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SOME REMARKS ON THE GENERALIZED ORDER AND GENERALIZED TYPE OF ENTIRE FUNCTIONS OF SEVERAL COMPLEX MATRICES

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Abstract. The main aim of this paper is to introduce the definitions of generalized order and generalized type of the entire function of several complex matrix variables in hyperspherical region and then study some of their properties. By considering the concepts of generalized order and generalized type, we will extend some results of Abul-Ez et al. [1].

1. Introduction

The study of the asymptotic mode of increase behavior of entire functions in one and several complex variables is one of the traditional central topics in complex analysis. It deals with various aspects of the behavior of entire functions in one and several complex variables, one of which is the study of their comparative growth. Basic tools to study the comparative growth properties of holomorphic functions are growth indicators such as growth order, the growth type, the maximum term, and the central index. During the past decades, several developments are made in this direction. For examples, we refer to ([8, 13, 15, 25, 23, 24]). Their results turned out to be very helpful in the study of partial differential equations and somewhere else. Generalization to higher dimensions has been given by many other authors (see, e.g., [16, 20, 26]) where they have contributed to the study of the order and type of entire functions of several complex variables. As Clifford analysis offers another possibility of generalizing complex function theory to higher dimensions, many authors introduced a study on the mode of increase of entire monogenic functions (see [3] to [6]). In recent years there has been a significant impulse to the study of the theory of matrix functions. For the most comprehensive applications of the theory of matrix functions, we refer the reader to see [[4] to [6], [9]]. During the last several years, different problems concerning functions in several complex matrix variables have been brought from different aspects and many important results have been gained (see [10, 11, 14]). In this connection, Kishka et al. [10] obtained the order and type of entire functions of two complex matrices in complete Reinhardt domains. Recently, Abul-Ez et al. [1] introduced the notion of

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order and type of entire functions of several complex matrices and established an explicit relation between the growth of the maximum modulus and the Taylor coefficients of entire functions in several complex matrix variables in hyperspherical regions. For details, one may see [1]. The main purpose of this present paper is to introduce the definitions of generalized order and generalized type of the entire function of several complex matrices and then study some of their properties which considerably extend some earlier results of Abul-Ez et al. [1]. To prove our main results we have followed some of the techniques as used by Abul-Ez et al. [1].

Following [14], we give some notations and associated properties in the framework of several complex variables. Assume $\mathbf{m} = (m_1, m_2, \ldots, m_k)$ and $\mathbf{n} = (n_1, n_2, \ldots, n_k)$ belonging to $\mathbb{N}_0^k$ are k-dimensional multi-indices, and

$$\begin{aligned}
[\mathbf{m}] &= m_1 + m_2 + \ldots + m_k, \\
[\mathbf{n}] &= n_1 + n_2 + \ldots + n_k, \\
\mathbf{z}^{\mathbf{m}} &= \left(z_1^{m_1} \cdot z_2^{m_2} \cdot \ldots \cdot z_k^{m_k}\right), \\
\mathbf{t}^{\mathbf{m}} &= \left(t_1^{m_1} \cdot t_2^{m_2} \cdot \ldots \cdot t_k^{m_k}\right), \\
0 &= (0, 0, \ldots, 0).
\end{aligned}$$

where $z_1, z_2, \ldots, z_k \in \mathbb{C}$ and $t_1, t_2, \ldots, t_k$ are nonnegative numbers. For the k-dimensional space, we write $\mathbf{z} = \left(z_1, z_2, \ldots, z_k\right)$ and $\mathbf{t} = \left(t_1, t_2, \ldots, t_k\right)$, and for $r > 0$, we set

$$\begin{aligned}
\mathbf{S}_r &= \{\mathbf{z} \in \mathbb{C}^k : |\mathbf{z}| < r\} \text{ and} \\
\overline{\mathbf{S}}_r &= = \{\mathbf{z} \in \mathbb{C}^k : |\mathbf{z}| \leqslant r\},
\end{aligned} \tag{1.1}$$

where $\mathbf{S}_r$ is an open spherical region of radius r and $\overline{\mathbf{S}}_r$ is its closure. Consider the function $f(\mathbf{z})$, which is regular in $\overline{\mathbf{S}}_r$; then, (see [11, 14]

$$f(\mathbf{z}) = \sum_{[\mathbf{m}]=0}^{+\infty} a_{\mathbf{m}} \mathbf{z}^{\mathbf{m}}. \tag{1.2}$$

The maximum modulus of $f(\mathbf{z})$ is represented by

$$M[f; \overline{\mathbf{S}}_r] = \sup_{\mathbf{z} \in \overline{\mathbf{S}}_r} |f(\mathbf{z})|.$$

Note that (1.1) leads to

$$\{|z_s| \leqslant r t_s : |\mathbf{t}| = 1; s = 1, 2, \ldots, k\} \subset \overline{\mathbf{S}}_r.$$

Nassif (see [14]) introduced the Cauchy's inequality for functions of several complex variables in the following way

$$|a_{\mathbf{m}}| = \sigma_{\mathbf{m}} \frac{M[f; \overline{\mathbf{S}}_r]}{r^{[\mathbf{m}]}}, \tag{1.3}$$

where

$$\sigma_{\mathbf{m}} = \inf_{|\mathbf{t}|=1} \frac{1}{\mathbf{t}^{\mathbf{m}}} = \frac{\{[\mathbf{m}]\}^{\frac{[\mathbf{m}]}{2}}}{\prod_{s=1}^{k} m_s^{\frac{m_s}{2}}} \tag{1.4}$$

and $1 \leqslant \sigma_{\mathbf{m}} \leqslant (k)^{\frac{[\mathbf{m}]}{2}}$ on the assumption that $m_s^{\frac{m_s}{2}} = 1$, whenever $m_s = 0$.

