\varphi = \ker B_\varphi^*$. Due to the fact that

$$(\ker T)^\perp = \mathrm{Im}(T^*), \ \forall\, T \in M_{i,j},$$

where $\perp$ denotes the orthogonal complement, we have $\mathrm{Im}(B_\varphi) \subset \mathrm{Im}(F_\varphi)$. Then by Douglas' Factorization Theorem [3], there exists $V \in M_{m,r}$ such that $B_\varphi = F_\varphi V$. Write V as $\big(v_{i,j}\big)$, by the correspondence between the Kraus matrix pairs and the decomposition of the generalized Choi matrix. We have

$$B_i^* = \sum_{j=1}^m \overline{v_{j,i}} F_j^*, \forall 1 \leqslant i \leqslant r.$$

Setting $R = V^* \otimes I_n : \mathrm{Im}(F_\varphi^* \otimes I_n) \to \mathbb{C}^{nr}$ we get

$$R\Big(\sum c_k \pi_2(X_k) F^* x_k\Big) = \Big(\sum c_k \pi_1(X_k) B^* x_k\Big).$$

It is easy to verify that $K_2 = \ker R$. Define the quotient map $\overline{R}$ by

$$\overline{R} : \mathrm{Im}(F_\varphi^* \otimes I_n)/K_2 \to \mathbb{C}^{nr}, \quad \Big[\sum c_k \pi_2(X_k) F^* x_k\Big] \mapsto \sum c_k \pi_1(X_k) B^* x_k.$$

Then we can check that $\overline{R}$ is a well-defined bijection, moreover

$$\overline{R}\, \overline{\pi_2(X)} = \pi_1(X)\, \overline{R}, \quad \overline{R}\, \overline{F^*} = B, \quad A\overline{R} = \overline{E}, \ \forall X \in M_n,$$

where $\overline{E},\overline{F^*},\overline{\pi_2(X)}$ are corresponding induced maps. Thus the principle dilation $(\pi_1, A, B^*, \mathbb{C}^{nr})$ is equivalent to the reduced dilation of $(\pi_2, E, F^*, \operatorname{Im}(F_\varphi^*\otimes I_n))$ with respect to $K_2=\ker(E_\varphi\otimes I)\bigcap \operatorname{Im}(F_\varphi^*\otimes I_n)$. $\square$

With the same notations as above, if $\operatorname{Rank}(F_\varphi^*)=nd$, then it follows that $K_1=\{0\}$. If $\operatorname{Rank}(F_\varphi^*)=r$, as $G_r(\varphi)=r=\operatorname{Rank}(G_\varphi)=\operatorname{Rank}(E_\varphi F_\varphi^*)$, then by the Sylvester Theorem [24] again we have $\ker E_\varphi\cap \operatorname{Im}F_\varphi^*=\{0\}$, and thus $K_2=\{0\}$. Consequently, we have the following:

COROLLARY 3.4. *Let $\varphi\in L(M_n,M_d)$ with $Gr(\varphi)=r$. If $(\pi,E,F^*,\operatorname{Im}(F_\varphi^*\otimes I_n))$ is the linearly minimal generalized Choi-Kraus dilation induced by $\varphi(X)=\sum_{i=1}^m E_iXF_i^*$. Then*

(1) If $\operatorname{Rank}(F_\varphi^*)=nd$, then $(\pi,E,F^*,\operatorname{Im}(F_\varphi^*\otimes I_n))$ is equivalent to the universal dilation.

(2) If $\operatorname{Rank}(F_\varphi^*)=r$, then $(\pi,E,F^*,\operatorname{Im}(F_\varphi^*\otimes I_n))$ is a principle dilation.

Suppose $(\pi,A,B,\mathbb{C}^{nm})$ is the generalized Choi-Kraus dilation induced by $\varphi(X)=\sum_{i=1}^m A_iXB_i^*$. As the linearly minimal dilation space is $\operatorname{Im}(B_\varphi^*\otimes I_n)$. We have the following corollary.

