Operators and Matrices
Volume 16, Number 4 (2022), 1239–1242 doi:10.7153/oam-2022-16-80

ERRATUM/ADDENDUM TO "*POWERS OF POSINORMAL OPERATORS*", OPERATORS AND MATRICES 10 (2016), 15—27

C. S. KUBRUSLY, P. C. M. VIEIRA AND J. ZANNI

(*Communicated by M. Omladič*)

Abstract. Erratum/Addendum to the paper *Powers of posinormal operators*, Operators and Matrices **10** (2016), 15–27.

The statement of Lemma 2 in the above paper is incomplete (in that we overlooked the necessary assumption of closed range to prove the second part of it – we referred to [2, Proof of Lemma 5.31] overlooking that that result is stated for Fredholm operators where ranges are closed). The corrected statement and proof go as follows (notation and terminology as in [4]).

LEMMA 2. *Take any operator $A \in \mathscr{B}[\mathscr{H}]$ and an arbitrary integer $k \geqslant 1$. If*

$$asc(A) \leqslant k \ \text{ and } \ dsc(A) < \infty \quad \text{ or } \quad asc(A) < \infty \ \text{ and } \ dsc(A) \leqslant k,$$

then

$$dsc(A) = asc(A) \leqslant k,$$

and so

$$\mathscr{R}(A^n) = \mathscr{R}(A^k) \quad \text{ and } \quad \mathscr{N}(A^n) = \mathscr{N}(A^k) \quad \text{ for each integer } n \geqslant k.$$

If, in addition, $\mathscr{R}(A^n)$ is closed for every n, then

$$dsc(A^*) = asc(A^*) \leqslant k,$$

and so

$$\mathscr{R}(A^{*n}) = \mathscr{R}(A^{*k}) \quad \text{ and } \quad \mathscr{N}(A^{*n}) = \mathscr{N}(A^{*k}) \quad \text{ for each integer } n \geqslant k.$$

Proof. Take an arbitrary $A \in \mathscr{B}[\mathscr{H}]$. Consider the following auxiliary results.

Claim (i). $\quad \operatorname{asc}(A) < \infty$ and $\operatorname{dsc}(A) < \infty \implies \operatorname{asc}(A) = \operatorname{dsc}(A)$.

Mathematics subject classification (2020): Primary 47B20; Secondary 47A53.
Keywords and phrases: Hyponormal operators, posinormal operators, quasiposinormal operators.

Proof of Claim (i). This is a well-known result, see e.g., [6, Theorem 6.2]. $\square$

Claim (ii).

(a) $\operatorname{dsc}(A^*) < \infty \implies \operatorname{asc}(A) < \infty$,

(b) $\operatorname{asc}(A) < \infty \implies \operatorname{dsc}(A^*) < \infty$ if $\mathscr{R}(A^n)$ is closed for every integer $n \geqslant 1$,

(c) $\operatorname{asc}(A) < \infty \not\Longrightarrow \operatorname{dsc}(A^*) < \infty$ if $\mathscr{R}(A^n)$ is not closed for some integer $n \geqslant 1$.

Proof of Claim (ii). Take an arbitrary positive integer n.

(a) If $\operatorname{asc}(A) = \infty$, then $\mathscr{N}(A^n) \subset \mathscr{N}(A^{n+1})$ so that $\mathscr{N}(A^{n+1})^\perp \subset \mathscr{N}(A^n)^\perp$ (since $\mathscr{N}(\cdot)$ is closed – indeed, $\mathscr{M} \subset \mathscr{N} \implies \mathscr{N}^\perp \subseteq \mathscr{M}^\perp$ and $\mathscr{N}^\perp = \mathscr{M}^\perp \implies \mathscr{M}^- = \mathscr{N}^-$). Equivalently, $\mathscr{R}(A^{*(n+1)})^- \subset \mathscr{R}(A^{*n})^-$. As $\mathscr{R}(A^{*n+1}) \subseteq \mathscr{R}(A^{*n})$, the above proper inclusion ensures the proper inclusion $\mathscr{R}(A^{*(n+1)}) \subset \mathscr{R}(A^{*n})$. So $\operatorname{dsc}(A^*) = \infty$, and