The number $\sigma_{\mathbf{m}}$ in (1.4) is considered to be a generalization to the number $\sigma_{h,k}$ in the two-complex variable case (cf. [17]), where

$$\sigma_{h,k} = \begin{cases} \dfrac{(h+k)^{\frac{(h+k)}{2}}}{h^{\frac{h}{2}} k^{\frac{k}{2}}}, & h,k \geqslant 1 \\ 0, & h \text{ or } k = 0. \end{cases}$$

Now, the radius of convergence of the power series (1.2) is defined in the open sphere $\mathbf{S}_r$ by

$$R_f = \frac{1}{\limsup\limits_{[\mathbf{m}]\to+\infty} \left\{ \frac{|a_{\mathbf{m}}|}{\sigma_{\mathbf{m}}} \right\}^{\frac{1}{[\mathbf{m}]}}}.$$

Therefore, the function $f(\mathbf{z})$ is an entire function if $R_f = +\infty$. The order of the function $f(\mathbf{z})$ is given in [14] in the form

$$\rho = \limsup_{r\to+\infty} \frac{\ln^{[2]} M[f;\overline{\mathbf{S}}_r]}{\ln r} = \limsup_{[\mathbf{m}]\to+\infty} \frac{[\mathbf{m}]\ln[\mathbf{m}]}{\ln\left(\frac{\sigma_{\mathbf{m}}}{|a_{\mathbf{m}}|}\right)},$$

where

$$\ln^{[0]} r = r, \quad \ln^{[2]} r = \ln(\ln r).$$

If $0 < \rho < +\infty$, then using the same way as in the single complex variable case (see [7, 12, 18]), one can easily prove that the type θ of $f(\mathbf{z})$ can be given in the form

$$\theta = \limsup_{r\to+\infty} \frac{\ln M[f;\overline{\mathbf{S}}_r]}{r^{\rho}} = \frac{1}{e\rho} \limsup_{[\mathbf{m}]\to+\infty} [\mathbf{m}] \left\{ \frac{|a_{\mathbf{m}}|}{\sigma_{\mathbf{m}}} \right\}^{\frac{\rho}{[\mathbf{m}]}}.$$

2. Functions of several complex matrix variables

Here, in this section we discuss some preliminaries and notations which will be needed in the sequel (see [21]).

2.1. Preliminaries and notations

Let $\mathcal{M}_N(\mathbb{C})$ be the space of $N \times N$ matrices whose entries are complex numbers. Let $A \in \mathcal{M}_N(\mathbb{C})$, $A = (a_{uv})$, $u,v = 1,2,\ldots,N$. The multiplication law of matrices takes the form

$$\{A^2\}_{uv} = \sum_{s=1}^{N} \{A\}_{us}\{A\}_{sv},$$

and in general,

$$\{A^{\mu}\}_{uv} = \sum_{j_1, j_2, \dots, j_{\mu-1}} \{A\}_{uj_1}\{A\}_{j_1 j_2} \dots \{A\}_{j_{\mu-2} j_{\mu-1}}\{A\}_{j_{\mu-1} v}\ , \mu \in \mathbb{N},$$

where the summation includes all symbols j_v independently, from 1 to N. Following [21], suppose that $A, B \in \mathcal{M}_N(\mathbb{C})$, $A = (a_{uv})$, $B = (b_{uv})$, $u, v = 1, 2, \dots, N$. We shall use the following notation:

$$\{|A|\}_{uv} = |\{A\}_{uv}|,$$

which means that a matrix A whose each of its elements have been taken to be moduli of the elements.

If a matrix B has positive elements which are greater than the elements of the matrix $|A|$, we have $|A| < B$. Alternatively, this inequality is equivalent to the following system of $N \times N$ inequalities:

$$|\{A\}_{uv}| = \{B\}_{uv}, \quad u, v = 1, 2, \dots, N.$$

Moreover referring to [21], the notation $\|d\|$ shall mean that a matrix in which all its elements are equal to the number d and determine its positive integral powers as follows:

$$\left\{\|d\|^2\right\}_{uv} = dd + dd + \dots + dd = Nd^2, \quad u, v = 1, 2, \dots, N.$$

Therefore, $\|d\|^2 = \|Nd\|^2$, and generally for positive integral powers we get that

$$\|d\|^{\mu} = \left\|N^{\mu-1} d^{\mu}\right\|, \quad \mu \in N.$$

2.2. Convergence property of functions of several complex matrix variables

In the light of previous subsection, we discuss convergence property of a power series of several complex matrix variables in hyperspherical regions by the convergence of a power series of several complex variables without any restrictions on the coefficients.