COROLLARY 3.5. *Let $\varphi\in L(M_n,M_d)$ with generalized Choi-Kraus representation $\varphi(X)=\sum_{i=1}^m A_iXB_i^*$. If $\operatorname{Rank}B_\varphi=m$, then $(\pi,A,B,\mathbb{C}^{mn})$ is a linearly minimal dilation.*

By Theorem 2.7, two linearly minimal dilations are equivalent if and only if the reduced spaces associated with them are same. Thus by Theorem 3.3 we get

COROLLARY 3.6. *For $\varphi\in L(M_n,M_d)$ with $\varphi(X)=\sum_{i=1}^m A_iXB_i^*=\sum_{j=1}^n E_jXF_j^*$. Then the linearly minimal generalized Choi-Kraus dilations $(\pi_1,A,B^*,\operatorname{Im}(B_\varphi^*\otimes I_n))$ and $(\pi_2,E,F^*,\operatorname{Im}(F_\varphi^*\otimes I_n))$ induced by its representations respectively are equivalent if and only if $\ker B_\varphi^*=\ker F_\varphi^*$.*

For a CP map, all the Choi-Kraus dilations are Stinespring dilations. By Theorem 3.3, the reduced subspace associated with a linearly minimal Choi-Kraus dilation $(\pi,A,A^*,\operatorname{Im}(A_\varphi^*\otimes I_n))$ induced by $\varphi(X)=\sum_{i=1}^m A_iXA_i^*$ is $\ker(A_\varphi^*\otimes I_n)$, as a notable fact about a linear equation system is that $\ker T=\ker T^*T$, it follows that the reduced space is $\ker(C_\varphi\otimes I_n)$ because of $A_\varphi A_\varphi^*=C_\varphi$, by the previous Remark 2, meaning all the linearly minimal Choi-Kraus dilations are principle dilations hence are equivalent. The following shows that they are actually also unitarily equivalent.

PROPOSITION 3.7. *For $\phi\in CP(M_n,M_d)$, all the linearly minimal Choi-Kraus dilations of ϕ are unitarily equivalent.*

Proof. Suppose $\operatorname{Cr}(\varphi)=r$ and $\varphi(X)=\sum_{i=1}^r A_iXA_i^*=\sum_{j=1}^m B_jXB_j^*$ are two Choi-Kraus representations of φ. $(\pi_1,A,A^*,\mathbb{C}^{nr})$ and $(\pi_2,B,B^*,\mathbb{C}^{nm})$ are Choi-Kraus dilations induced by $\varphi(X)=\sum_{i=1}^r A_iXA_i^*=\sum_{j=1}^m B_jXB_j^*$, respectively. Then by Proposition

3.2, $(\pi_1, A, A^*, \mathbb{C}^{nr})$ is a principle dilation. And $(\pi_2, B, B^*, \mathrm{Im}(B_\varphi^* \otimes I_n))$ is a linearly minimal Choi-Kraus dilation.

Due to Choi [2, 4], there exists a unique matrix $U = (u_{i,j}) \in M_{m,r}$ such that $U^*U = I_r, B_i = \sum_{j=1}^r u_{i,j} A_j$. Define $\mathscr{U} = (\overline{u_{i,j}}) \otimes I_n : \mathbb{C}^{nr} \to \mathbb{C}^{nm}$, then it can be checked that $B^* = \mathscr{U} A^*, B\mathscr{U} = A$ and $\pi_2(X)\mathscr{U} = \mathscr{U}\pi_1(X)$. Meanwhile, for any $\sum_k c_k \pi(X_k) A^* x_k \in \mathbb{C}^{nr}$ where $k \in \mathbb{N}, \{c_k\} \subset \mathbb{C}, \{X_k\} \subset M_n, \{x_k\} \subset \mathbb{C}^d$, we have

$$\mathscr{U}\left(\sum_k c_k \pi_1(X_k) A^* x_k\right) = \sum_k c_k \pi_2(X_k) B^* x_k.$$

Furthermore, as $\mathscr{U}^*\mathscr{U} = I_{nr}$ and $\mathscr{U} : \mathbb{C}^{nr} \to \mathrm{Im}(B_\varphi^* \otimes I_n)$ is a surjective isometry, we get that $\mathscr{U}$ is unitary. Hence $(\pi_1, A, A^*, \mathbb{C}^{nr})$ and $(\pi_2, B, B^*, \mathrm{Im}(B_\varphi^* \otimes I_n))$ are unitarily equivalent. By the transitivity of equivalence, we have all the linearly minimal Choi-Kraus dilations are unitarily equivalent. $\square$

