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APPROXIMATE EQUIVALENCE IN VON NEUMANN ALGEBRAS

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Abstract. Suppose $\mathcal{A}$ is a separable unital ASH C*-algebra, $\mathcal{M}$ is a sigma-finite II_∞ factor von Neumann algebra, and $\pi,\rho:\mathcal{A}\to\mathcal{M}$ are unital $*$-homomorphisms such that, for every $a\in\mathcal{A}$, the range projections of $\pi(a)$ and $\rho(a)$ are Murray von Neuman equivalent in $\mathcal{M}$. We prove that π and ρ are approximately unitarily equivalent modulo $\mathcal{K}_{\mathcal{M}}$, where $\mathcal{K}_{\mathcal{M}}$ is the norm closed ideal generated by the finite projections in $\mathcal{M}$. We also prove a very general result concerning approximate equivalence in arbitrary finite von Neumann algebras.

1. Introduction

In 1977 D. Voiculescu [15] proved a remarkable theorem concerning approximate (unitary) equivalence for representations of a separable unital C*-algebra on a separable Hilbert space. The beauty of the theorem is that the characterization was in purely algebraic terms. This was made explicit in the reformulation of Voiculescu's theorem [7] in terms of rank.

THEOREM 1. [15] *Suppose $B(H)$ is the set of operators on a separable Hilbert space H and $\mathcal{K}(H)$ is the ideal of compact operators. Suppose $\mathcal{A}$ is a separable unital C*-algebra, and $\pi,\rho:\mathcal{A}\to B(H)$ are unital $*$-homomorphisms. The following are equivalent:*

1. *There is a sequence $\{U_n\}$ of unitary operators in $B(H)$ such that*
   (a) $U_n\pi(a)U_n^*-\rho(a)\in\mathcal{K}(H)$ *for every $n\in\mathbb{N}$ and every $a\in\mathcal{A}$.*
   (b) $\|U_n\pi(a)U_n^*-\rho(a)\|\to 0$ *for every $a\in\mathcal{A}$.*
2. *There is a sequence $\{U_n\}$ of unitary operators in $B(H)$ such that, for every $a\in\mathcal{A}$,*
$$\|U_n\pi(a)U_n^*-\rho(a)\|\to 0.$$

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3. *For every* $a \in \mathcal{A}$,
$$rank(\pi(a)) = rank(\rho(a)).$$
4. $\ker\pi = \ker\rho$, *and* $\pi|_{span^{-\|\|}(\cup\{ran\pi(a):\pi(a)\in\mathcal{K}(H)\})}$ *is unitarily equivalent to* $\rho|_{span^{-\|\|}(\cup\{ran\rho(a):\rho(a)\in\mathcal{K}(H)\})}$.

If $\pi : \mathcal{A} \to B(H)$ is a unital $*$-homomorphism, we will write $\pi \sim_a \rho$ in $B(H)$ to mean that statement (2) in the preceding theorem holds and we will write $\pi \sim_a \rho$ $(\mathcal{K}(H))$ in $B(H)$ to indicate statements (1) and (2) hold. When the C*-algebra $\mathcal{A}$ is not separable, $\pi \sim_a \rho$ means that there is a *net* of unitaries $\{U_\lambda\}$ such that, for every $a \in \mathcal{A}$, $\|U_\lambda \pi(a) U_\lambda^* - \rho(a)\| \to 0$. It was shown in [7] that $\pi \sim_a \rho$ if and only if $\operatorname{rank}(\pi(a)) = \operatorname{rank}(\rho(a))$ always holds even when $\mathcal{A}$ or H is not separable, where, for $T \in B(H)$, $\operatorname{rank}(T)$ is the Hilbert-space dimension of the projection $\mathfrak{R}(T)$ onto the closure of the range of T.

Later Huiru Ding and the second author [4] extended the notion of rank to operators in a von Neumann algebra $\mathcal{M}$, i.e., if $T \in \mathcal{M}$, then $\mathcal{M}$-rank(T) is the Murray von Neumann equivalence class of the projection $\mathfrak{R}(T)$ onto the closure of the range of T. If p and q are projections in a C*-algebra $\mathcal{W}$, we say that p and q are Murray-von Neumann equivalent in $\mathcal{W}$, written $p \sim q$, if there is a partial isometry $v \in \mathcal{W}$ such that $v^*v = p$ and $vv^* = q$. Thus $\mathcal{M}$-rank$(T) = \mathcal{M}$-rank(S) if and only if $\mathfrak{R}(S) \sim \mathfrak{R}(T)$. In [4] they extended Voiculescu's theorem for representations of a separable AH C*-algebra into a von Neumann algebra on a separable Hilbert space, i.e., $\pi \sim_a \rho$ in $\mathcal{M}$ if and only if, for every $a \in \mathcal{A}$,
$$\mathcal{M}\text{-rank}(\pi(a)) = \mathcal{M}\text{-rank}(\rho(a)).$$
When the algebra $\mathcal{A}$ is ASH, their characterization works when the von Neumann algebra is a II_1 factor [4]. (See Theorem 4.) In [2] A. Ciuperca, T. Giordano, P. W. Ng, and Z. Niu found a limit for the results in [4]. We say that two representations $\pi, \rho : \mathcal{A} \to \mathcal{M}$ are *weak*-approximately equivalent* if and only if, there are nets $\{U_\lambda\}$ and $\{V_\lambda\}$ of unitary operators in $\mathcal{M}$ such that, for every $a \in \mathcal{A}$,
$$\text{weak*-}\lim U_\lambda^* \pi(a) U_\lambda = \rho(a) \text{ and weak*-}\lim V_\lambda^* \rho(a) V_\lambda = \pi(a).$$
They proved that a separable unital C*-algebra $\mathcal{A}$ is nuclear if and only if, for every von Neumann algebra $\mathcal{M}$, and all representations $\pi, \rho : \mathcal{A} \to \mathcal{M}$, we have that for all $a \in \mathcal{A}$, $\mathcal{M}$-rank$(\pi(a)) = \mathcal{M}$-rank$(\rho(a))$, implies that π and ρ are weak*-approximately equivalent.

Therefore the central questions in this subject are:

QUESTION 1. Are the results in [4] true whenever $\mathcal{A}$ is nuclear?

Another important question involves the analogue of part $1(a)$ of Theorem 1 holds when $\mathcal{M}$ is a semifinite and $\mathcal{K}(H)$ is replaced with the norm closed ideal $\mathcal{K}_{\mathcal{M}}$ generated by the finite projections in $\mathcal{M}$.

QUESTION 2. If $\pi, \rho : \mathcal{A} \to \mathcal{M}$ are approximately equivalent representations from a separable unital C*-algebra $\mathcal{A}$ into a semifinite von Neumann algebra $\mathcal{M}$ acting on a separable Hilbert space, does there exist a sequence $\{U_n\}$ of unitary operators in $\mathcal{M}$ such that

1. $\lim_{n\to\infty} \|U_n^* \pi(a) U_n - \rho(a)\| = 0$ for every $a \in \mathcal{A}$, and
2. $U_n \pi(a) U_n^* - \rho(a) \in \mathcal{K}_{\mathcal{M}}$ for every $n \in \mathbb{N}$ and every $a \in \mathcal{A}$?

If these two conditions hold, we write $\pi \sim_a \rho \ (\mathcal{K}_{\mathcal{M}})$.

When $\mathcal{A}$ is abelian the second author and Rui Shi [9] proved that Question 2 has an affirmative answer when $\mathcal{M}$ is a sigma-finite II_∞ factor. This was extended to the case of AF C*-algebras by Shilin Wen, Junsheng Fang and Rui Shi [5], and to the case when $\mathcal{A}$ is an AH C*-algebra, and by Junhao Shen and Rui Shi [14].

In this paper we show (Theorem 5) that Question 1 has an affirmative answer when $\mathcal{M}$ is a finite von Neumann algebra and $\mathcal{A}$ is satisfies the property that, for every finite subset F of $\mathcal{A}$ and every $\varepsilon > 0$, there is a type I von Neumann algebra $\mathcal{B}$ contained in the second dual $\mathcal{A}^{\#\#}$ such that, for every $x \in F$,

$$\operatorname{dist}(x, \mathcal{B}) < \varepsilon.$$

If this happens we say that $\mathcal{A}$ is *approximately type I* in $\mathcal{A}^{\#\#}$. This class of C*-algebras contains the ASH algebras and algebras that are direct limits of GCR C*-algebras. For these theorems there are no assumptions on $\mathcal{A}$ being separable or $\mathcal{M}$ acting on a separable Hilbert space. We say that $\mathcal{A}$ is *approximately finite type I* in $\mathcal{A}^{\#\#}$ if the type I algebra $\mathcal{B}$ can always be chosen to be a finite type I von Neumann algebra. It is clear that this latter property implies that $\mathcal{A}$ is strongly quasidiagonal. We do not know if this property is equivalent to strong quasidiagonality.

In [7] the second author extended Voiculescu's theorem (Theorem 1) in another way:

THEOREM 2. [7] *Suppose $\mathcal{A}$ is a separable unital C*-algebra, H is a separable Hilbert space, and $\pi, \rho : \mathcal{A} \to B(H)$ are unital representations. The following are equivalent:*

1. *For every $a \in \mathcal{A}$,*
$$\mathit{rank}\,\pi(a) \leqslant \mathit{rank}(\rho(a))$$
2. *There is a representation σ such that*
$$\rho \sim_a \pi \oplus \sigma.$$

An analogue of this result was proved in [9] when $\mathcal{M}$ is a II_1 factor and $\mathcal{A}$ is abelian. This result was further extended to the case when $\mathcal{A}$ is AF by Shilin Wen, Junsheng Fang and Rui Shi [5]. We extend this result to the case when there is an LF C*-algebra $\mathcal{D}$ such that $\mathcal{A} \subset \mathcal{D} \subset \mathcal{A}^{\#\#}$. This class of algebras includes the ASH C*-algebras.

The proof of Voiculescu's theorem (Theorem 1) have two parts.

The "easy part" involves the compact operators. Suppose $\mathcal{A}$ is a separable unital C*-algebra and $\pi : \mathcal{A} \to \mathcal{B}(\ell^2)$ is a unital $*$-homomorphism. Then $\sup\{\mathfrak{R}(\pi(a)) : \pi(a) \in \mathcal{K}(\ell^2)\}$ reduces π and leads to a decomposition

$$\pi = \pi_{\mathcal{K}(H)} \oplus \pi_1.$$

The "easy part" says that if $\pi \sim_a \rho$, then $\pi_{\mathcal{K}(H)}$ and $\rho_{\mathcal{K}(H)}$ must be unitarily equivalent. Using descriptions of C*-algebras of compact operators and their representations (see [1]), and it is not too hard to show that the equality of rank conditions imply that $\pi_{\mathcal{K}(H)}$ and $\rho_{\mathcal{K}(H)}$ are unitarily equivalent. When $B(H)$ is replaced with a sigma-finite type II_∞ factor von Neumann algebra $\mathcal{M}$ and $\mathcal{K}(H)$ is replaced with the closed ideal $\mathcal{K}_\mathcal{M}$ generated by the finite projections, the hard part is harder (and unsolved) and the easy part is not true. For example, if $\mathcal{M}$ is the set of all bounded operator matrices (A_{ij}) with each A_{ij} in the free group factor $\mathcal{L}_{\mathbb{F}_2} \subset B\left(\ell^2(\mathbb{F}_2)\right)$, and U, V are the unitary generators of $\mathcal{L}_{\mathbb{F}_2}$, then $A = \operatorname{diag}(U, 0, 0, \ldots)$ and $B = \operatorname{diag}(V, 0, 0, \ldots)$ are in $\mathcal{K}_\mathcal{M}$ and are approximately equivalent, but not unitarily equivalent. If $\mathcal{A} = C^*(A)$, $\pi(A) = A$ and $\rho(A) = B$, then $\pi \sim_a \rho$ in $\mathcal{M}$, but $\pi_{\mathcal{K}_\mathcal{M}}$ and $\rho_{\mathcal{K}_\mathcal{M}}$ are not unitarily equivalent in $\mathcal{M}$. However, $\pi_{\mathcal{K}_\mathcal{M}}$ and $\rho_{\mathcal{K}_\mathcal{M}}$ are approximately equivalent. So the analogue of the "easy" part must look something like

$$\pi_{\mathcal{K}_\mathcal{M}} \sim_a \rho_{\mathcal{K}_\mathcal{M}} \ (\mathcal{K}_\mathcal{M}).$$

In Theorem 7 we prove that this holds in a very general setting when $\mathcal{A}$ is a separable unital ASH algebra. One of our main results (Theorem 8) gives an affirmative answer to both Questions 1 and 2 when $\mathcal{A}$ is a separable ASH C*-algebra and $\mathcal{M}$ is a semifinite von Neumann algebra acting on a separable Hilbert space.