Let $\mathbf{X} = ([x_{s;ij}])$; $s = 1, 2, \dots, k$ be commutative matrices in $s; ij$, and then the function $\mathcal{F}(\mathbf{X})$ of several complex matrices can be written as a power series in the form

$$\begin{aligned} \mathcal{F}(\mathbf{X}) &= \sum_{[\mathbf{n}]=0}^{+\infty} a_{n_1, n_2, \dots, n_k,} X_1^{n_1}, X_2^{n_2}, \dots, X_k^{n_k}, \\ &= \sum_{[\mathbf{n}]=0}^{+\infty} a_{\mathbf{n}} \mathbf{X}^{\mathbf{n}} = \sum_{[\mathbf{n}]=0}^{+\infty} a_{\mathbf{n}} Z. \end{aligned} \tag{2.1}$$

As $Z = [\{Z\}_{ij}]_{1 \leqslant i,j \leqslant N}$ and $Z \in \mathcal{M}_N(\mathbb{C})$, therefore Z may enjoy the same notation given in previous subsection. Hence, we write that

$$
\begin{aligned}
Z &= X_1^{n_1}, X_2^{n_2}, \ldots, X_k^{n_k}, \\
\{Z\}_{ij} &= \sum_{s_1, s_2, \ldots, s_{k-1}} \{X_1^{n_1}\}_{is_1}, \{X_2^{n_2}\}_{s_1 s_2}, \ldots, \{X_k^{n_k}\}_{s_{k-1} j} \\
&= \sum_{s_1, s_2, \ldots, s_{k-1}} \sum_{j_1^{(1)}, j_2^{(1)}, \ldots, j_{n_1-1}^{(1)}} x_{1; i j_1^{(1)}}, x_{1; j_1^{(1)} j_2^{(1)}}, \ldots, x_{1; j_{n_1-1}^{(1)} s_1}, \\
&\qquad \sum_{j_1^{(2)}, j_2^{(2)}, \ldots, j_{n_2-1}^{(2)}} x_{2; s_1 j_1^{(2)}}, x_{2; j_1^{(2)} j_2^{(2)}}, \ldots, x_{2; j_{n_2-1}^{(2)} s_2}, \\
&\qquad \cdots \sum_{j_1^{(k)}, j_2^{(k)}, \ldots, j_{n_k-1}^{(k)}} x_{k; s_{k-1} j_1^{(k)}}, x_{k; j_1^{(k)} j_2^{(k)}}, \ldots, x_{k; j_{n_k-1}^{(k)} j}.
\end{aligned}
$$

Therefore, $\mathcal{F}(\mathbf{X}) = [f_{ij}]_{1 \leqslant i,j \leqslant N}$, where

$$
f_{ij} = \sum a_{\mathbf{n}} \{Z\}_{ij}, \quad i, j = 1, 2, \ldots, N. \tag{2.2}
$$

For this purpose examining the convergence of series (2.2) in a domain which is a subset of the space $\mathcal{M}_N(\mathbb{C})$ determined by the following inequalities:

$$
|X_1| < \|rt_1\|, |X_2| < \|rt_2\|, \ldots, |X_k| < \|rt_k\|, \tag{2.3}
$$

let us suppose that

$$
|f_{ij}| = |\sum a_{\mathbf{n}} \{Z\}_{ij}|, \quad i, j = 1, 2, \ldots, N. \tag{2.4}
$$

The convergence of this series guarantees the convergence of series (2.2), and in this case, series (2.2) will be absolutely convergent.

In order to show the sufficient condition for the absolute convergence of series (2.2), assume that the scalar function $F(\mathbf{z}) = \sum\limits_{[\mathbf{n}]=0} a_{\mathbf{n}} \mathbf{z}^{\mathbf{n}}$ of several complex variables associated with the matrix function in (2.1) is an analytic function in the region $\overline{\mathbf{S}}_{NR}$, where NR is the radius of convergence of this series, N is the common order of our matrices, and R is a positive number. As

$$
\begin{aligned}
F(\mathbf{z}) &= \sum_{[\mathbf{n}]} a_{\mathbf{n}} \mathbf{z}^{\mathbf{n}}, \\
M[\mathbf{F}; \overline{\mathbf{S}}_{NR}] &= \sup_{\mathbf{z} \in \overline{\mathbf{S}}_{NR}} |F(\mathbf{z})|,
\end{aligned} \tag{2.5}
$$

using similar procedure in deriving (1.3), one can easily deduce the following inequality for the coefficients of series (2.5), taking into account the common order of matrices N. Thus,

$$
|a_{\mathbf{n}}| \leqslant \frac{\sigma_{\mathbf{n}} M[\mathbf{F}; \overline{\mathbf{S}}_{NR}]}{(NR)^{[\mathbf{n}]}}, \tag{2.6}
$$

where

$$\sigma_{\mathbf{n}} = \inf_{|\mathbf{t}|=1} \frac{1}{\mathbf{t}^{\mathbf{n}}} = \frac{\{[\mathbf{n}]\}^{\frac{[\mathbf{n}]}{2}}}{\prod_{s=1}^{k} n_s^{\frac{n_s}{2}}}.$$