For $\varphi \in L(M_n, M_d)$ with $\mathrm{Gr}(\varphi) = r$, a basis fact about the dimension k of the linearly minimal dilation space of a generalized Choi-Kraus dilation of φ is nm where $r \leqslant m \leqslant nd$. This leads to some natural questions. For example, (i) Under what condition is there only one (two, three, $\cdots$) equivalent class of linearly minimal generalized Choi-Kraus dilations? (ii) Clearly equivalent linearly minimal dilations have the same dilation dimension. Is the converse is also true? For question (i) we have

PROPOSITION 3.8. *Let $\varphi \in L(M_n, M_d)$ with $\mathrm{Gr}(\varphi) = r$.*

- *There is only one equivalent class of linearly minimal generalized Choi-Kraus dilations for φ if and only if $r = nd$.*
- *There are two equivalent classes of linearly minimal generalized Choi-Kraus dilations for φ if and only if $r = nd - 1$.*
- *There are infinite many equivalent classes of linearly minimal generalized Choi-Kraus dilations for φ if and only if $r \leqslant nd - 2$.*

Proof. For $\varphi \in L(M_n, M_d)$, by Theorem 3.3, the reduced subspace associated with the linearly minimal generalized Choi-Kraus dilation $(\pi, A, B^*, \mathrm{Im}(B_\varphi^* \otimes I_n))$ induced by $\varphi(X) = \sum_{i=1}^m A_i X B_i^*$ is $\ker(B_\varphi^* \otimes I_n)$, that is, $\ker B_\varphi^* \otimes \mathbb{C}^n$. While the reduced subspace associated with any principle generalized Choi-Kraus dilation is $\ker(G_\varphi \otimes I_n)$, which can be verified to be the largest reduced subspace, then

$$\ker(B_\varphi^* \otimes I_n) \subset \ker(G_\varphi \otimes I_n) = \ker G_\varphi \otimes \mathbb{C}^n.$$

Thus $\ker B_\varphi^* \subset \ker G_\varphi$. Moreover, by Theorem 2.7, each reduced subspace determines an equivalent class of linearly minimal dilations, then the classification of all linearly minimal generalized Choi-Kraus dilations of φ can be reduced to the structure of the linear subspaces of $\ker G_\varphi$. Thus we have

- There is only one equivalent class of linearly minimal generalized Choi-Kraus dilations if and only if $\ker G_\varphi$ has only one linear subspace if and only if $\ker G_\varphi = \{0\}$, that is, $\dim \ker G_\varphi = 0$.

- There are two equivalent classes of linearly minimal generalized Choi-Kraus dilations if and only if $\ker G_\varphi$ has only two different linear subspaces if and only if $\dim\ker G_\varphi = 1$.
- There are infinite equivalent classes of linearly minimal generalized Choi-Kraus dilations if and only if $\ker G_\varphi$ has infinite different linear subspaces if and only if $\dim\ker G_\varphi \geqslant 2$.

As seen in Equation (4), $\operatorname{Rank}(G_\varphi) = \operatorname{Gr}(\varphi)$ and $\dim\ker G_\varphi + \dim\operatorname{Im}(G_\varphi) = nd$, that is,

$$\dim\ker G_\varphi = nd - \operatorname{Gr}(\varphi).$$

Thus those statements are a direct consequence of the above characterizations. $\square$

We first give an example with full rank.

EXAMPLE 3.9. Let $\varphi : M_n \to M_n$ be the transpose map, i.e. $\varphi(X) = X^T, \forall X \in M_n$. One of the generalized Choi-Kraus representations is $\varphi(X) = \sum_{1\leqslant i,j\leqslant n} E_{i,j} X E_{i,j}$. Define

$$\pi(X) = \begin{pmatrix} X & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X \end{pmatrix} = I_{n^2} \otimes X,\ \forall\, X \in M_n$$

and

$$A = (E_{1,1}, \cdots, E_{1,n}, E_{2,1}, \cdots, \cdots, E_{n,n}), \quad B = (E_{1,1}, \cdots, E_{n,1}, E_{1,2}, \cdots, \cdots, E_{n,n}).$$