The "hard" part of the proof of Voiculescu's theorem is showing that if $\mathcal{A} \subset B\left(\ell^2\right)$ is a separable unital C*-algebra, $\pi : \mathcal{A} \to B\left(\ell^2\right)$ is a unital $*$-homomorphism such that $\mathcal{K}\left(\ell^2\right) \cap \mathcal{A} \subset \ker \pi$, then

$$id_\mathcal{A} \oplus \pi \sim_a id_\mathcal{A} \ (\mathcal{K}\left(\ell^2\right)),$$

where $id_\mathcal{A}$ denotes the identity representation on $\mathcal{A}$.

In a deep and beautiful paper [12], Qihui Li, Junhao Shen, and Rui Shi proved the best-to-date version of the "hard" part.

THEOREM 3. [12] *Suppose $\mathcal{A}$ is a separable nuclear C*-algebra, $\mathcal{M}$ is a sigma-finite type II_∞ factor von Neumann algebra and $\mathcal{K}_\mathcal{M}$ is the closed ideal generated by the finite projections in $\mathcal{M}$. If $\pi, \sigma : \mathcal{A} \to \mathcal{M}$ are unital $*$-homomorphisms such that*

$$\pi^{-1}(\mathcal{K}_\mathcal{M}) \subset \ker \rho,$$

then

$$\pi \sim_a \pi \oplus \sigma \ (\mathcal{K}_\mathcal{M}).$$

2. Finite von Neumann algebras

A separable C*-algebra is *AF* if it is a direct limit of finite-dimensional C*-algebras. A separable C*-algebra is *homogeneous* if it is a finite direct sum of algebras of the form $\mathbb{M}_n(C(X))$, where X is a compact Hausdorff space. A unital C*-algebra $\mathcal{A}$ is *subhomogeneous* if there is an $n \in \mathbb{N}$, such that every irreducible representation is on a Hilbert space of dimension at most n; equivalently, if $x^n = 0$ for every nilpotent $x \in \mathcal{A}$. Every subhomogeneous algebra is a subalgebra of a homogeneous one. Every subhomogeneous von Neumann algebra is homogeneous; in particular, if $\mathcal{A}$ is subhomogeneous, then $\mathcal{A}^{\#\#}$ is homogeneous, i.e., $\mathcal{A}^{\#\#}$ is a finite direct sum of algebras of the form $\mathbb{M}_n(L^\infty(X,\Sigma,\mu))$ with (X,Σ,μ) a measure space. A C*-algebra is approximately subhomogeneous (ASH) if it is a direct limit of subhomogeneous C*algebras. A C*-algebra $\mathcal{A}$ is GCR (Type I) if for every irreducible representation $\pi:\mathcal{A}\to B(H)$ we have $\mathcal{K}(H)\subset\pi(\mathcal{A})$. Thus every subhomogeneous C*-algebra is GCR and every ASH C*-algebra is a direct limit of GCR C*-algebras. It was proved by Glimm [6] that a C*-algebra $\mathcal{A}$ is GCR if and only if, for every representation $\pi:\mathcal{A}\to B(H)$, $\pi(\mathcal{A})''$ is a type I von Neumann algebra. This is equivalent to saying $\mathcal{A}^{\#\#}$ is a type I von Neumann algebra.

There has been a lot of work determining which separable C*-algebras are AF-embeddable. A (possibly nonseparable) C*-algebra $\mathcal{B}$ is *LF* if, for every finite subset $F\subset\mathcal{B}$ and every $\varepsilon>0$ there is a finite-dimensional C*-algebra $\mathcal{D}$ of $\mathcal{B}$ such that, for every $b\in F$, $\operatorname{dist}(b,\mathcal{D})<\varepsilon$. Every separable unital C*-subalgebra of a LF C*-algebra is contained in a separable AF subalgebra [3]. A C*-algebra $\mathcal{A}$ is *AL* if, for every finite subset $F\subset\mathcal{A}$ and every $\varepsilon>0$, there is a finite-dimensional C*-subalgebra $\mathcal{D}$ of $\mathcal{A}$ such that, for every $x\in F$, $\operatorname{dist}(b,\mathcal{D})<\varepsilon$. We say that a unital C*-subalgebra $\mathcal{B}$ of a unital C*-algebra $\mathcal{E}$ is *relatively LF in* $\mathcal{E}$ if and only if, for every finite subset $F\subset\mathcal{B}$ and every $\varepsilon>0$ there is a finite-dimensional C*-algebra $\mathcal{D}$ of $\mathcal{E}$ such that, for every $b\in F$, $\operatorname{dist}(b,\mathcal{D})<\varepsilon$.

We are interested in the property that a C*-algebra $\mathcal{A}$ is relatively LF in $\mathcal{A}^{\#\#}$. If $\mathcal{A}$ is subhomogeneous, then $\mathcal{A}^{\#\#}$ is a finite direct sum of algebras of the form $\mathbb{M}_n(L^\infty(\Omega,\Sigma,\mu))$ with (Ω,Σ,μ) a measure space. If $\{E_1,\ldots,E_s\}$ is a measurable partition of Ω, then the set of matrices of the form (f_{ij}) with each f_{ij} in the linear span of $\{\chi_{E_1},\ldots,\chi_{E_s}\}$ is an sn^2-dimensional C*-subalgebra of $\mathbb{M}_n(L^\infty(\Omega,\Sigma,\mu))$. Since the set of $n\times n$ matrices of simple functions is dense in $\mathbb{M}_n(L^\infty(\Omega,\Sigma,\mu))$, we see that $\mathbb{M}_n(L^\infty(\Omega,\Sigma,\mu))$ is LF. If $\mathcal{A}$ is ASH, then there is a sequence $\{\mathcal{A}_n\}$ of subhomogeneous C*-subalgebras of $\mathcal{A}$ such that

$$\mathcal{A}_1\subset\mathcal{A}_2\subset\cdots \text{ and } \mathcal{A}=(\cup_{n\in\mathbb{N}}\mathcal{A}_n)^{-\|\|}.$$

It follows that $\mathcal{A}\subset\left(\cup_{n\in\mathbb{N}}\mathcal{A}_n^{\#\#}\right)^{-\|\|}\subset\mathcal{A}^{\#\#}$ and $(\cup_{n\in\mathbb{N}}\mathcal{A}_n)^{-\|\|}$ is LF. Thus every subhomogeneous C*-algebra is relatively LF in its second dual.

For LF C*-algebras we can prove an approximate equivalence theorem for representation into an arbitrary unital C*-algebra.

LEMMA 1. *Suppose $\mathcal{B}$ is a unital LF C*-algebra and $\mathcal{D}=\mathbb{M}_{n_1}(\mathbb{C})\oplus\cdots\oplus\mathbb{M}_{n_k}(\mathbb{C})$ and $\mathcal{W}$ is a unital C*-algebra.*

1. *If $\pi,\rho:\mathcal{D}\to\mathcal{W}$ are unital $*$-homomorphisms and $\pi(e_{11,s})\sim\rho(e_{11,s})$ for $1\leqslant s\leqslant k$, where $\{e_{ij,s}\}$ is the system of matrix units for $\mathbb{M}_{n_s}(\mathbb{C})$, then π and ρ are unitarily equivalent in $\mathcal{W}$.*

2. *If $\pi,\rho:\mathcal{B}\to\mathcal{W}$ are unital $*$-homomorphisms such that $\pi(p)\sim\rho(p)$ in $\mathcal{W}$ for every projection $p\in\mathcal{B}$, then $\pi\sim_a\rho$ in $\mathcal{W}$.*

Proof. (1) Since $e_{ii,s} \sim e_{11,s}$ in $\mathcal{D}$ for $1 \leqslant i \leqslant n_s$ and $1 \leqslant s \leqslant k$, we see that $\pi(e_{ii,s}) \sim \rho(e_{ii,s})$ in $\mathcal{W}$ for $1 \leqslant i \leqslant n_s$ and $1 \leqslant s \leqslant k$. It follows from [4, Theorem 2] that π and ρ are unitarily equivalent in $\mathcal{W}$.

(2) Suppose Λ is the set of all pairs $\lambda = (F_\lambda, \varepsilon_\lambda)$ with F_λ a finite subset of $\mathcal{B}$ and $\varepsilon_\lambda > 0$. Clearly Λ is directed by $(\subset, \geqslant)$. For $\lambda \in \Lambda$, we can choose a finite-dimensional algebra $\mathcal{D}_\lambda \subset \mathcal{B}$ such that, for every $x \in F_\lambda$, $\operatorname{dist}(x, \mathcal{D}_\lambda) < \varepsilon_\lambda$. It follows from part (1) that there is a unitary operator $U_\lambda \in \mathcal{W}$ such that, for every $x \in \mathcal{D}_\lambda$, $U\pi(x)U^* = \rho(x)$. For each $a \in F_\lambda$, we can choose $x_a \in \mathcal{D}_\lambda$ such that $\|a - x_a\| < \varepsilon_\lambda$. Hence, for every $a \in F_\lambda$

$$\left\|U_\lambda \pi(a) U_\lambda^* - \rho(a)\right\| = \left\|U_\lambda \pi(a - x_a) U_\lambda^* - \rho(a - x_a)\right\| < 2\varepsilon_\lambda.$$

It follows that, for every $a \in \mathcal{A}$,

$$\lim_\lambda \left\|U_\lambda \pi(a) U_{gl}^* - \rho(a)\right\| = 0. \quad \square$$

A key property of a finite von Neumann algebra $\mathcal{M}$ is that there is a faithful normal tracial conditional expectation $\Phi_\mathcal{M}$ from $\mathcal{M}$ to its center $\mathcal{Z}(\mathcal{M})$, and that for projections p and q in $\mathcal{M}$, we have p and q are Murray-von Neumann equivalent if and only if $\Phi_\mathcal{M}(p) = \Phi_\mathcal{M}(q)$. (See [11].) The map $\Phi_\mathcal{M}$ is called the *center-valued trace* on $\mathcal{M}$. Note that in the next lemma and the theorem that follows, there is no separability assumption on the C*-algebra $\mathcal{A}$ or the dimension of the Hilbert space on which $\mathcal{M}$ acts. This lemma appears in [2] and [8].

LEMMA 2. *Suppose $\mathcal{A}$ is a (possibly nonunital) C*-algebra, $\mathcal{M}$ is a finite von Neumann algebra. If $\pi, \rho : \mathcal{A} \to \mathcal{M}$ are $*$-homomorphisms, the following are equivalent:*

1. *For every $a \in \mathcal{A}$,*
$$\mathcal{M}\text{-}rank(\pi(a)) = \mathcal{M}\text{-}rank(\rho(a)),$$
2. $\Phi_\mathcal{M} \circ \pi = \Phi_\mathcal{M} \circ \rho$.