Therefore we obtain that

$$|X_1|^{n_1} < \left\| N^{n_1-1}(rt_1)^{n_1} \right\|, |X_2|^{n_2} < \left\| N^{n_2-1}(rt_2)^{n_2} \right\|, \ldots, |X_k|^{n_k} < \left\| N^{n_k-1}(rt_k)^{n_k} \right\|. \tag{2.7}$$

Consequently, using (2.6) and (2.7) in (2.4), it follows that

$$\begin{aligned}
|f_{ij}| &= |\sum_{[\mathbf{n}]=0} a_{\mathbf{n}}\{Z\}_{ij}| \leqslant \sum_{\mathbf{n}} |a_{\mathbf{n}}| \sum_{s_1,s_2,\ldots,s_{k-1}} \left|\{X_1^{n_1}\}_{is_1}\right|, \left|\{X_2^{n_2}\}_{s_1s_2}\right|, \ldots, \left|\{X_k^{n_k}\}_{s_{k-1}j}\right| \\
&\leqslant \sum_{[\mathbf{n}]=0} |a_{\mathbf{n}}| N^{n_1-1+n_2-1+\ldots+n_k-1+(k-1)} r^{[\mathbf{n}]} t^{\mathbf{n}} = \frac{M}{N} \sum_{[\mathbf{n}]=0} \left(\frac{r}{R}\right)^{[\mathbf{n}]} \\
&= \frac{M}{N} \sum_{\mu=0}^{+\infty} \left(\frac{r}{R}\right)^{\mu} = \frac{M}{N}\left(1 - \frac{r}{R}\right), \quad R > r,
\end{aligned}$$

that is, the power series in (2.1) will be absolutely convergent. Thus, one may have the following theorem.

THEOREM 1. (see [1]) *If the radius of convergence of the series in* (2.5) *is equal to NR, then series* (2.1) *will be absolutely convergent for all matrices situated in the neighborhood of domain* (2.3).

3. Order and type of functions of several complex matrix variables

Let $\mathcal{F}(\mathbf{X})$ be an entire function of several complex matrix variables of common order N with Taylor expansion:

$$\mathcal{F}(\mathbf{X}) = \sum_{[\mathbf{n}]=0}^{+\infty} a_{\mathbf{n}} \mathbf{X}^{\mathbf{n}}, \tag{3.1}$$

and the maximum modulus

$$M[\mathcal{F}; \overline{\mathbf{S}}_r] = \max_{ij} \max_{|X_1|<\|rt_1\| \ldots |X_k|<\|rt_k\|} |\mathcal{F}(\mathbf{X})|.$$

So, Cauchy's inequality for the matrix function $\mathcal{F}(\mathbf{X})$ can be given in the form

$$|a_{\mathbf{n}}| \leqslant \frac{NM[\mathcal{F}; \overline{\mathbf{S}}_r]}{(rN)^{[\mathbf{n}]}} \sigma_{\mathbf{n}}. \tag{3.2}$$

In this connection, we recall the following two definitions.

DEFINITION 1. [1] Let $\mathcal{F}(\mathbf{X})$ be an entire function of several complex matrices, then the order of growth of the maximum modulus of an entire function of several complex matrices is described by

$$\rho(\mathcal{F}) = \limsup_{r\to+\infty} \frac{\ln^{[2]} M[\mathcal{F};\overline{\mathbf{S}}_r]}{\ln r}.$$

DEFINITION 2. [1] For any entire function of several complex matrices $\mathcal{F}(\mathbf{X})$ of order $\rho(\mathcal{F})$ $(0<\rho(\mathcal{F})<+\infty)$, the growth type $\tau(\mathcal{F})$ is given by

$$\tau(\mathcal{F}) = \limsup_{r\to+\infty} \frac{\ln M[\mathcal{F};\overline{\mathbf{S}}_r]}{r^{\rho(\mathcal{F})}}.$$

Now we state the following two results due to Abul-Ez et al. [1] concerning the entire function of several complex matrices:

THEOREM 2. [1] *A necessary and sufficient condition that an entire function in several complex matrix variables with a Taylor series representation of the form* (3.1) *should be of order* $\rho(\mathcal{F})$ *is that*

$$\rho(\mathcal{F}) = \limsup_{[\mathbf{n}]\to+\infty} \frac{[\mathbf{n}]\ln([\mathbf{n}])}{-\ln|a_{\mathbf{n}}/K_{\mathbf{n}}|} \quad \textit{where} \quad K_{\mathbf{n}} = \frac{\sigma_{\mathbf{n}}}{N^{[\mathbf{n}]}}.$$

THEOREM 3. [1] *If* $\mathcal{F}(\mathbf{X})$ *is an entire function of several complex matrices of finite growth order* $\rho(\mathcal{F})$ $(0<\rho(\mathcal{F})<+\infty)$ *and growth type* $\tau(\mathcal{F})$, *then*

$$\tau(\mathcal{F}) = \frac{N^{\rho(\mathcal{F})}}{e\rho(\mathcal{F})} \limsup_{[\mathbf{n}]\to+\infty} ([\mathbf{n}]) \left\{\frac{|a_{\mathbf{n}}|}{\sigma_{\mathbf{n}}}\right\}^{\frac{\rho(\mathcal{F})}{[\mathbf{n}]}}.$$

The above two theorems are very useful to evaluate the exact value of the growth order and the growth type for some elementary matrix generalizations of the classical exponential, trigonometric and other special functions. For details about the illustrations and examples for basic applications, one may see [1].