Then $(\pi, A, B^*, \mathbb{C}^{n^3})$ is a linearly minimal generalized Choi-Kraus dilation. It can be checked that $(\pi, A, B^*, \mathbb{C}^{n^3})$ is a universal dilation as well as a principle dilation, by Theorem 2.7, the largest reduced subspace is $\{0\}$. Thus any linearly minimal generalized Choi-Kraus dilation is equivalent to the universal dilation. This also follows from Proposition 3.8 since its generalized Choi matrix $G_\varphi = \big(\varphi(E_{i,j})\big) = \big(E_{j,i}\big)$ is a unitary and hence has full rank. Therefore all the linearly minimal dilations of φ are equivalent. $\square$

Here is an example with exactly two equivalent classes of linearly minimal generalized Choi-Kraus dilations.

EXAMPLE 3.10. Let $\varphi : M_2 \to M_2$ be defined by

$$\varphi\left(\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}\right) = \begin{pmatrix} a_{1,1} + a_{2,2} & 0 \\ 0 & -a_{1,1} \end{pmatrix}.$$

We can decompose the generalized Choi-matrix G_φ into the sum of rank one matrices and rearrange them into Kraus matrix pairs. Define

$$\pi_1(X) = \begin{pmatrix} X & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & X \end{pmatrix} = I_3 \otimes X,\ \forall\, X \in M_2$$

and

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then $(\pi_1, A, B^*, \mathbb{C}^6)$ is a principle generalized Choi-Kraus dilation. And the universal dilation $\pi_u : M_2 \to M_8$ is given by

$$\pi_u(X) = \begin{pmatrix} X & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X \end{pmatrix} = I_4 \otimes X, \ \forall\, X \in M_2$$

with

$$S_u = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad T_u = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T.$$

Clearly, $(\pi_1, A, B^*, \mathbb{C}^6)$ is not equivalent to $(\pi_u, S_u, T_u, \mathbb{C}^8)$. Moreover, it can be verified that $\{0\}$ and $\ker(G_\varphi \otimes I_2)$ are the only two reduced subspaces and hence there are only two equivalent classes of linearly minimal generalized Choi-Kraus dilations. $\square$

The next example shows that there are infinitely many inequivalent classes of linearly minimal dilations even with the same dilation dimension. This answers question (ii) negatively.

EXAMPLE 3.11. Let $A = \left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right)$. Define $\varphi \in L(M_2, M_2)$ by $\varphi(X) = A \circ X$, $\forall X \in M_2$, where $A \circ B$ denotes the Hadamard (entrywise) product of A and B (also called Schur product). Specifically,

$$\varphi\left(\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}\right) = \begin{pmatrix} 1 \cdot a_{1,1} & 2 \cdot a_{1,2} \\ 3 \cdot a_{2,1} & 4 \cdot a_{2,2} \end{pmatrix}.$$

Set

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \pi_1(X) = \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}, \ \forall X \in M_2.$$

Then $(\pi_1, A, B^*, \mathbb{C}^4)$ is a principle dilation.

Define

$$\pi_u(X) = \begin{pmatrix} X & 0 & 0 & 0 \\ 0 & X & 0 & 0 \\ 0 & 0 & X & 0 \\ 0 & 0 & 0 & X \end{pmatrix}, \ \forall X \in M_2$$

with

$$S_u = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}, \quad T_u = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T.$$

Then $(\pi_u, S_u, T_u, \mathbb{C}^8)$ is the universal dilation.

Besides, if we define

$$\pi_3(X) = \begin{pmatrix} X & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & X \end{pmatrix}, \ \forall X \in M_2$$

with

$$C = \begin{pmatrix} 1 & 0 & 1 & 0 & -2 & 0 \\ 0 & 3 & 0 & 1 & 0 & -4 \end{pmatrix}, \quad D_\lambda = \begin{pmatrix} 1 & \lambda & 0 & \lambda & 0 & \lambda \\ 1-\lambda & 0 & 1-\lambda & 0 & 1-\lambda & -1 \end{pmatrix},$$

then $(\pi_3, C, D_\lambda^*, \mathbb{C}^6)$ is a linearly minimal generalized Choi-Kraus dilation.