Proof. $(1) \Rightarrow (2)$. We can extend π and ρ to weak*-weak* continuous *-homomorphisms $\hat{\pi}, \hat{\rho} : \mathcal{A}^{\#\#} \to \mathcal{M}$. Suppose $x \in \mathcal{A}$ and $0 \leqslant x \leqslant 1$. Suppose $0 < \alpha < 1$ and define $f_\alpha : [0,1] \to [0,1]$ by

$$f(t) = \operatorname{dist}(t, [0, \alpha]).$$

Since $f(0) = 0$, we see that $f(x) \in \mathcal{A}$, and $\chi_{(\alpha,1]}(x) = \text{weak*-}\lim_{n\to\infty} f(x)^{1/n} \in \mathcal{A}^{\#\#}$, so

$$\mathfrak{R}(f(x)) = \chi_{(\alpha,1]}(x).$$

It follows that

$$\hat{\pi}\left(\chi_{(\alpha,1]}(x)\right) = \mathfrak{R}(\pi(f_\alpha(x))) = \chi_{(\alpha,1]}(\pi(x))$$

and

$$\hat{\rho}\left(\chi_{(\alpha,1]}(x)\right) = \Re\left(\rho\left(f_\alpha(x)\right)\right) = \chi_{(\alpha,1]}\left(\rho(x)\right).$$

Hence

$$\Phi_{\mathcal{M}}\left(\hat{\pi}\left(\chi_{(\alpha,1]}(x)\right)\right) = \Phi_{\mathcal{M}}\left(\hat{\rho}\left(\chi_{(\alpha,1]}(x)\right)\right).$$

Suppose $0 < \alpha < \beta < 1$. Since $\chi_{(\alpha,\beta]} = \chi_{(\alpha,1]} - \chi_{(\beta,1]}$, we see that

$$\Phi_{\mathcal{M}}\left(\hat{\pi}\left(\chi_{(\alpha,\beta]}(x)\right)\right) = \Phi_{\mathcal{M}}\left(\hat{\rho}\left(\chi_{(\alpha,\beta]}(x)\right)\right).$$

Thus, for all $n \in \mathbb{N}$,

$$\Phi_{\mathcal{M}}\left(\hat{\pi}\left(\sum_{k-1}^{n-1} \frac{k}{n}\chi_{(\frac{k}{n},\frac{k+1}{n}]}(x)\right)\right) = \Phi_{\mathcal{M}}\left(\hat{\rho}\left(\sum_{k-1}^{n-1} \frac{k}{n}\chi_{(\frac{k}{n},\frac{k+1}{n}]}(x)\right)\right).$$

Since, for every $n \in \mathbb{N}$,

$$\left\| x - \sum_{k-1}^{n-1} \frac{k}{n}\chi_{(\frac{k}{n},\frac{k+1}{n}]}(x) \right\| \leqslant 1/n,$$

it follows that

$$\Phi_{\mathcal{M}}\left(\pi(x)\right) = \Phi_{\mathcal{M}}\left(\hat{\pi}(x)\right) = \Phi_{\mathcal{M}}\left(\hat{\rho}(x)\right) = \Phi_{\mathcal{M}}\left(\rho(x)\right).$$

Since $\mathcal{A}$ is the linear span of its positive contractions, $\Phi_{\mathcal{M}} \circ \pi = \Phi_{\mathcal{M}} \circ \rho$.

$(2) \Rightarrow (1)$. This is contained in [4]. $\square$

THEOREM 4. *Suppose $\mathcal{A}$ is relatively LF in $\mathcal{A}^{\#\#}$ and $\mathcal{M}$ is a finite von Neumann algebra. If $\pi, \rho : \mathcal{A} \to \mathcal{M}$ are unital $*$-homomorphisms, then the following are equivalent:*

1. *$\pi \sim_a \rho$ in $\mathcal{M}$.*
2. *$\mathcal{M}$-rank$(\pi(a)) = \mathcal{M}$-rank$(\rho(a))$ for every $a \in \mathcal{A}$.*
3. *$\Phi_{\mathcal{M}} \circ \pi = \Phi_{\mathcal{M}} \circ \rho$.*

Proof. $(3) \Rightarrow (1)$. We can extend π and ρ to weak*-weak* continuous $*$-homomorphisms $\hat{\pi}, \hat{\rho} : \mathcal{A}^{\#\#} \to \mathcal{M}$. Since $\Phi_{\mathcal{M}}$ is weak*-weak* continuous, it follows that $\Phi_{\mathcal{M}} \circ \hat{\pi} = \Phi_{\mathcal{M}} \circ \hat{\rho}$.

Let

$$\Lambda = \{(F, \varepsilon) : F \subset \mathcal{A}, F \text{ is finite}, \varepsilon > 0\},$$

ordered by the relation $(\subset, \geqslant)$. Suppose $\lambda = (F, \varepsilon) \in \Lambda$. Since $\mathcal{A}$ is relatively LF in $\mathcal{A}^{\#\#}$, there is a finite-dimensional algebra $\mathcal{B} \subset \mathcal{A}^{\#\#}$ such that, for every $x \in F$,

$$\operatorname{dist}(x, \mathcal{B}) < \varepsilon.$$

Thus, for each $x \in F$ there is a $b_x \in \mathcal{B}$ such that

$$\|x - b_x\| < \varepsilon/2.$$

We know from Lemma 1 that $\hat{\pi}|_{\mathcal{B}}$ and $\hat{\rho}|_{\mathcal{B}}$ are unitarily equivalent in $\mathcal{M}$. Hence, there is a unitary $U_\lambda \in \mathcal{M}$ such that, for every $b \in \mathcal{B}$,

$$U_\lambda^* \hat{\pi}(B) U_\lambda = \hat{\rho}(b).$$

Thus, for every $x \in F$,

$$\left\| U_\lambda^* \pi(x) U_\lambda - \rho(x) \right\| \leqslant \left\| U_\lambda^* \hat{\pi}(x - b_x) U_\lambda \right\| + \|\hat{\rho}(b_x - x)\| < \varepsilon.$$

Hence, for every $x \in \mathcal{A}$

$$\lim_\lambda \left\| U_\lambda^* \pi(x) U_\lambda - \rho(x) \right\| = 0.$$

Thus $\pi \sim_a \rho \ (\mathcal{M})$.

$(1) \Rightarrow (3)$. Suppose $\{U_\lambda\}$ is a net of unitaries in $\mathcal{M}$ such that, for every $a \in \mathcal{A}$,

$$\left\| U_\lambda \pi(a) U_\lambda^* - \rho(a) \right\| \to 0.$$

Thus, since $\Phi_{\mathcal{M}}$ is tracial and continuous,

$$\Phi_{\mathcal{M}}(\rho(a)) = \lim_\lambda \Phi_{\mathcal{M}}\left(U_\lambda \pi(a) U_\lambda^*\right) = \Phi_{\mathcal{M}}(\pi(a)).$$

$(3) \Rightarrow (2)$. Assume (3). Then, for any $a \in \mathcal{A}$,

$$\begin{aligned} \Phi_{\mathcal{M}}(\mathfrak{R}(\pi(a))) &= \lim_{n\to\infty} \Phi_{\mathcal{M}}\left(\pi\left((aa^*)^{1/n}\right)\right) = \lim_{n\to\infty} \Phi_{\mathcal{M}}\left(\rho\left((aa^*)^{1/n}\right)\right) \\ &= \Phi_{\mathcal{M}}(\mathfrak{R}(\pi(a))). \end{aligned}$$

Hence $\mathfrak{R}(\pi(a)) \sim \mathfrak{R}(\rho(a))$. Thus $\mathcal{M}$-rank$(\pi(a)) = \mathcal{M}$-rank$(\rho(a))$.

$(2) \Rightarrow (3)$. This is Lemma 2. $\square$

REMARK 1. It is important to note that the proof of $(2) \Rightarrow (3)$ in Theorem 4 holds even when $\mathcal{A}$ is not unital.

Here is our main theorem of this section.

THEOREM 5. *Suppose $\mathcal{A}$ is a unital C*-algebra that is approximately type I in $\mathcal{A}^{\#\#}$, $\mathcal{M}$ is a finite von Neumann algebra, and $\pi, \rho : \mathcal{A} \to \mathcal{M}$ are unital $*$-homomorphisms such that*

$$(\mathcal{M}\text{-}rank) \circ \pi = (\mathcal{M}\text{-}rank) \circ \rho.$$

Then $\pi \sim_a \rho$ in $\mathcal{M}$.

Proof. Let $\Phi_{\mathcal{M}} : \mathcal{M} \to \mathcal{Z}(\mathcal{M})$ be the center-valued trace on $\mathcal{M}$. Let $\hat{\pi}, \hat{\rho} : \mathcal{A}^{\#\#} \to \mathcal{M}$ be the weak*-continuous extensions of π and ρ. Then $\Phi_{\mathcal{M}} \circ \hat{\pi} = \Phi_{\mathcal{M}} \circ \hat{\rho}$, or

$$(\mathcal{M}\text{-rank}) \circ \hat{\pi} = (\mathcal{M}\text{-rank}) \circ \hat{\rho}.$$

In particular, $\ker \hat{\pi} = \ker \hat{\rho}$ is a weak*-closed ideal in $\mathcal{A}^{\#\#}$, so there is a projection $Q \in \mathcal{Z}\left(\mathcal{A}^{\#\#}\right)$ such that

$$\ker \hat{\pi} = \ker \hat{\rho} = (1 - Q)\mathcal{A}^{\#\#}.$$

Thus $\hat{\pi}, \hat{\rho} : Q\mathcal{A}^{\#\#} \to \mathcal{M}$ is an embedding. Since $Q\mathcal{A}^{\#\#}$ is isomorphic to a subalgebra of $\mathcal{M}$, we know that $Q\mathcal{A}^{\#\#}$ is a finite von Neumann algebra and a summand of $\mathcal{A}^{\#\#}$. Suppose $\mathcal{N}$ is a type I von Neumann subalgebra of $\mathcal{A}^{\#\#}$. Then $Q\mathcal{N}$ is a type I von Neumann subalgebra of $Q\mathcal{A}^{\#\#}$. Since $Q\mathcal{A}^{\#\#}$ is finite, $Q\mathcal{N}$ is a finite type I von Neumann algebra. Thus there is an orthogonal sequence $\{e_n\}$ of projections in the center of $Q\mathcal{N}$ whose sum is Q such that

$$Q\mathcal{N} = \sum_{k\in\mathbb{N}}^{\oplus} e_k Q\mathcal{N}$$

and each $e_k Q\mathcal{N}$ is a type I_k von Neumann algebra and is isomorphic to $\mathbb{M}_k\left(L^\infty(\mu_k)\right)$ acting on

$$L^2(\mu_k)^{(n)} = L^2(\mu_k) \oplus \cdots \oplus L^2(\mu_k)$$

for some measure space (X_k, Σ_k, μ_k). Clearly, $e_k Q\mathcal{N} = \mathbb{M}_k\left(L^\infty(\mu_k)\right)$ is an AL C*-algebra. Since $\hat{\pi}(Q) = \hat{\rho}(Q) = 1$, it follows that

$$1 = \sum_{n\in\mathbb{N}} \hat{\pi}(e_n) = \sum_{n\in\mathbb{N}} \hat{\rho}(e_n).$$