4. Main results

First of all, let L be a class of continuous non-negative functions α defined on $(-\infty,+\infty)$ such that $\alpha(x)=\alpha(x_0)\geqslant 0$ for $x\leqslant x_0$ and $\alpha(x)\uparrow+\infty$ as $x_0\leqslant x\to+\infty$. We say that $\alpha\in L_1$, if $\alpha\in L$ and $\alpha((1+o(1))x)=(1+o(1))\alpha(x)$ as $x\to+\infty$. Finally, $\alpha\in L_{si}$, if $\alpha\in L$ and $\alpha(cx)=(1+o(1))\alpha(x)$ as $x\to+\infty$ for each fixed $c\in(0,+\infty)$, i.e., α is slowly increasing function. Clearly $L_{si}\subset L_1$.

Considering this, Sheremeta [19] in 1967, introduced the concept of generalized order of entire functions in complex context taking two functions belonging to L. For details about the generalized order of entire functions, one may see [19]. However, during the past decades, several authors made close investigations on the properties of entire functions related to generalized order in some different directions. For the purpose of further applications, here in this paper we introduce the definitions of the generalized order and the generalized type the entire function of several complex matrices in the following way:

DEFINITION 3. Let $\mathcal{F}(\mathbf{X})$ be an entire function of several complex matrices. Then, the generalized order of growth of the maximum modulus of an entire function of several complex matrices is defined by

$$\rho(\mathcal{F})=\limsup_{r\to+\infty}\frac{\alpha(\ln^{[2]}M[\mathcal{F};\overline{\mathbf{S}}_r])}{\beta(\ln r)}\quad(\alpha\in L,\ \beta\in L).$$

DEFINITION 4. The generalized type $\lambda(\mathcal{F})$ of the entire function of several complex matrices $\mathcal{F}(\mathbf{X})$ with generalized order $\rho(\mathcal{F})\in(0,+\infty)$ is given by

$$\lambda(\mathcal{F})=\limsup_{r\to+\infty}\frac{\exp(\alpha(\ln^{[2]}M[\mathcal{F};\overline{\mathbf{S}}_r]))}{(\exp(\beta(\ln r)))^{\rho(\mathcal{F})}}\quad(\alpha\in L,\ \beta\in L).$$

REMARK 1. If $\alpha(r)=\beta(r)=r$, then Definition 1 and Definition 2 are special cases of Definition 3 and Definition 4 respectively.

Now we add two conditions on α and β: (i) α and β always denote the functions belonging to L_1 and (ii) $\alpha(\ln x)=o(\beta(x))$ as $x\to+\infty$. Henceforth, we assume that α and β always satisfy the above two conditions.

Now we present the main results of this paper. In the sequel, we use the following notations due to Sato [22]:

$$\exp^{[0]}x=x,\quad \exp^{[2]}x=\exp(\exp x).$$

THEOREM 4. *For an entire function in several complex matrix variables with a Taylor series representation of form* (3.1)*, let*

$$\Omega=\limsup_{[\mathbf{n}]\to+\infty}\frac{\alpha(\ln([\mathbf{n}]))}{\beta\left(\frac{-\ln|a_{\mathbf{n}}/K_{\mathbf{n}}|}{[\mathbf{n}]}\right)}\quad \textit{where}\quad K_{\mathbf{n}}=\frac{\sigma_{\mathbf{n}}}{N^{[\mathbf{n}]}}.\tag{4.1}$$

Then

$$\rho(\mathcal{F})=\Omega.$$

Proof. Case I. Let $\rho(\mathcal{F})\geqslant\Omega$. For $\Omega=0$, this inequality is trivial. So, let us assume that $0<\Omega\leqslant+\infty$. In view of (4.1), there exist infinitely many $\mathbf{n}\in\mathbb{N}_0^k$ with

$$\frac{\alpha(\ln([\mathbf{n}]))}{\beta\left(\frac{-\ln|a_{\mathbf{n}}/K_{\mathbf{n}}|}{[\mathbf{n}]}\right)}\geqslant b,\tag{4.2}$$

where b is a real constant to be chosen such that $b = \Omega - \varepsilon > 0$ with $\varepsilon > 0$ if $\Omega < +\infty$. In the case where $\Omega = +\infty$, one can take for b any arbitrary positive real number. Now from (4.2), we obtain that

$$\ln\left(\left|\frac{a_{\mathbf{n}}}{K_{\mathbf{n}}}\right|\right) \geqslant -[\mathbf{n}]\beta^{-1}\left(\frac{\alpha(\ln([\mathbf{n}]))}{b}\right). \tag{4.3}$$

The coefficients of a matrix Taylor series (3.1) satisfy Cauchy's inequality in the form

$$|a_n| \leqslant \min_{r_0<r} N K_{\mathbf{n}} r^{-[\mathbf{n}]} M[\mathcal{F};\overline{\mathbf{S}}_r]. \tag{4.4}$$