Let

$$R_\lambda = \begin{pmatrix} I_2 & (1-\lambda)I_2 & \lambda I_2 & 0 \\ 0 & (1-\lambda)I_2 & \lambda I_2 & 0 \\ 0 & (1-\lambda)I_2 & \lambda I_2 & -I_2 \end{pmatrix}.$$

The reduced subspace associated with $(\pi_3, C, D_\lambda^*, \mathbb{C}^6)$ is the $\ker R_\lambda$. Taking $\lambda_1 \neq \lambda_2$, clearly $\ker R_{\lambda_1} \neq \ker R_{\lambda_2}$. Then $(\pi_3, C, D_{\lambda_1}^*, \mathbb{C}^6)$ and $(\pi_3, C, D_{\lambda_2}^*, \mathbb{C}^6)$ are not equivalent. In summary, there are infinitely many inequivalent classes of 6-dimensional dilations. $\square$

Although none of the above three examples is completely positive. It is also a trivial exercise to construct similar examples for quantum channels (i.e., trace-preserving CP maps). Here is a one that admits infinitely many equivalent classes of linearly minimal generalized Choi-Kraus dilations.

EXAMPLE 3.12. Let $\phi \in CP(M_2, M_2)$ be

$$\phi\left(\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}\right) = \begin{pmatrix} a_{1,1} & 0 \\ 0 & a_{2,2} \end{pmatrix}.$$

Define

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \pi_1(X) = \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}, \ \forall X \in M_2$$

and

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \quad \pi_2(X) = \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}, \ \forall X \in M_2.$$

Then $(\pi_1, A, A^*, \mathbb{C}^4)$ and $(\pi_2, B, B^*, \mathbb{C}^4)$ are linearly minimal Choi-Kraus dilations hence unitarily equivalent.

The universal dilation $(\pi_u, S_u, T_u, \mathbb{C}^8)$ is given by

$$\pi_u(X) = \begin{pmatrix} X & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X \end{pmatrix} = I_4 \otimes X, \ \forall\, X \in M_2$$

with

$$S_u = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad T_u = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T.$$

Besides, if we set

$$\pi(X) = \begin{pmatrix} X & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & X \end{pmatrix} = I_3 \otimes X, \ \forall\, X \in M_2.$$

As the generalized Choi matrix $(\varphi(E_{i,j}))$ has the following decomposition.

$$\begin{pmatrix} 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ 0&0&0 \\ 0&0&0 \\ 0&0&1 \end{pmatrix} \begin{pmatrix} \frac{3}{5} & \frac{4(1-\alpha)}{5} & \frac{4\alpha}{5} & 0 \\ \frac{4}{5} & \frac{-3(1-\alpha)}{5} & \frac{-3\alpha}{5} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

thus we can arrange those matrices into the following maps

$$E = \begin{pmatrix} \frac{3}{5} & 0 & \frac{4}{5} & 0 & 0 & 0 \\ 0&0&0&0&0&1 \end{pmatrix}, \quad F_\alpha = \begin{pmatrix} \frac{3}{5} & \frac{4\alpha}{5} & \frac{4}{5} & -\frac{3\alpha}{5} & 0 & 0 \\ \frac{4(1-\alpha)}{5} & 0 & -\frac{3(1-\alpha)}{5} & 0 & 0 & 1 \end{pmatrix},$$

then it follows that $(\pi, E, F_\alpha^*, \mathbb{C}^6)$ is a linearly minimal generalized Choi-Kraus dilation.

By Corollary 3.6, if $\alpha_1 \neq \alpha_2$, then $(\pi, E, F_{\alpha_1}^*, \mathbb{C}^6)$ and $(\pi, E, F_{\alpha_2}^*, \mathbb{C}^6)$ are not equivalent. Thus there are infinitely many equivalent classes of linearly minimal generalized Choi-Kraus dilations. $\square$

Acknowledgement. The authors would like to thank the referee for carefully reading the manuscript and providing many helpful comments and suggestions that help improve the quality of its presentation.

REFERENCES

[1] W. ARVESON, *Dilation theory yesterday and today*, In: A glimpse at Hilbert space operators, vol. 207 of Oper. Theory Adv. Appl., pages 99–123, Birkhäuser Verlag, Basel, 2010.

[2] M. D. CHOI, *Completely positive linear maps on complex matrices*, Linear Algebra Appl. **10**, 3 (1975), 285–290.

[3] R. G. DOUGLAS, *On majorization, factorization, and range inclusion of operators on Hilbert space*, Proc. Am. Math. Soc. **17**, 2 (1966), 413–415.