Since, for each $n \in \mathbb{N}$, $(\mathcal{M}\text{-rank}) \circ \hat{\pi}(e_n) = (\mathcal{M}\text{-rank}) \circ \hat{\rho}(e_n)$ we see that the projections $\hat{\pi}(e_n)$ and $\hat{\rho}(e_n)$ are unitarily equivalent in $\mathcal{M}$. Thus there is a unitary operator $U \in \mathcal{M}$ such that, for every $n \in \mathbb{N}$,

$$U\hat{\pi}(e_n)U^* = \hat{\rho}(e_n).$$

By replacing π with $U\pi(\cdot)U^*$, we can assume, for every $n \in \mathbb{N}$, that

$$\hat{\pi}(e_n) = \hat{\rho}(e_n).$$

We now have $\hat{\pi}|_{e_n Q\mathcal{N}}, \hat{\rho}|_{e_n Q\mathcal{N}} : e_n Q\mathcal{N} \to \hat{\pi}(e_n)\mathcal{M}\hat{\pi}(e_n)$. Since $e_n Q\mathcal{N}$ is AL and $\hat{\pi}(e_n)\mathcal{M}\hat{\pi}(e_n)$ is a finite von Neumann algebra, it follows from Theorem 4 that $\hat{\pi}_{e_n Q\mathcal{N}}$ and $\hat{\rho}|_{e_n Q\mathcal{N}}$ are approximately equivalent in $\hat{\pi}(e_n)\mathcal{M}\hat{\pi}(e_n)$ for each $n \in \mathbb{N}$. Since $\hat{\pi}|_{Q\mathcal{N}}, \hat{\rho}|_{Q\mathcal{N}} : Q\mathcal{N} \to \sum_{n\in\mathbb{N}}^{\oplus} \hat{\pi}(e_n)\mathcal{M}\hat{\pi}(e_n)$ and

$$\hat{\pi}|_{Q\mathcal{N}} = \sum_{n\in\mathbb{N}}^{\oplus} \hat{\pi}|_{e_n Q\mathcal{N}} \text{ and } \hat{\rho}|_{Q\mathcal{N}} = \sum_{n\in\mathbb{N}}^{\oplus} \hat{\rho}|_{e_n Q\mathcal{N}},$$

we easily see that $\hat{\pi}|_{Q\mathcal{N}}$ and $\hat{\rho}|_{Q\mathcal{N}}$ are approximately equivalent in $\mathcal{M}$. Since $\hat{\pi}|_{(1-Q)\mathcal{N}} = \hat{\rho}_{(1-Q)\mathcal{N}} = 0$, we see that $\hat{\pi}|\mathcal{N}$ and $\hat{\rho}|_{\mathcal{N}}$ are approximately equivalent in $\mathcal{M}$.

Let $\Lambda = \{(F,\varepsilon) : F \subset \mathcal{A} \text{ is finite}, \varepsilon > 0\}$ directed by the partial order $(\subset, >)$. Suppose $\lambda = (F,\varepsilon) \in \Lambda$. Since $\mathcal{A}$ approximately type I in $\mathcal{A}^{\#\#}$, we know that there is a type I von Neumann subalgebra $\mathcal{N}$ of $\mathcal{A}^{\#\#}$ such that, for every $T \in F$,

$$\operatorname{dist}(T,\mathcal{N}) < \varepsilon/2.$$

Thus, for each $T \in F$, there is an $x_T \in \mathcal{N}$ such that $\|T - x_T\| < \varepsilon/37$.

Thus $\|\hat{\pi}(x_T) - \pi(T)\| < \varepsilon/37$ and $\|\hat{\rho}(x_T) - \rho(T)\| < \varepsilon/37$ whenever $T \in \mathcal{F}$. Since $\{x_T : T \in F\}$ is finite and $\hat{\pi}|\mathcal{N}$ and $\hat{\rho}|_{\mathcal{N}}$ are approximately equivalent in $\mathcal{M}$, there is a unitary $U_\lambda \in \mathcal{M}$ such that

$$\left\|U_\lambda \hat{\pi}(x_T) U_\lambda^* - \hat{\rho}(x_T)\right\| < \varepsilon/37$$

for every $T \in F$. Thus

$$\begin{aligned} &\left\|U_\lambda \pi(T) U_\lambda^* - \rho(T)\right\| \\ \leqslant &\left\|U_\lambda \hat{\pi}(x_T) U_\lambda^* - \hat{\rho}(x_T)\right\| + \|U_\lambda \hat{\pi}(T - x_T) U^*\| + \|\hat{\rho}(T - x_T)\| < \varepsilon \end{aligned}$$

Thus, for every $T \in \mathcal{A}$,

$$\lim_\lambda \left\|U_\lambda \pi(T) U_\lambda^* - \rho(T)\right\| = 0.$$

Hence π and ρ are approximately equivalent in $\mathcal{M}$. $\square$

In [7] it was shown that if $\mathcal{A}$ is a separable unital C*-algebra and π and ρ are representations on a separable Hilbert space such that, for every $x \in \mathcal{A}$

$$\operatorname{rank}\pi(x) \leqslant \operatorname{rank}\rho(x),$$

then there is a representation σ such that

$$\pi \oplus \sigma \sim_a \rho.$$

In [9], Rui Shi and the first author proved an analogue for representations of separable abelian C*-algebras into II_1 factor von Neumann algebras. This result was extended by Shilin Wen, Junsheng Fang and Rui Shi [5] to separable AF C*-algebras. We extend this result further, including separable ASH C*-algebras.

THEOREM 6. *Suppose $\mathcal{A}$ is a separable C*-algebra and there is an LF C*-algebra $\mathcal{D}$ such that $\mathcal{A} \subset \mathcal{D} \subset \mathcal{A}^{\#\#}$. Suppose also that $\mathcal{M}$ is a II_1 factor von Neumann algebra with a faithful normal tracial state τ. Suppose P is a projection in $\mathcal{M}$ and $\pi : \mathcal{A} \to P\mathcal{M}P$ and $\rho : \mathcal{A} \to \mathcal{M}$ are unital $*$-homomorphisms such that, for every $a \in \mathcal{A}$,*

$$\mathcal{M}\text{-}rank(\pi(a)) \leqslant \mathcal{M}\text{-}rank(\rho(a)).$$

Then there is a unital $*$-homomorphism $\sigma : \mathcal{A} \to P^\perp \mathcal{M} P^\perp$ such that

$$\pi \oplus \sigma \sim_a \rho \ (\mathcal{M}).$$

Proof. As in the proof of Theorem 4 choose a separable AF C*-algebra $\mathcal{B}$ such that $\mathcal{A} \subset \mathcal{B} \subset \mathcal{D}$, and extend π and ρ to unital weak*-weak* continuous $*$-homomorphisms $\hat{\pi}$ and $\hat{\rho}$ with domain $\mathcal{A}^{\#\#}$. It was shown in [4] that the condition on π and ρ is equivalent to: for every $a \in \mathcal{M}$ with $0 \leqslant a$, $\tau(\pi(a)) \leqslant \tau(\rho(a))$. It follows from weak* continuity that, for every $a \in \mathcal{A}^{\#\#}$ with $0 \leqslant a$, $\tau(\hat{\pi}(a)) \leqslant \tau(\hat{\rho}(a))$. In particular this holds for $0 \leqslant a \in \mathcal{B}$. However, since $\mathcal{B}$ is AF, it follows from [5] that there is a unital $*$-homomorphism $\gamma : \mathcal{B} \to P^{\perp}\mathcal{A}P^{\perp}$ such that

$$(\hat{\pi}|_{\mathcal{B}}) \oplus \gamma \sim_a \hat{\rho}|_{\mathcal{B}} \ (\mathcal{M}).$$

If we let $\sigma = \gamma|_{\mathcal{A}}$, we see $\pi \oplus \sigma \sim_a \rho \ \ (\mathcal{M})$. $\square$

3. Representations of ASH algebras relative to ideals

In this section we prove (Theorem 8) a version of Voiculescu's theorem for representations of a separable ASH C*-algebra into a semifinite von Neumann algebra acting on a separable Hilbert space.

We first prove a more general result. If $\mathcal{J}$ is a norm closed two-sided ideal in a von Neumann algebra $\mathcal{M}$, we let $\mathcal{J}_0$ denote the ideal in $\mathcal{M}$ generated by the projections in $\mathcal{J}$. We begin with a probably well-known lemma.

LEMMA 3. *Suppose $\mathcal{J}$ is a norm closed two-sided ideal in a von Neumann algebra $\mathcal{M}$ and $\mathcal{A}$ is a C*-algebra and $\pi, \rho : \mathcal{A} \to \mathcal{M}$ are unital $*$-homomorphisms. Then*

1. *$\mathcal{J}$ is the norm closed linear span of the set of projections in $\mathcal{J}$, i.e.,*
$$\mathcal{J}_0^{-\|\|} = \mathcal{J},$$
2. *$\mathcal{J}_0 = \{T \in \mathcal{M} : T = PTP \text{ for some projection } P \in \mathcal{J}\}$,*
3. *$T \in \mathcal{J}_0$ if and only if $\chi_{(0,\infty)}(|T|) = \mathfrak{R}(T) \in \mathcal{J}_0$,*
4. *If P and Q are projections in $\mathcal{J}_0$ then $P \vee Q = \mathfrak{R}(P+Q) \in \mathcal{J}_0$,*
5. *$\pi^{-1}(\mathcal{J}_0)^{-\|\|} = \pi^{-1}(\mathcal{J})$,*
6. *If $\{\mathcal{A}_i : i \in I\}$ is an increasingly directed family of unital C*-subalgebras of $\mathcal{A}$ and $\mathcal{A} = [\cup_{i\in I}\mathcal{A}_i]^{-\|\|}$, then*
$$\left[\cup_{i\in I}\mathcal{A}_i \cap \pi^{-1}(\mathcal{J}_0)\right]^{-\|\|} = \pi^{-1}(\mathcal{J}).$$

Proof. (1), (2), (3) can be found in [11].

(4). Suppose $a \in \pi^{-1}(\mathcal{J})$. Suppose $\varepsilon > 0$ and define $g_\varepsilon : [0,\infty) \to [0,\infty)$ by

$$g_\varepsilon(t) = \begin{cases} t/\varepsilon \text{ if } 0 \leqslant t \leqslant \varepsilon \\ 1 \quad\quad \text{if } 1 < t \end{cases}.$$

Then $\pi(a) \in \mathcal{J}$, so

$$\pi(g_\varepsilon(|a|)) = g_\varepsilon(|\pi(a)|)\chi_{(\varepsilon,\infty)}(|\pi(a)|) \in \mathcal{J}_0,$$

and

$$\|a - ag_\varepsilon(|a|)\| \leqslant \varepsilon.$$

(5). Let $\eta : \mathcal{M} \to \mathcal{M}/\mathcal{J}$ be the quotient map. Suppose $a \in \pi^{-1}(\mathcal{J})$ and $\varepsilon > 0$. Then there is an $i \in I$ and a $b \in \mathcal{A}_i$ such that $\|a-b\| < \varepsilon$. Thus

$$\|(\eta \circ (\pi|_{\mathcal{A}_i}))(b)\| = \|(\eta \circ \pi)(b)\| = \|(\eta \circ \pi)(b-a)\| \leqslant \varepsilon,$$

so there is a $w \in \mathcal{A}_i$ so that

$$\|w\| = \|(\eta \circ (\pi|_{\mathcal{A}_i}))(w)\| = \|(\eta \circ (\pi|_{\mathcal{A}_i}))(b)\| \leqslant \varepsilon.$$

$z = b - w \in \ker(\eta \circ (\pi|_{\mathcal{A}_i})) = \pi^{-1}(\mathcal{J}) \cap \mathcal{A}_i$, and $\|b-z\| = \|w\| < \varepsilon$. It follows from part (2) that there is a $v \in \pi^{-1}(\mathcal{J}_0) \cap \mathcal{A}_i$ such that $\|z-v\| \leqslant \varepsilon$. Hence $\|a-v\| \leqslant \|a-b\| + \|b-z\| + \|z-v\| \leqslant 3\varepsilon$.