So from (4.4) we get that

$$\ln M[\mathcal{F};\overline{\mathbf{S}}_r] \geqslant \ln\left(\left|\frac{a_{\mathbf{n}}}{K_{\mathbf{n}}}\right|\frac{r^{[\mathbf{n}]}}{N}\right) = \ln\left(\left|\frac{a_{\mathbf{n}}}{K_{\mathbf{n}}}\right|\right) + [\mathbf{n}]\ln\left(\frac{r}{N^{\frac{1}{[\mathbf{n}]}}}\right). \tag{4.5}$$

Now from (4.3) and (4.5) we find that

$$\ln M[\mathcal{F};\overline{\mathbf{S}}_r] \geqslant [\mathbf{n}]\ln\left(\frac{r}{N^{\frac{1}{[\mathbf{n}]}}}\right) - [\mathbf{n}]\beta^{-1}\left(\frac{\alpha(\ln([\mathbf{n}]))}{b}\right). \tag{4.6}$$

Now, let $r = \exp\left(\frac{1}{b} + \left(\beta^{-1}\left(\frac{\alpha(\ln([\mathbf{n}]))}{b}\right)\right)\right)$ in (4.6). Therefore, for this arbitrarily large r, we obtain the following inequality:

$$\begin{aligned}\ln M[\mathcal{F};\overline{\mathbf{S}}_r] &\geqslant [\mathbf{n}]\ln\left(\frac{\exp\left(\frac{1}{b} + \left(\beta^{-1}\left(\frac{\alpha(\ln([\mathbf{n}]))}{b}\right)\right)\right)}{N^{\frac{1}{[\mathbf{n}]}}}\right) - [\mathbf{n}]\beta^{-1}\left(\frac{\alpha(\ln([\mathbf{n}]))}{b}\right)\\ &= [\mathbf{n}]\left(\frac{1}{b} - \ln N^{\frac{1}{[\mathbf{n}]}}\right) = \frac{[\mathbf{n}]}{b}\left(1 - \ln N^{\frac{b}{[\mathbf{n}]}}\right),\end{aligned} \tag{4.7}$$

where

$$[\mathbf{n}] = \exp\left(\alpha^{-1}\left(b\beta\left(\log r - \frac{1}{b}\right)\right)\right). \tag{4.8}$$

Since $\alpha, \beta \in L_1$, then in view of (4.7) and (4.8), one can easily derive that

$$\rho(\mathcal{F}) = \limsup_{r\to+\infty}\frac{\alpha(\ln^{[2]} M[\mathcal{F};\overline{\mathbf{S}}_r])}{\beta(\ln r)} \geqslant b = \Omega - \varepsilon. \tag{4.9}$$

As $\varepsilon(>0)$ is arbitrary we obtain from (4.9) that

$$\rho(\mathcal{F}) \geqslant \Omega. \tag{4.10}$$

Case II. Let $\Omega \geqslant \rho(\mathcal{F})$. If $\Omega = +\infty$, then there is nothing to prove. So, let us assume without loss of generality that $0 < \Omega \leqslant +\infty$. Since $\mathcal{F}(\mathbf{X})$ is an entire matrix function, we have that $\lim_{[\mathbf{n}]\to+\infty} |a_{\mathbf{n}}| = 0$. Because of this property and in view of (4.1), one can find that, for all $\varepsilon > 0$, $\kappa \in \mathbb{N}$ such that for all multi-indices $[\mathbf{n}]$ with $[\mathbf{n}] \geqslant \kappa$,

$$0 \leqslant \frac{\alpha(\ln([\mathbf{n}]))}{\beta\left(\frac{-\ln|a_{\mathbf{n}}/K_{\mathbf{n}}|}{[\mathbf{n}]}\right)} \leqslant \Omega + \varepsilon \tag{4.11}$$

For all these multi-indices with $[\mathbf{n}] \geqslant \kappa$, we get from (4.11) that

$$|a_{\mathbf{n}}/K_{\mathbf{n}}| \leqslant \exp\Big(-[\mathbf{n}]\beta^{-1}\Big(\frac{\alpha(\ln([\mathbf{n}]))}{(\Omega+\varepsilon)}\Big)\Big). \tag{4.12}$$

Therefore,

$$M[\mathcal{F};\overline{\mathbf{S}}_r] \leqslant \max_{ij} \max_{\overline{\mathbf{S}}_r} \sum_{[\mathbf{n}]=0} |a_n||X^n| \leqslant \frac{1}{N}\sum_{[\mathbf{n}]=0}^{+\infty} (Nr)^{[\mathbf{n}]} \frac{|a_n|}{\sigma_n}. \tag{4.13}$$