[4] V. P. GUPTA, P. V. MANDAYAM AND S. SUNDER, *The functional analysis of quantum information theory*, Lecture Notes in Physics., Cham, Switzerland, Springer, 2015.

[5] D. HAN, D. R. LARSON, B. LIU AND R. LIU, *Operator-valued measures, dilations, and the theory of frames*, Mem. Amer. Math. Soc. Vol. 229, No. 1075, 2014.

[6] D. HAN, D. R. LARSON, B. LIU AND R. LIU, *Structural Properties of Homomorphism Dilation Systems*, Chin. Ann. Math. Ser. B. **41**, 4 (2020), 585–600.

[7] J. HOU, C. K. LI, Y. T. POON, X. QI AND N. S. SZE, *A new criterion and a special class of k-positive maps*, Linear Algebra Appl. **470**, 2015, 51–69.

[8] J. HOU AND X. QI, *Linear maps preserving separability of pure states*, Linear Algebra Appl. **439**, 5 (2013), 1245–1257.

[9] K. M. KRISHNA AND P. S. JOHNSON, *Dilations of linear maps on vector spaces*, Oper. Matrices. **16**, 2 (2022), 465–477.

[10] Y. LI AND H. DU, *Interpolations of entanglement breaking channels and equivalent conditions for completely positive maps*, J. Funct. Anal. **268**, 11 (2015), 3566–3599.

[11] D. MCLAREN, S. PLOSKER AND C. RAMSEY, *On Operator Valued Measures*, Houst. J. Math., **46**, 1 (2020), 201–226.

[12] M. A. NIELSEN AND I, NIELSEN, *Quantum computation and quantum information*, Cambridge, UK, Cambridge University Press, 2002.
[13] B. S. NAGY, C. FOIAS, H. BERCOVICI AND L. KÉRCHY, *Harmonic analysis of operators on Hilbert space*, Universitext, Springer Science and Business Media, New York, second edition, 2010.
[14] V. PAULSEN, *Completely bounded maps and operator algebras*, Cambridge, UK, Cambridge University Press, 2002.
[15] V. PAULSEN, *A survey of completely bounded maps*, Notes at the Banff International Research Station, 2007.
[16] X. QI AND J. HOU, *Positive finite rank elementary operators and characterizing entanglement of states*, J. Phys. A: Math. Theor. **44**, 21 (2011), 215305.
[17] X. QI AND J. HOU, *Detecting entanglement of states by entries of their density matrices*, Int. J. Theor. Phys. **51**, 7 (2012), 2003–2014.
[18] C. RAMSEY AND A. REEVES, *L^p-spaces of operator-valued functions*, Integr. Equ. Oper. Theory. **93,** (2021), 54.
[19] X. SUN AND Y. LI, *Some characterizations of quantum channel in infinite Hilbert spaces*, J. Math. Phys. **55**, 5 (2014), 053511.
[20] R. R. SMITH, *Completely bounded maps between C^*-algebras*, J. Lond. Math. Soc. **2**, 1 (1983), 157–166.
[21] W. STINESPRING, *Positive functions on C^*-algebras*, Proc. Amer. Math. Soc. **6**, 2 (1955), 211–216.
[22] G. WITTSTOCK, *Ein operatorwertiger Hahn-Banach Satz*, J. Funct. Anal. **40**, (1981), 127–150.
[23] G. WITTSTOCK, *On matrix order and convexity*, in Functional Analysis: Surveys and Recent Results, Math. Studies, Vol. 90, North-Holland, Amsterdam, 1984, 175–188.
[24] F. ZHANG, *Matrix theory: basic results and techniques*, New York (NY), Springer, 2011.

(Received September 23, 2022)

Deguang Han
Department of Mathematics
University of Central Florida
Orlando, USA
e-mail: deguang.han@ucf.edu

Qianfeng Hu
School of Science
Hebei University of Technology
Tianjin 300400, China
and
School of Mathematical Sciences and LPMC
Nankai University
Tianjin, China
e-mail: qianfenghu@mail.nankai.edu.cn

Rui Liu
School of Mathematical Sciences and LPMC
Nankai University
Tianjin, China
e-mail: ruiliu@nankai.edu.cn