(6). Let $\eta : \mathcal{M} \to \mathcal{M}/\mathcal{J}$ be the quotient map. Suppose $a \in \pi^{-1}(\mathcal{J})$ and $\varepsilon > 0$. Then there is an $i \in I$ and a $b \in \mathcal{A}_i$ such that $\|a-b\| < \varepsilon$. Thus

$$\|(\eta \circ (\pi|_{\mathcal{A}_i}))(b)\| = \|(\eta \circ \pi)(b)\| = \|(\eta \circ \pi)(b-a)\| \leqslant \varepsilon,$$

so there is a $w \in \mathcal{A}_i$ so that

$$\|w\| = \|(\eta \circ (\pi|_{\mathcal{A}_i}))(w)\| = \|(\eta \circ (\pi|_{\mathcal{A}_i}))(b)\| \leqslant \varepsilon.$$

$z = b - w \in \ker(\eta \circ (\pi|_{\mathcal{A}_i})) = \pi^{-1}(\mathcal{J}) \cap \mathcal{A}_i$, and $\|b-z\| = \|w\| < \varepsilon$. It follows from part (5) that there is a $v \in \pi^{-1}(\mathcal{J}_0) \cap \mathcal{A}_i$ such that $\|z-v\| \leqslant \varepsilon$. Hence $\|a-v\| \leqslant \|a-b\| + \|b-z\| + \|z-v\| \leqslant 3\varepsilon$. $\square$

Suppose $\mathcal{A}$ is a unital C*-algebra, $\mathcal{M} \subset B(H)$ is a von Neumann algebra with a norm-closed ideal $\mathcal{J}$ and $\pi : \mathcal{A} \to \mathcal{M}$ is a unital $*$-homomorphism. We define

$$H_{\pi,\mathcal{J}} = \mathrm{sp}^{-\|\|}(\cup\{\mathrm{ran}\pi(a) : a \in \mathcal{A} \text{ and } \pi(a) \in \mathcal{J}\}).$$

It is clear that $H_{\pi,\mathcal{J}}$ is a reducing subspace for π and we call the summand $\pi(\cdot)|_{H_{\pi,\mathcal{J}}} = \pi_{\mathcal{J}}$.

The following is a fairly general version of the analogue of the "easy part" of the proof of Voiculescu's theorem when the C*-algebra is ASH. In particular, there is no assumption that the von Neumann algebra $\mathcal{M}$ is sigma-finite (e.g., acts on a separable Hilbert space).

THEOREM 7. *Suppose $\mathcal{A}$ is a separable unital ASH C*-algebra, $\mathcal{M} \subset B(H)$ is a von Neumann algebra with a norm closed two-sided ideal $\mathcal{J}$. Suppose $\pi, \rho : \mathcal{A} \to \mathcal{M}$ are unital $*$-homomorphisms such that*

1. *Every projection in $\mathcal{J}$ is finite,*

2. *$\mathcal{M}$-rank$(\pi(a)) = \mathcal{M}$-rank$(\rho(a))$ for every $a \in \mathcal{A}$.*

Then there is a sequence $\{W_n\}$ of partial isometries in $\mathcal{M}$ such that

(3) $W_n^* W_n$ is the projection onto $H_{\pi,\mathcal{J}}$ and $W_n W_n^*$ is the projection onto $H_{\rho,\mathcal{J}}$,,

(4) $W_n \pi_{\mathcal{J}}(a) W_n^* - \rho_{\mathcal{J}}(a) \in \mathcal{J}$ for every $n \in \mathbb{N}$ and every $a \in \mathcal{A}$,

(5) $\lim_{n\to\infty} \|W_n \pi_{\mathcal{J}}(a) W_n^* - \rho_{\mathcal{J}}(a)\| = 0$ for every $a \in \mathcal{A}$.

Proof. First, suppose $x \in \mathcal{A}$ and $x = x^*$. It follows from [4] that there is a sequence $\{U_n\}$ of unitary operators in $\mathcal{M}$ such that

$$\|U_n \pi(x) U_n^* - \rho(x)\| \to 0.$$

It follows that $\pi(x) \in \mathcal{J}$ if and only if $\rho(x) \in \mathcal{J}$ when $x = x^*$. However, for any $a \in \mathcal{A}$, we get $\pi(a) \in \mathcal{J}$ if and only if $\pi(|a|) \in \mathcal{J}$. Hence $\pi^{-1}(\mathcal{J}) = \rho^{-1}(\mathcal{J})$. Also, $\pi(a) \in \mathcal{J}_0$ if and only if $\mathfrak{R}(\pi(a)) \in \mathcal{J}_0$. Since $\mathfrak{R}(\pi(a))$ and $\mathfrak{R}(\rho(a))$ are Murray von Neumann equivalent (from (2)), we see that $\pi(a) \in \mathcal{J}_0$ if and only if $\rho(a) \in \mathcal{J}_0$. It follows that $\pi^{-1}(\mathcal{J}_0) \cap \mathcal{A}_n = \rho^{-1}(\mathcal{J}_0) \cap \mathcal{A}_n$ for each $n \in \mathbb{N}$, and, from Lemma 3,

$$\left[\bigcup_{n=1}^{\infty} \pi^{-1}(\mathcal{J}_0) \cap \mathcal{A}_n\right]^{-\|\|} = \left[\bigcup_{n=1}^{\infty} \rho^{-1}(\mathcal{J}_0) \cap \mathcal{A}_n\right]^{-\|\|} = \pi^{-1}(\mathcal{J}) = \rho^{-1}(\mathcal{J}).$$

Since $\mathcal{A}$ is an ASH algebra, we can assume that there is a sequence

$$\mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots$$

of subalgebras of $\mathcal{A}$ such that $\cup_{n=1}^{\infty} \mathcal{A}_n$ is norm dense in $\mathcal{A}$ such that, for each $n \in \mathbb{N}$,

$$\mathcal{A}_n^{\#\#} = \mathcal{M}_{k(n,1)}(C(X_{n,1})) \oplus \cdots \oplus \mathcal{M}_{k(n,s_n)}(C(X_{n,s_n}))$$

with $X_{n,1}, \ldots, X_{n,s_n}$ compact Hausdorff spaces.

Suppose $T = (f_{ij}) \in \mathbb{M}_k(C(X))$ is a $k \times k$ matrix of functions. We define $T^{\maltese} = \operatorname{diag}(f, f, \ldots, f)$ where $f = \sum_{i,j=1}^{k} |f_{ij}|^2$. If $\{e_{ij} : 1 \leqslant i, j \leqslant n\}$ is the system of matrix units for $\mathbb{M}_n(\mathbb{C})$, then $T = \sum_{i,j=1}^{n} f_{ij} e_{ij}$. It is clear that if $T \geqslant 0$, then $\mathfrak{R}(T) \leqslant \mathfrak{R}(T^{\maltese})$. Since $f_{ij} e_{ss} = e_{si} T e_{js}$, we have

$$|f_{ij}|^2 e_{ss} = (e_{si} T e_{js})^* (e_{si} T e_{js}) = e_{sj} T^* e_{is} e_{si} T e_{js} = e_{js}^* T^* e_{ii} T e_{js}.$$

Thus

$$T^{\maltese} = \sum_{s=1}^{g} \sum_{i,j=1}^{k} |f_{ij}|^2 e_{ss} = \sum_{s=1}^{g} \sum_{i,j=1}^{k} e_{js}^* T^* e_{ii} T e_{js}.$$

Suppose $A = A_1 \oplus \cdots \oplus A_{s_n} \in \mathcal{A}_n^{\#\#}$, with each $A_j \in \mathcal{M}_{k(n,j)}(C(X_{n,j}))$. We define $\Delta_n : \mathcal{A}_n^{\#\#} \to \mathcal{Z}(\mathcal{A}_n^{\#\#})$ by

$$\Delta_n(A) = A_1^{\maltese} \oplus \cdots \oplus A_{s_n}^{\maltese}.$$

Thus if $A \in \mathcal{A}_n^{\#\#}$, then $\Delta_n(A)$ has the form

$$\Delta_n(A) = \sum_{k=1}^{m} B_k A C_k,$$

with $B_1, C_1, \ldots, B_m, C_m \in \mathcal{A}_n^{\#\#}$.

It is clear that

a. $\Delta_n\left(\mathcal{A}_n^{\#\#}\right)$ is contained in the center $\mathcal{Z}\left(\mathcal{A}_n^{\#\#}\right)$ of $\mathcal{A}_n^{\#\#}$, and

b. If $A \geqslant 0$, then $\mathfrak{R}(A) \leqslant \mathfrak{R}(\Delta_n(A)) \in \mathcal{Z}\left(\mathcal{A}_n^{\#\#}\right)$.

We call a projection $Q \in \mathcal{A}_n^{\#\#}$ *good* if

c. $\hat{\pi}(Q), \hat{\rho}(Q) \in \mathcal{J}_0$

d. $Q \in \left[\mathcal{A}_n \cap \pi^{-1}(\mathcal{J}_0)\right]^{-\text{weak}*}$

e. For all $T \in Q\mathcal{A}^{\#\#}Q$, $\mathcal{M}$-rank$(\hat{\pi}(T)) = \mathcal{M}$-rank$(\hat{\rho}(T))$.

Our proof is based on four claims.

CLAIM 0. Suppose $Q_1, Q_2 \in \mathcal{A}_n^{\#\#}$ are good projections and $Q_1 \perp Q_2$. Then $Q = Q_1 + Q_2$ is a good projection.

Proof of Claim 0. It is clear that Q satisfies (c) and (d). Let $P = \hat{\pi}(Q) \vee \hat{\rho}(Q) \in \mathcal{J}_0$. Thus P is a finite projection in $\mathcal{M}$, so $P\mathcal{M}P$ is a finite von Neumann algebra. Let $\Phi_P : P\mathcal{M}P \to \mathcal{Z}(P\mathcal{M}P)$ be the center-valued trace. Since Q_1 and Q_2 are good, we know from Lemma 2 that

$$\Phi_P \circ \hat{\pi}|_{Q_k\mathcal{A}^{\#\#}Q_k} = \Phi_P \circ \hat{\rho}|_{Q_k\mathcal{A}^{\#\#}Q_k}$$

for $k = 1, 2$. Since $Q_1 \perp Q_2$, we know $\hat{\pi}(Q_1) \perp \hat{\pi}(Q_2)$ and $\hat{\rho}(Q_1) \perp \hat{\rho}(Q_2)$. Since Φ_P is tracial, we know that if $1 \leqslant i \neq j \leqslant 2$ and $A \in \mathcal{A}^{\#\#}$, then

$$\Phi_P(\hat{\pi}(Q_i A Q_j)) = \Phi_P\left(\hat{\pi}(Q_i)\hat{\pi}(A)\hat{\pi}(Q_j)^2\right)$$

$$= \Phi_P(\hat{\pi}(Q_j)\hat{\pi}(Q_i)\hat{\pi}(A)\hat{\pi}(Q_j)) = 0.$$

Similarly,

$$\Phi_P(\hat{\rho}(Q_i A Q_j)) = 0.$$

Thus

$$\begin{aligned}\Phi_P(\hat{\pi}(QAQ)) &= \Phi_P(\hat{\pi}(Q_1 A Q_1)) + \Phi_P(\hat{\pi}(Q_2 A Q_2)) \\ &= \Phi_P(\hat{\rho}(Q_1 A Q_1)) + \Phi_P(\hat{\rho}(Q_2 A Q_2)).\end{aligned}$$

Thus, by Lemma 2, Q satisfies (e). Hence Q is a good projection. This proves the claim. A simple induction proof implies that the sum of a finite family of pairwise orthogonal good projections is good. $\square$

CLAIM 1. If $Q \in \mathcal{A}_n^{\#\#}$ is a good projection, then there is a good projection $P \in \mathcal{Z}\left(\mathcal{A}_n^{\#\#}\right)$ such that $Q \leqslant P$.