Hence from (4.12) and (4.13) we get that

$$\begin{aligned} M[\mathcal{F};\overline{\mathbf{S}}_r] &\leqslant \frac{1}{N}\left\{ \sum_{[\mathbf{n}]=0}^{\kappa} + \sum_{[\mathbf{n}]=\kappa+1}^{+\infty} \right\} (Nr)^{[\mathbf{n}]} \frac{|a_n|}{\sigma_n} \\ &\leqslant \frac{1}{N}\left\{ \mathcal{L}_1 + \sum_{[\mathbf{n}]=\kappa+1}^{+\infty} (r)^{[\mathbf{n}]} \exp\Big(-[\mathbf{n}]\beta^{-1}\Big(\Big(\frac{\alpha(\ln([\mathbf{n}]))}{(\Omega+\varepsilon)}\Big)\Big)\right\}, \end{aligned} \tag{4.14}$$

where $\mathcal{L}_1$ is a positive real constant. Choose the number $r_0 > 1$ such that

$$\exp(\alpha^{-1}((\Omega+\varepsilon)\beta(\ln(2r_0)))) > \mu \text{ for } r > r_0,\ \mu \in \mathbb{N},$$

and then fix the positive integer ε such that

$$\kappa < \varepsilon \leqslant \exp(\alpha^{-1}((\Omega+\varepsilon)\beta(\ln(2r_0)))) < \varepsilon+1;\ \ r > r_0,$$

we get from (4.14) that

$$\begin{aligned} M[\mathcal{F};\overline{\mathbf{S}}_r] \leqslant \frac{1}{N}\Bigg\{ &\mathcal{L}_1 + \sum_{[\mathbf{n}]=\kappa+1}^{\varepsilon} (r)^{[\mathbf{n}]} \exp\Big(-[\mathbf{n}]\beta^{-1}\Big(\frac{\alpha(\ln([\mathbf{n}]))}{(\Omega+\varepsilon)}\Big)\Big) \\ &+ \sum_{[\mathbf{n}]=\varepsilon+1}^{+\infty} (r)^{[\mathbf{n}]} \exp\Big(-[\mathbf{n}]\beta^{-1}\Big(\frac{\alpha(\ln([\mathbf{n}]))}{(\Omega+\varepsilon)}\Big)\Big)\Bigg\}. \end{aligned} \tag{4.15}$$

Now

$$\begin{aligned} &\sum_{[\mathbf{n}]=\kappa+1}^{\varepsilon} (r)^{[\mathbf{n}]} \exp\Big(-[\mathbf{n}]\beta^{-1}\Big(\frac{\alpha(\ln([\mathbf{n}]))}{(\Omega+\varepsilon)}\Big)\Big) \\ &< (r)^{\varepsilon} \sum_{[\mathbf{n}]=\kappa}^{\varepsilon} \exp\Big(-[\mathbf{n}]\beta^{-1}\Big(\frac{\alpha(\ln(\kappa+1))}{(\Omega+\varepsilon)}\Big)\Big) \\ &< (r)^{\varepsilon} \sum_{[\mathbf{n}]=0}^{\varepsilon} \exp\Big(-[\mathbf{n}]\beta^{-1}\Big(\frac{\alpha(\ln(\kappa+1))}{(\Omega+\varepsilon)}\Big)\Big) \\ &< \mathcal{L}_2 r^{\exp(\alpha^{-1}((\Omega+\varepsilon)\beta(\ln(2r_0))))}. \end{aligned} \tag{4.16}$$

and

$$\begin{aligned}
&\sum_{[\mathbf{n}]=\varepsilon+1}^{+\infty} (r)^{[\mathbf{n}]} \exp\Big(-[\mathbf{n}]\beta^{-1}\Big(\frac{\alpha(\ln([\mathbf{n}]))}{(\Omega+\varepsilon)}\Big)\Big) \\
\leqslant &\sum_{[\mathbf{n}]=\varepsilon+1}^{+\infty} (r)^{[\mathbf{n}]} \exp\Big(-[\mathbf{n}]\beta^{-1}\Big(\frac{\alpha(\ln(\varepsilon+1))}{(\Omega+\varepsilon)}\Big)\Big) \\
< &\sum_{[\mathbf{n}]=0}^{+\infty} \Big(\frac{1}{2}\Big)^{[\mathbf{n}]} = \mathcal{L}_3.
\end{aligned} \tag{4.17}$$

Therefore summarizing (4.15), (4.16) and (4.17) we get that

$$M[\mathcal{F};\overline{\mathbf{S}}_r] \leqslant \mathcal{L}_4 r^{\exp(\alpha^{-1}((\Omega+\varepsilon)\beta(\ln(2r))))}, \tag{4.18}$$

where $\mathcal{L}_2$, $\mathcal{L}_3$, and $\mathcal{L}_4$ are constants. Hence from (4.18) we obtain that

$$\ln^{[2]} M[\mathcal{F};\overline{\mathbf{S}}_r] \leqslant \alpha^{-1}((\Omega+\varepsilon)\beta(\ln(2r))) + \ln^{[2]} r + o(1).$$

Since $\frac{\alpha(\ln r)}{\beta(r)} \to 0$ as $r \to +\infty$ and $\beta \in L_1$, so it follows from above that

$$\begin{aligned}
&\alpha(\ln^{[2]} M[\mathcal{F};\overline{\mathbf{S}}_r]) \leqslant (1+o(1))(\Omega+\varepsilon)\beta(\ln(2r)) \\
i.e.,\ &\frac{\alpha(\ln^{[2]} M[\alpha_s r])}{(1+o(1))\beta(\ln(r))} \leqslant (1+o(1))(\Omega+\varepsilon).
\end{aligned} \tag{4.19}$$

Making r tends to infinity, we get from (4.19) that

$$\rho(\mathcal{F}) = \limsup_{r\to+\infty} \frac{\alpha(\ln^{[2]} M[\alpha_s r])}{\beta(\ln r)} \leqslant \Omega+\varepsilon. \tag{4.20}$$

As $\varepsilon(>0)$ is arbitrary, we get from (4.20) that

$$\rho(\mathcal{F}) \leqslant \Omega. \tag{4.21}$$

Hence the theorem follows from (4.10) and (4.21). $\square$

Now we prove the following lemma which will be needed in the sequel.