$$\operatorname{asc}(A) = \infty \implies \operatorname{dsc}(A^*) = \infty.$$

(b) If $\operatorname{dsc}(A) = \infty$, then $\mathscr{R}(A^{n+1}) \subset \mathscr{R}(A^n)$. Suppose $\mathscr{R}(A^n)$ is closed so that $\mathscr{R}(A^{n+1}) \subset \mathscr{R}(A^n)$ implies $\mathscr{R}(A^n)^\perp \subset \mathscr{R}(A^{n+1})^\perp$. That is, $\mathscr{N}(A^{*n}) \subset \mathscr{N}(A^{*(n+1)})$. Hence $\operatorname{asc}(A^*) = \infty$. Therefore

$$\operatorname{dsc}(A) = \infty \implies \operatorname{asc}(A^*) = \infty \quad \text{if } \mathscr{R}(A^n) \text{ is closed for every integer } n \geqslant 1.$$

Dually (as $A^{**} = A$ and $\mathscr{R}(A^n)$ closed $\iff$ $\mathscr{R}(A^{*n})$ closed),

$$\operatorname{dsc}(A^*) = \infty \implies \operatorname{asc}(A) = \infty \quad \text{if } \mathscr{R}(A^n) \text{ is closed for every integer } n \geqslant 1,$$

(c) To verify (c) consider the following example. Take A such that $\mathscr{N}(A^*) = \{0\}$ and $\mathscr{R}(A^*) \neq \mathscr{R}(A^*)^- = \mathscr{H}$. Then $\mathscr{N}(A) = \mathscr{R}(A^*)^\perp = \{0\}$, and hence $\operatorname{asc}(A) = 0$. We show that $\operatorname{dsc}(A^*) = \infty$.

Since $\mathscr{R}(A^*) \neq \mathscr{R}(A^*)^- = \mathscr{H}$, take $v \in \mathscr{H} \backslash \mathscr{R}(A^*)$. Suppose $\operatorname{dsc}(A^*) < \infty$, say, suppose $\operatorname{dsc}(A^*) = n$. Then $\mathscr{R}(A^{*n}) = \mathscr{R}(A^{*n+1})$, and so there exists $w \in \mathscr{H}$ such that $A^{*n+1}w = A^{*n}v$. Thus $A^{*n}(A^{*n}w - v) = 0$ so that $A^*w = v$ (since $\operatorname{asc}(A^*) = 0 \implies \mathscr{N}(A^{*n}) = \{0\}$). Hence $v \in \mathscr{R}(A^*)$, which is a contradiction. Thus $\operatorname{dsc}(A^*) = \infty$. $\square$

Claim (iii). $\operatorname{dsc}(A) < \infty \implies \operatorname{asc}(A^*) \leqslant \operatorname{dsc}(A)$.

Proof of Claim (iii). Consider the argument in the proof of Claim (ii-a). So $\operatorname{dsc}(A) = n_0$ implies $\mathscr{R}(A^n) = \mathscr{R}(A^{n_0})$ for every $n \geqslant n_0$. Thus $\mathscr{R}(A^n)^- = \mathscr{R}(A^{n_0})^-$. Equivalently, $\mathscr{N}(A^{*n}) = \mathscr{N}(A^{*n_0})$ (as $\mathscr{R}(\cdot)^\perp = \mathscr{N}(\cdot\,^*)$), which implies $\operatorname{asc}(A^n) \leqslant n_0$. $\square$

If $\operatorname{asc}(A) \leqslant k$ and $\operatorname{dsc}(A) < \infty$ (or if $\operatorname{asc}(A) < \infty$ and $\operatorname{dsc}(A) \leqslant k$), then

$$\operatorname{dsc}(A) = \operatorname{asc}(A) \leqslant k$$

by Claim (i). Moreover, this implies that $\operatorname{asc}(A^*) \leqslant \operatorname{dsc}(A) \leqslant k$ by Claim (iii). Now suppose $\mathscr{R}(A^n)$ is closed for every n. Since $\operatorname{asc}(A) \leqslant k$, we get $\operatorname{dsc}(A^*) < \infty$ by Claim (ii-b). Then, since $\operatorname{asc}(A^*) \leqslant k$, Claim (i) ensures that

$$\operatorname{dsc}(A^*) = \operatorname{asc}(A^*) \leqslant k.$$

The range and kernel identities follow from the definition of ascent and descent. □

Consequently, Theorem 1 and Corollary 1 are to be modified, whose proofs follow the same argument as before, now applying the correct version of Lemma 2.