Proof of Claim 1. Suppose $Q \in \mathcal{A}_n^{\#\#}$ is a good projection. Choose $B_1, C_1, \ldots, B_k, C_k$ in $\mathcal{A}_n^{\#\#}$ such that

$$E \underset{\text{def}}{=} \sum_{k=1}^{m} B_k Q C_k = \Delta_n(Q) \in \mathcal{Z}\left(\mathcal{A}_n^{\#\#}\right).$$

Since $\mathfrak{R}(E) \in \mathcal{Z}\left(\mathcal{A}_n^{\#\#}\right)$ and $E \geqslant 0$, we see that

$$E = \mathfrak{R}(E) E \mathfrak{R}(E) = \sum_{k=1}^{m} [\mathfrak{R}(E) B_k \mathfrak{R}(E)] \, Q \, [\mathfrak{R}(E) C_k \mathfrak{R}(E)].$$

Hence we can assume, for $1 \leqslant k \leqslant m$, that $B_k, C_k \in \mathfrak{R}(E) \mathcal{A}^{\#\#} \mathfrak{R}(E)$.

Since $\hat{\pi}(Q)$, $\hat{\rho}(Q) \in \mathcal{J}_0$, we see that $\hat{\pi}(E)$ and $\hat{\rho}(E) \in \mathcal{J}_0$, which, in turn, implies $\hat{\pi}(\mathfrak{R}(E))$ and $\hat{\rho}(\mathfrak{R}(E)) \in \mathcal{J}_0$. Then $F = \hat{\pi}(\mathfrak{R}(E)) \vee \hat{\rho}(\mathfrak{R}(E)) \in \mathcal{J}_0$ is a finite projection. Thus $F \mathcal{M} F$ is a finite von Neumann algebra. Also, since, for $1 \leqslant k \leqslant m$, $B_k, C_k \in \mathfrak{R}(E) \mathcal{A}_n^{\#\#} \mathfrak{R}(E)$, we see that $\hat{\pi}(B_k Q C_k), \hat{\rho}(B_k Q C_k) \in F \mathcal{M} F$. Let Φ_F be the center-valued trace on $F \mathcal{M} F$. Since Q is a good projection and in $E \mathcal{A}^{\#\#} E$, we know from Lemma 2, that for every $A \in \mathcal{A}^{\#\#}$,

$$\Phi_F(\hat{\pi}(QAQ)) = \Phi_F(\hat{\rho}(QAQ)).$$

Now $\hat{\pi}, \hat{\rho} : E \mathcal{A}^{\#\#} E \to F \mathcal{M} F$ are $*$-homomorphisms, and, since Φ_F is tracial, we see for $A \in \mathcal{A}^{\#\#}$,

$$\Phi_F(\hat{\pi}(EAE)) =$$

$$= \sum_{j,k=1}^{m} \Phi_F([\hat{\pi}(B_k) \hat{\pi}(Q)] [\hat{\pi}(Q) \hat{\pi}(C_k) \hat{\pi}(A) \hat{\pi}(B_j) \hat{\pi}(Q) \hat{\pi}(C_j)])$$

$$= \sum_{j,k=1}^{m} \Phi_F([\hat{\pi}(Q) \hat{\pi}(C_k) \hat{\pi}(A) \hat{\pi}(B_j) \hat{\pi}(Q) \hat{\pi}(C_j)] [\hat{\pi}(B_k) \hat{\pi}(Q)])$$

$$= \sum_{j,k=1}^{m} \Phi_F(\hat{\pi}(Q C_k A B_j Q C_j B_k Q)) = \sum_{j,k=1}^{m} \Phi_F(\hat{\rho}(Q C_k A B_j Q C_j B_k Q))$$

$$= \sum_{j,k=1}^{m} \Phi_F([\hat{\rho}(Q) \hat{\rho}(C_k) \hat{\rho}(A) \hat{\rho}(B_j) \hat{\rho}(Q) \hat{\rho}(C_j)] [\hat{\rho}(B_k) \hat{\rho}(Q)])$$

$$= \Phi_F(\hat{\rho}(EAE)).$$

Thus $\Phi_F \circ \hat{\pi} = \Phi_F \circ \hat{\rho}$ on $E \mathcal{A}^{\#\#} E$, and since $\hat{\pi}, \hat{\rho}$, and Φ_F are weak* continuous, we have $\Phi_F \circ \hat{\pi} = \Phi_F \circ \hat{\rho}$ on $\left(E \mathcal{A}^{\#\#} E\right)^{-\text{weak}*} = \mathfrak{R}(E) \mathcal{A}^{\#\#} \mathfrak{R}(E)$.

Finally, since $\left[\mathcal{A}_n \cap \pi^{-1}(\mathcal{J}_0)\right]^{-\text{weak}*}$ is a weak* closed $*$-algebra, and an ideal for $\mathcal{A}_n^{\#\#}$, we see that

$$E = \Delta_n(Q) = \sum_{k=1}^{m} B_k Q C_k \in \left[\mathcal{A}_n \cap \pi^{-1}(\mathcal{J}_0)\right]^{-\text{weak}*},$$

so $P=\mathfrak{R}(E)\in\left[\mathcal{A}_n\cap\pi^{-1}(\mathcal{J}_0)\right]^{-\text{weak}^*}$. Thus $P=\mathfrak{R}(E)\in\mathcal{Z}\left(\mathcal{A}_n^{\#\#}\right)$ is a good projection and $Q\leqslant P$. This proves Claim 1. $\square$

CLAIM 2. If $Q_1,Q_2\in\mathcal{A}_n^{\#\#}$ are good projections, then there is a good projection $Q\in\mathcal{Z}\left(\mathcal{A}_n^{\#\#}\right)$ such that $Q_1,Q_2\leqslant Q$.

Proof of Claim 2. By Claim 1 we can choose good projections $P_1,P_2\in\mathcal{Z}\left(\mathcal{A}_n^{\#\#}\right)$ such that $Q_1\leqslant P_1$ and $Q_2\leqslant P_2$. Since P_1 and P_2 commute and $P_1(1-P_2)\leqslant P_1$, $P_1P_2\leqslant P_1$ and $(1-P_1)P_2\leqslant P_2$, we see that $\{P_1(1-P_2),P_1P_2,(1-P_1)P_2\}$ is an orthogonal family of good projections. Thus, by Case 0,

$$Q=P_1\vee P_2=P_1(1-P_2)+P_1P_2+(1-P_1)P_2$$

is a good projection in $\mathcal{Z}\left(\mathcal{A}^{\#\#}\right)$. Thus Claim 2 is proved. $\square$

CLAIM 3. If $0\leqslant x\in\mathcal{A}_n\cap\pi^{-1}(\mathcal{J}_0)$, then $\mathfrak{R}(\Delta_n(x))\in\mathcal{Z}\left(\mathcal{A}_n^{\#\#}\right)$ is good.

Proof of Claim 3. We know that $\hat{\pi}(\mathfrak{R}(x))$ and $\hat{\rho}(\mathfrak{R}(x))$ are Murray von Neumann equivalent and $\mathcal{M}$-rank$(\pi(x))$ and $\mathcal{M}$-rank$(\rho(x))$ are equal. Since $\pi(x)\in\mathcal{J}_0$, we know $\hat{\pi}(\mathfrak{R}(x)),\hat{\rho}(\mathfrak{R}(x))\in\mathcal{J}_0$. Arguing as in the proof of Claim 1, we see that $F=\hat{\pi}(\mathfrak{R}(x))\vee\hat{\rho}(\mathfrak{R}(x))\in\mathcal{J}_0$ and that

$$\hat{\pi},\hat{\rho}:[x\mathcal{A}x]^{-\|\|}\to F\mathcal{M}F$$

satisfy $\Phi_{F\mathcal{M}F}\circ\hat{\pi}=\Phi_{F\mathcal{M}F}\circ\hat{\rho}$. Thus $\Phi_{F\mathcal{M}F}\circ\hat{\pi}=\Phi_{F\mathcal{M}F}\circ\hat{\rho}$ on $[x\mathcal{A}x]^{-\text{weak}^*}=\mathfrak{R}(x)\mathcal{A}^{\#\#}\mathfrak{R}(x)$. Thus $\mathfrak{R}(x)$ is a good projection. This proves Claim 3. $\square$

We can choose a countable dense set $\{b_1,b_2,\ldots\}$ of $\cup_{n=1}^{\infty}\left(\mathcal{A}_n\cap\pi^{-1}(\mathcal{J}_0)\right)$ whose closure is $\pi^{-1}(\mathcal{J})$.

We now want to define a sequence $0=P_0\leqslant P_1\leqslant P_2\leqslant\cdots$ of good projections such that

1. $P_n\in\mathcal{Z}\left(\mathcal{A}_n^{\#\#}\right)$ for all $n\in\mathbb{N}$,
2. If $1\leqslant k\leqslant n$ and $b_k\in\mathcal{A}_n$, then $\mathfrak{R}(b_k)\leqslant P_n$, i.e.,

$$b_k=P_nb_k$$

Define $P_0=0$. Suppose $n\in\mathbb{N}$ and P_k has been defined for $0\leqslant k\leqslant n$. We let $x_n=\sum_{k\leqslant n+1,b_k\in\mathcal{A}_{n+1}}b_kb_k^*\in\mathcal{A}_{n+1}\cap\pi^{-1}(\mathcal{J}_0)$. Thus, by Claim 3, P_n and $\mathfrak{R}(\Delta_{n+1}(x_n))$ are good projections in $\mathcal{A}_n^{\#\#}$, and they commute since $\mathfrak{R}(\Delta_{n+1}(x_n))\in\mathcal{Z}\left(\mathcal{A}_{n+1}^{\#\#}\right)$. By Claim 2, there is a good projection $P_{n+1}\in\mathcal{Z}\left(\mathcal{A}_{n+1}^{\#\#}\right)$ such that $P_n\leqslant P_{n+1}$ and $\mathfrak{R}(\Delta_{n+1}(x_n))\leqslant P_{n+1}$. Clearly, if $1\leqslant k\leqslant n$ and $b_k\in\mathcal{A}_n$, we have $\mathfrak{R}(b_k)=\mathfrak{R}\left(b_kb_k^*\right)\leqslant\mathfrak{R}(x_n)\leqslant P_{n+1}$.

Since P_n is a good projection, $P_n\in\left[\mathcal{A}_n\cap\pi^{-1}(\mathcal{J}_0)\right]^{-\text{weak}^*}$. Thus

$$P_n\leqslant\sup\left\{\mathfrak{R}(x):x\in\mathcal{A}_n\cap\pi^{-1}(\mathcal{J}_0)\right\}\in\mathcal{A}_n^{\#\#}.$$

Thus $\hat{\pi}(P_n) \leqslant P_{\pi,\mathcal{J}}$ (the projection onto $H_{\pi,\mathcal{J}}$) and $\hat{\rho}(P_n) \leqslant P_{\rho,\mathcal{J}}$ (the projection onto $H_{\rho,\mathcal{J}}$). Let $P_e = \lim_{n\to\infty} P_n$ (weak*). Thus $\hat{\pi}(P_e) \leqslant P_{\pi,\mathcal{J}}$ and $\hat{\rho}(P_e) \leqslant P_{\rho,\mathcal{J}}$. On the other hand, for every $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \|b_k - P_n b_k\| = 0.$$

This implies

$$P_e b = b \text{ for every } b \in \left[\pi^{-1}(\mathcal{J})\right]^{-\|\|}.$$

Thus $\hat{\pi}(P_e) = P_{\pi,\mathcal{J}}$ and $\hat{\rho}(P_e) = P_{\rho,\mathcal{J}}$. Thus $P_{\pi,\mathcal{J}}$ and $P_{\rho,\mathcal{J}}$ are Murray von Neumann equivalent.

Since $P_n \in \mathcal{A}_n'$ for each $n \in \mathbb{N}$, we have of every $A \in \cup_{k=1}^{\infty} \mathcal{A}_k$,

$$\lim_{n\to\infty} \|AP_n - P_n A\| = 0.$$

Hence,

$$\lim_{n\to\infty} \|AP_n - P_n A\| = 0$$

holds for every $A \in \mathcal{A}$.

Choose a dense subset $\{A_1, A_2, \ldots\}$ of $\mathcal{A}$. Suppose and $m \in \mathbb{N}$. It follows that we can choose a subsequence $\left\{P_{n_k}\right\}$ of $\{P_n\}$ such that, for all $1 \leqslant n < \infty$,

$$\sum_{k=1}^{\infty} \left\|A_n P_{n_k} - P_{n_k} A_n\right\| < \infty,$$

and, for $1 \leqslant n \leqslant m$,

$$\sum_{k=1}^{\infty} \left\|A_n P_{n_k} - P_{n_k} A_n\right\| < \frac{1}{8m}.$$

Define $e_k = P_{n_k} - P_{n_{k-1}}$ (with $P_{n_0} = 0$) and define $\varphi : \mathcal{A} \to \sum_{1\leqslant k<\infty}^{\oplus} e_k \mathcal{A} e_k$ by

$$\varphi(T) = \sum_{k=1}^{\infty} e_k T e_k.$$

It follows from [10, page 903] that the above conditions on $\left\|A_n P_{n_k} - P_{n_k} A_n\right\|$ that, for all $k \in \mathbb{N}$,

$$A_k - \varphi(A_k) \in \hat{\pi}^{-1}(\mathcal{J}) \cap \hat{\rho}^{-1}(\mathcal{J})$$

and

$$\|P_e A_n - \varphi(A_n)\| < \frac{1}{4m}.$$

for $1 \leqslant n \leqslant m$.

Suppose $k \in \mathbb{N}$. For each $n \geqslant n_k$, $e_k \mathcal{A}_n e_k \subset \mathcal{A}_n^{\#\#}$, which is homogeneous. Hence $C^*(e_k \mathcal{A}_n e_k)$ is subhomogeneous. Thus $C^*(e_k \mathcal{A} e_k)$ is ASH. If we let $E_k = \hat{\pi}(e_k) \vee \hat{\rho}(e_k)$ for each $k \in \mathbb{N}$, we have E_k is a finite projection, $E_k \mathcal{M} E_k$ is a finite von Neumann algebra,

$$\hat{\pi}, \hat{\rho} : C^*(e_k \mathcal{A} e_k) \to E_k \mathcal{M} E_k,$$

and, if Φ_{E_k} is the center-valued trace on $E_k\mathcal{M}E_k$, then

$$\Phi_{E_k}\circ\left(\hat{\pi}|_{C^*(e_k\mathcal{A}e_k)}\right)=\Phi_{E_k}\circ\left(\hat{\rho}|_{C^*(e_k\mathcal{A}e_k)}\right),$$

and $C^*(e_k\mathcal{A}e_k)$ is ASH, it follows from Theorem 4 that

$$\hat{\pi}|_{C^*(e_k\mathcal{A}e_k)}\sim_a\hat{\rho}|_{C^*(e_k\mathcal{A}e_k)}\ \ (E_k\mathcal{M}E_k).$$

Since $\hat{\pi}(e_k)$ and $\hat{\rho}(e_k)$ are projections, then by [16, Proposition 5.2.6], any unitary that conjugates $\hat{\pi}(e_k)$ to a projection that is really close to $\hat{\rho}(e_k)$ is close to a unitary that conjugates $\hat{\pi}(e_k)$ exactly to $\hat{\rho}(e_k)$. We can therefore, for each $k\in\mathbb{N}$, choose a unitary $U_k\in E_k\mathcal{M}E_k$ such that

$$\|U_k\hat{\pi}(e_ka_ne_k)U_k^*-\hat{\rho}(e_ka_ne_k)\|<\frac{1}{4km}$$

when $1\leqslant n\leqslant k+m<\infty$, and such that

$$U_k\hat{\pi}(e_k)U_k^*=\rho(e_k).$$

For each $k\in\mathbb{N}$, let $V_k=U_k\hat{\pi}(e_k)$. Then V_k is a partial isometry whose initial projection is $\hat{\pi}(e_k)=V_k^*V_k$ and final projection is $\hat{\rho}(e_k)=V_kV_k^*$. Also

$$\|V_k\hat{\pi}(e_k)\pi(a_n)\hat{\pi}(e_k)V_k^*-\hat{\rho}(e_k)\rho(a_n)\hat{\rho}(e_k)\|<\frac{1}{4km}$$

for $1\leqslant n\leqslant k+m<\infty$. Then $W_m=\sum_{k=1}^{\infty}V_k$ is a partial isometry in $\mathcal{M}$ with initial projection $\hat{\pi}(P_e)=P_{\pi,\mathcal{J}}$ and final projection $\hat{\rho}(P_e)=P_{\rho,\mathcal{J}}$. Moreover,

$$W_m\hat{\pi}(\varphi(a_n))W_m^*=\sum_{1\leqslant k<\infty}^{\oplus}V_k\hat{\pi}(e_ka_ne_k)V_k^*,$$

and

$$\hat{\rho}(\varphi(a_n))=\sum_{1\leqslant k<\infty}^{\oplus}\hat{\rho}(e_ka_ne_k).$$

Since $V_k\hat{\pi}(e_ka_ne_k)V_k^*$, $\hat{\rho}(e_ka_ne_k)\in\mathcal{J}$ for each $n,k\in\mathbb{N}$ and since

$$\lim_{k\to\infty}\|V_k\hat{\pi}(e_ka_ne_k)V_k^*-\hat{\rho}(e_ka_ne_k)\|=0,$$

we see that

$$W_m\hat{\pi}(\varphi(a_n))W_m^*-\hat{\rho}(\varphi(a_n))\in\mathcal{J}$$

for every $n\in\mathbb{N}$. Also,

$$\|W_m\hat{\pi}(\varphi(a_n))W_m^*-\hat{\rho}(\varphi(a_n))\|<\frac{1}{4m}$$

for $1\leqslant n\leqslant m$.

Also
$$\hat{\pi}(\varphi(a_n)) - \pi(a_n) = \hat{\pi}(\varphi(a_n) - a_n) \in \mathcal{J}$$
and
$$\hat{\pi}(\varphi(a_n)) - \rho(a_n) = \hat{\rho}(\varphi(a_n) - a_n) \in \mathcal{J}$$
for every $n \in \mathbb{N}$ and
$$\|\hat{\pi}(\varphi(a_n)) - \pi(a_n)\| < \frac{1}{4m} \text{ and } \|\hat{\rho}(\varphi(a_n)) - \rho(a_n)\| < \frac{1}{4m}$$
for $1 \leqslant n \leqslant m$.

For each $n \in \mathbb{N}$,
$$\begin{gathered} W_m \pi(a_n) W_m^* - \rho(a_n) \\ = [W_m(\pi(a_n) - \hat{\pi}(\varphi(a_n))) W_m^*] + [W_m \hat{\pi}(\varphi(a_n)) W_m^* - \hat{\rho}(\varphi(a_n))] \\ + \hat{\rho}(\varphi(a_n)) - \rho(a_n). \end{gathered}$$
Thus, for every $n \in \mathbb{N}$,
$$W_m \pi(a_n) W_m^* - \rho(a_n) \in \mathcal{J}.$$
Also, for $1 \leqslant n \leqslant m$,
$$\|W_m \pi(a_n) W_m^* - \rho(a_n)\| < \frac{1}{m}.$$
It follows, for every $a \in \mathcal{A}$, that
$$W_m \hat{\pi}(\varphi(a)) W_m^* - \hat{\rho}(\varphi(a)) \in \mathcal{J}$$
and
$$\lim_{m \to \infty} \|W_m \pi(a) W_m^* - \rho(a)\| = 0. \quad \square$$

REMARK 2. In two cases, namely, when $H_{\pi,\mathcal{J}} = H_{\rho,\mathcal{J}} = H$, or when $\pi(\cdot)|_{H_{\pi,\mathcal{J}}^{\perp}}$ and $\rho(\cdot)|_{H_{\rho,\mathcal{J}}^{\perp}}$ are unitarily equivalent, the conclusion in Theorem 7 becomes
$$\pi \sim_a \rho \ (\mathcal{J}).$$

When $\mathcal{A}$ is a separable ASH C*-algebra and $\mathcal{M}$ is a sigma-finite II_∞ factor von Neumann algebra, we can use Theorems 7 and 3 to have both parts of Voiculescu's theorem, including an extension of results in [4]. If σ is a representation of a C*-algebra, we let $\sigma^{(\infty)}$ denote $\sigma \oplus \sigma \oplus \cdots$.

COROLLARY 1. *Suppose $\mathcal{A}$ is a separable ASH C*-algebra, $\mathcal{M}$ is a sigma-finite type II_∞ factor von Neumann algebra on a Hilbert space H. Suppose $\pi, \rho : \mathcal{A} \to \mathcal{M}$ are unital $*$-homomorphisms such that, for every $a \in \mathcal{A}$*
$$\mathcal{M}\text{-}rank(\pi(a)) = \mathcal{M}\text{-}rank(\rho(a)).$$
Then $\pi \sim_a \rho \ (\mathcal{K}_{\mathcal{M}})$.

Proof. We can write $\pi = \pi_{\mathcal{K}_{\mathcal{M}}} \oplus \pi_1$ and $\rho = \rho_{\mathcal{K}_{\mathcal{M}}} \oplus \rho_1$. It follows from Theorem 3 that

$$\pi \sim_a \pi_{\mathcal{K}_{\mathcal{M}}} \oplus \pi_1^{(\infty)} \oplus \rho_1^{(\infty)} \ (\mathcal{K}_{\mathcal{M}}) \text{ and } \rho \sim_a \rho_{\mathcal{K}_{\mathcal{M}}} \oplus \pi_1^{(\infty)} \oplus \rho_1^{(\infty)} \ (\mathcal{K}_{\mathcal{M}}).$$

It follows from Theorem 7 that

$$\pi_{\mathcal{K}_{\mathcal{M}}} \oplus \pi_1^{(\infty)} \oplus \rho_1^{(\infty)} \sim_a \rho_{\mathcal{K}_{\mathcal{M}}} \oplus \pi_1^{(\infty)} \oplus \rho_1^{(\infty)} \ (\mathcal{K}_{\mathcal{M}}).$$

Thus $\pi \sim_a \rho \ (\mathcal{K}_{\mathcal{M}})$. $\square$

We have now arrived at our main result concerning semifinite von Neumann algebras.

THEOREM 8. *Suppose $\mathcal{M} \subset B(H)$ is a semifinite von Neumann algebra, H is separable, and $\mathcal{A}$ is a separable unital ASH C*-algebra. Also suppose $\pi, \rho : \mathcal{A} \to \mathcal{M}$ are unital $*$-homomorphisms such that, for every $a \in \mathcal{A}$*

$$\mathcal{M}\text{-}rank(\pi(a)) = \mathcal{M}\text{-}rank(\rho(a)) .$$

Then $\pi \sim_a \rho \ (\mathcal{K}_{\mathcal{M}})$.