NOTE. Posinormal operators were introduced in [5] (see also [3]) – an operator is posinormal if its range is included in the range of its adjoint.

THEOREM 1. *Take $T \in \mathscr{B}[\mathscr{H}]$. Suppose $\mathscr{R}(T^n)$ is closed for every $n \geqslant 1$.*

(a) *If T^k is posinormal for some $k \geqslant 1$ and $\operatorname{dsc}(T^m) < \infty$ for some $m \geqslant 1$, then T^n is posinormal for every $n \geqslant k$.*

(b) *If T^k is posinormal for some $k \geqslant 1$ and T^{*m} is posinormal for some $m \geqslant k$, then T^n is posinormal for every $n \geqslant k$ and coposinormal for every $n \geqslant m$.*

COROLLARY 1. *Take $T \in \mathscr{B}[\mathscr{H}]$. Suppose $\mathscr{R}(T^n)$ is closed for every $n \geqslant 1$.*

(a) *If T is posinormal and $\operatorname{dsc}(T) < \infty$, then T^n is posinormal for every $n \geqslant 1$.*

(b) *If T is posinormal and coposinormal, then T^n is posinormal and coposinormal for every $n \geqslant 1$.*

In fact, the assumption "$\operatorname{dsc}(T) < \infty$" in Corollary 1(a) above can be dismissed, yielding a corrected version of Corollary 3:

COROLLARY 3. *If T is posinormal and $\mathscr{R}(T^n)$ is closed for every $n \geqslant 1$, then T^n is posinormal.*

In a subsequent paper [1], the above assumption "$\mathscr{R}(T^n)$ is closed for every $n \geqslant 1$" has been weakened, yielding a sharper result as follows.

THEOREM [1]. *If T is posinormal and has closed range, then T^n is posinormal and has closed range for every $n \geqslant 1$.*

Acknowledgement. We thank Paul S. Bourdon and Derek Thompson who pointed out an error in the previous version of Lemma 2 (at the previous version of Claim (ii-b)). The counterexample in Claim (ii-c) was communicated to us by Paul S. Bourdon.

REFERENCES

[1] P. S. BOURBON, C. S. KUBRUSLY, AND D. THOMPSON, *Powers of posinormal Hilbert-space operators*, available at https://arxiv.org/abs/2203.01473.
[2] C. S. KUBRUSLY, *Spectral Theory of Operators on Hilbert Spaces*, Birkhäuser-Springer, New York, 2012.
[3] C. S. KUBRUSLY AND B. P. DUGGAL, *On posinormal operators*, Adv. Math. Sci. Appl. **17** (2007), 131–148.
[4] C. S KUBRUSLY, P. C. M. VIEIRA, AND J. ZANNI, *Powers of posinormal operators*, Operators and Matrices **10** (2016), 15–27.
[5] H. C. RHALY, JR., *Posinormal operators*, J. Math. Soc. Japan **46** (1994), 587–605.
[6] A. E. TAYLOR AND D. C. LAY, *Introduction to Functional Analysis*, Wiley, New York, 1980.

(Received February 19, 2022)

C. S. Kubrusly
Catholic University of Rio de Janeiro
22453-900, Rio de Janeiro, RJ, Brazil
e-mail: carlos@ele.puc-rio.br

P. C. M. Vieira
National Laboratory for Scientific Computation
25651-070, Petrópolis, RJ, Brazil
e-mail: paulocm@lncc.br

J. Zanni
Catholic University of Rio de Janeiro
22453-900, Rio de Janeiro, RJ, Brazil
e-mail: jzanni@gmail.com