Proof. We can write $\mathcal{M} = \mathcal{F} \oplus \mathcal{N}$ where $\mathcal{F}$ is a finite von Neumann algebra and $\mathcal{N}$ has no finite direct summands, and $\mathcal{N}$ is a type II_∞ von Neumann algebra. Correspondingly, we can write $\pi = \pi_{\mathcal{F}} \oplus \pi_{\mathcal{N}}$ and $\rho = \rho_{\mathcal{F}} \oplus \rho_{\mathcal{N}}$. It is clear that $(\mathcal{F}\text{-rank}) \circ \pi_{\mathcal{F}} = (\mathcal{F}\text{-rank}) \circ \rho_{\mathcal{F}}$ and $(\mathcal{N}\text{-rank}) \circ \pi_{\mathcal{N}} = (\mathcal{N}\text{-rank}) \circ \rho_{\mathcal{N}}$. Since $\mathcal{F} \oplus 0 \subset \mathcal{K}_{\mathcal{M}}$ and $\pi_{\mathcal{F}} \sim_a \rho_{\mathcal{F}}$, by Theorem 5, there is a sequence $\{W_n\}$ of unitary operators in $\mathcal{F}$ such that, for every $a \in \mathcal{A}$,

$$\|W_n \pi(a) W_n - \rho(a)\| \to 0.$$

Clearly, for every $a \in \mathcal{A}$ and every $n \in \mathbb{N}$,

$$W_n \pi(a) W_n - \rho(a) \in \mathcal{F} \oplus 0 \subset \mathcal{K}_{\mathcal{M}}.$$

Hence we can assume that $\mathcal{M} = \mathcal{N}$ and $\pi = \pi_{\mathcal{N}}$. From the central decomposition for $\mathcal{M}$ there is a complete probability measure space (Ω, Σ, μ) so that we can write

$$H = \int_{\Omega}^{\oplus} \ell^2 d\mu(\omega)$$

and

$$\mathcal{M} = \int_{\Omega}^{\oplus} \mathcal{M}_\omega d\mu(\omega)$$

where each $\mathcal{M}_\omega$ is either a type I_∞ factor or a type II_∞ factor. Also there are families $\{\varphi_1, \varphi_2, \ldots\}$ and $\{\psi_1, \psi_2, \ldots\}$ of *SOT-measurable functions from Ω into the closed unit ball $\mathcal{B}$ of $B(\ell^2)$ such that, for every $\omega \in \Omega$,

$$\{\varphi_1(\omega), \varphi_2(\omega), \ldots\}^{-SOT} = \text{ball}(\mathcal{M}_\omega), \text{ and}$$

$$\{\psi_1(\omega), \psi_2(\omega), \ldots\}^{-SOT} = \operatorname{ball}\left(\mathcal{M}'_\omega\right).$$

Let $\mathcal{C}$ be the set of trace class operator $K \in B\left(\ell^2\right)$ such that $K \geqslant 0$ and $Trace(K) = 1$. With the trace norm $\|\|_1$, $\mathcal{C}$ is a complete separable metric space. Let $\mathcal{C}^{\&} = \prod_{(n,j,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}} \mathcal{C}$ with the product topology. Let $\mathcal{B}^{\&} = \prod_{n\in\mathbb{N}} \mathcal{B}$ with the product $*$-SOT topology, let $\mathcal{P}$ be the set of projections in $B\left(\ell^2\right)$ equipped with the $*$-SOT and let $\mathcal{P}^{\&} = \prod_{(n,j,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}} \mathcal{P}$ with the product topology. Let $\mathcal{U}$ be the set of unitary operators in $B\left(\ell^2\right)$ with the $*$-SOT and let $\mathcal{U}^{\&} = \prod_{n\in\mathbb{N}} \mathcal{U}$ with the product topology.

We now let X be the set of all (U,A,B,P,K,C,D) in $\mathcal{U}\times\mathcal{B}^{\&}\times\mathcal{P}^{\&}\times\mathcal{C}^{\&}\times\mathcal{B}^{\&}\times\mathcal{B}^{\&}$, with $U = \{U_n\}$, $A = \{A_n\}$, $P = \left\{P_{n,j,k}\right\}$, $K = \left\{K_{n,j,k}\right\}$, $C = \{C_n\}$, $D = \{D_n\}$, such that

1. $\|U_n^* A_k U_n - B_k\| \leqslant 1/n$ for $1 \leqslant k \leqslant n < \infty$
2. $\left\|(U_n^* A_k U_n - B_k)\left(1 - P_{n,j,k}\right)\right\| \leqslant 1/j$ for $(n,j,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$,
3. $K_{n,j,k} = P_{n,j,k} K_{n,j,k} P_{n,j,k}$ for $(n,j,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$,
4. $U_n D_j = D_j U_n$ for $j,n \in \mathbb{N}$
5. $Tr\left(K_{n,j,k} C_s P_{n,j,k} C_t P_j\right) = Tr\left(K_{n,j,k} C_t P_{n,j,k} C_s P_{n,j,k}\right)$ for $n,j,k,s,t \in \mathbb{N}$.

It is not hard to show that X is closed in $\mathcal{U}\times\mathcal{B}^{\&}\times\mathcal{P}^{\&}\times\mathcal{C}^{\&}\times\mathcal{B}^{\&}\times\mathcal{B}^{\&}$. Thus X is a complete separable metric space. Define

$$\Phi : X \to B^{\&} \times B^{\&} \times B^{\&} \times B^{\&}$$

by

$$\Phi((U,A,B,P,K,C,D)) = (A,B,C,D).$$

Then Φ is continuous and it follows from [1, Theorem 3.4.3] that $\Phi(X)$ is an absolutely measurable set and there is an absolutely measurable function $\gamma : \Phi(X) \to X$ such that $\Phi\circ\gamma = id_{\Phi(X)}$.

We can write $\pi = \int_\Omega^\oplus \pi_\omega d\mu(\omega)$ and $\rho = \int_\Omega^\oplus \rho_\omega d\mu(\omega)$ so that, for almost every $\omega\in\Omega$, $\pi_\omega, \rho_\omega : \mathcal{A} \to \mathcal{M}_\omega$ and, for every $a \in \mathcal{A}$,

$$\pi(a) = \int_\Omega^\oplus \pi_\omega(a)\, d\mu(\omega) \text{ and } \rho(a) = \int_\Omega^\oplus \rho_\omega(a)\, d\mu(\omega).$$

We know from [4, Theorem 4 (3)], that, for almost every $\omega\in\Omega$,

$$\mathcal{M}_\omega\text{-rank}(\pi_\omega(a)) = \mathcal{M}_\omega\text{-rank}(\rho_\omega(a)).$$

By throwing away a subset of Ω of measure 0, we can assume that all of the preceding statements that were true for almost every ω are now true for *every* $\omega\in\Omega$.

Let $\{a_1, a_2, \ldots\}$ be norm dense in the closed unit ball if $\mathcal{A}$. We now define a measurable map $\Gamma : \Omega \to B^{\&} \times B^{\&} \times B^{\&} \times B^{\&}$ by

$$\Gamma(\omega) = \left(\{\pi_\omega(a_n)\}_{n\in\mathbb{N}}, \{\rho_\omega(a_n)\}_{n\in\mathbb{N}}, \{\varphi_n(\omega)\}_{n\in\mathbb{N}}, \{\psi_\omega(\omega)\}_{n\in\mathbb{N}}\right).$$

Suppose $\omega \in \Omega$. Since $\mathcal{M}_\omega$ is a semifinite factor, it follows from Corollary 1 that $\pi_\omega \sim \rho_\omega \ (\mathcal{K}_{\mathcal{M}_\omega})$. Thus there is a sequence $\{W_n\}$ of unitary operators in $\mathcal{M}_\omega$ such that

(6) $\|W_n^* \pi_\omega(a_k) W_n - \rho_\omega(a_k)\| \leqslant 1/n$ for $1 \leqslant k \leqslant n < \infty$, and

(7) $W_n^* \pi_\omega(a_k) W_n - \rho_\omega(a_k) \in \mathcal{K}_{\mathcal{M}_\omega}$ for all $n, k \in \mathbb{N}$.

Since each $W_n^* \pi_\omega(a_k) W_n - \rho_\omega(a_k) \in \mathcal{K}_{\mathcal{M}_\omega}$, there are projections $P_{n,j,k} \in \mathcal{K}_{\mathcal{M}_\omega}$ such that, for $n, j, k \in \mathbb{N}$

$$\left\|\left(W_n^* \pi_\omega(a_k) W_n - \rho_\omega(a_k)\right)\left(1 - P_{n,j,k}\right)\right\| < 1/n.$$

Since $P_{n,j,k} \in \mathcal{K}_{\mathcal{M}}$, $P_{n,j,k}$ must be a finite projection, and since $\mathcal{M}_\omega$ is a semifinite factor in $B\left(\ell^2\right)$, $P_{n,j,k}\mathcal{M}_\omega P_{n,j,k}$ is a finite factor. Thus $P_{n,j,k}\mathcal{M}_\omega P_{n,j,k}$ has a faithful normal tracial state $\tau_{n,j,k}$. Thus there is a $K_{n,j,k} \in \mathcal{C}$ such that $P_{n,j,k} K_{n,j,k} P_{n,j,k} = K_{n,j,k}$ and, for every $S \in P_{n,j,k}\mathcal{M}_\omega P_{n,j,k}$,

$$\tau_{n,j,k}(S) = Tr\left(K_{n,j,k}\right)$$

Hence,

$$\left(\{W_n\}, \{\pi_\omega(a_n)\}, \{\rho_\omega(a_n)\}, \left\{P_{n,j,k}\right\}, \{\varphi_n(\omega)\}, \{\psi_n(\omega)\}\right) \in X,$$

and thus

$$\Gamma(\omega) \in \Phi(X).$$

Then

$$(\gamma \circ \Gamma)(\omega) = \left(\{U_n(\omega)\}, \{\pi_\omega(a_n)\}, \{\rho_\omega(a_n)\}, \left\{P_{n,j,k}(\omega)\right\}, \{\varphi_n(\omega)\}, \{\psi_n(\omega)\}\right)$$

is a measurable function from Ω to X. For $n, j, k \in \mathbb{N}$. Let

$$U_n = \int_\Omega^\oplus U_n(\omega)\, d\mu(\omega) \text{ and } P_{n,j,k} = \int_\Omega^\oplus P_{n,j,k}(\omega)\, d\mu(\omega).$$

Then each U_n is unitary in $\mathcal{M}$, and each $P_{n,j,k}$ is a finite projection in $\mathcal{M}$ and

(8) $\|U_n^* \pi(a_k) U_n - \rho(a_k)\| \leqslant 1/n$ for $1 \leqslant k \leqslant n < \infty$, and

(9) $\left\|\left(U_n^* \pi(a_k) U_n - \rho(a_k)\right)\left(1 - P_{n,j,k}\right)\right\| \leqslant 1/j$ for $n, j, k \in \mathbb{N}$.

Since $\{a_1, a_2, \ldots\}$ is dense in the closed unit ball of $\mathcal{A}$, we see that (8) and (9) hold when a_k is replaced with any $a \in \mathcal{A}$ with $\|a\| \leqslant 1$. It follows from (9) that, for every $a \in \mathcal{A}$ with $\|a\| \leqslant 1$ that $U_n^* \pi(a) U_n - \rho(a) \in \mathcal{K}_{\mathcal{M}}$. Therefore,

$$\pi \sim_a \rho \ (\mathcal{K}_{\mathcal{M}}). \quad \square$$

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