LEMMA 1. *Let $\mathcal{F}(\mathbf{X})$ be a function in several complex matrix variables of common order N which have a Taylor series expansion in (3.1). Suppose there are numbers $\eta > 0$ and $\xi > 0$ and an integer $\gamma = \gamma(\eta,\xi) > 0$ such that*

$$(|a_n|/\sigma_n) \leqslant \left(\frac{(e)^{\frac{1}{\eta}}}{N\cdot\left(\exp\left(\beta^{-1}\left(\ln\left(\left(\frac{\exp(\alpha(\ln([\mathbf{n}])))}{\eta\xi}\right)^{\frac{1}{\eta}}\right)\right)\right)\right)}\right)^{[\mathbf{n}]}$$

for all $[\mathbf{n}] > \gamma$*. Then,* $\mathcal{F}(\mathbf{X})$ *is an entire matrix function, and given any* $\varepsilon > 0$*, there is a number* $R = R(\varepsilon) > 0$ *such that*

$$M[\mathcal{F};\overline{\mathbf{S}}_r] < \exp\left(\frac{1}{\eta}\cdot\exp(\alpha^{-1}(\ln(\eta(\xi+\varepsilon)(\exp(\beta(\ln(r))))^{\eta})))\right) \quad \textit{for all} \quad r > R.$$

Proof. As $(|a_n|/\sigma_n) \leqslant \left(\frac{(e)^{\frac{1}{\eta}}}{N\cdot\left(\exp\left(\beta^{-1}\left(\ln\left(\left(\frac{\exp(\alpha(\ln([\mathbf{n}])))}{\eta\xi}\right)^{\frac{1}{\eta}}\right)\right)\right)\right)}\right)^{[\mathbf{n}]}$, for all these multi-indices with $[\mathbf{n}] > \gamma$, it holds that

$$\left(\frac{N^{[\mathbf{n}]}|a_n|}{\sigma_n}\right)^{\frac{1}{[\mathbf{n}]}} \leqslant \frac{(e)^{\frac{1}{\eta}}}{\left(\exp\left(\beta^{-1}\left(\ln\left(\left(\frac{\exp(\alpha(\ln([\mathbf{n}])))}{\eta\xi}\right)^{\frac{1}{\eta}}\right)\right)\right)\right)}.$$

Therefore, $\left(\frac{N^{[\mathbf{n}]}|a_n|}{\sigma_n}\right)^{\frac{1}{[\mathbf{n}]}} \to 0$ as $[\mathbf{n}] \to +\infty$, and $\mathcal{F}(\mathbf{X})$ is an entire matrix function. Moreover,

$$\left(\frac{(Nr)^{[\mathbf{n}]}|a_n|}{\sigma_n}\right)^{\frac{1}{[\mathbf{n}]}} \leqslant \frac{(e)^{\frac{1}{\eta}}}{\left(\exp\left(\beta^{-1}\left(\ln\left(\left(\frac{\exp(\alpha(\ln([\mathbf{n}])))}{\eta\xi}\right)^{\frac{1}{\eta}}\right)\right)\right)\right)} \quad r < \frac{1}{2},$$

if multi-indices with $[\mathbf{n}] > \kappa = \kappa(r) = \exp(\alpha^{-1}(\ln(\eta\xi(\exp(\beta(\ln(2((e)^{\frac{1}{\eta}}r)))))^{\eta})))$. Choosing $R_1 = R_1(\eta,\xi) > 1$, which is so large that $\kappa(r) > \gamma$, and if $r > R_1$, then

$$\left(\frac{(Nr)^{[\mathbf{n}]}|a_n|}{\sigma_n}\right) < \left(\frac{1}{2}\right)^{[\mathbf{n}]},$$

provided multi-indices with $[\mathbf{n}] > \kappa$. We now establish an upper bound for $M[\mathcal{F};\overline{\mathbf{S}}_r]$ in the following way:

$$\begin{aligned} M[\mathcal{F};\overline{\mathbf{S}}_r] &\leqslant \max_{ij}\max_{\overline{\mathbf{S}}_r}\sum_{[\mathbf{n}]=0}|a_n||X^n| \leqslant \frac{1}{N}\sum_{[\mathbf{n}]=0}^{+\infty}\frac{|a_n|}{\sigma_n}(Nr)^{[\mathbf{n}]} \\ &= \frac{1}{N}\sum_{[\mathbf{n}]=0}^{\kappa}\frac{|a_n|}{\sigma_n}(Nr)^{[\mathbf{n}]} + \frac{1}{N}\sum_{[\mathbf{n}]=\kappa+1}^{+\infty}\frac{|a_n|}{\sigma_n}(Nr)^{[\mathbf{n}]} \\ &< \sum_{[\mathbf{n}]=0}^{\kappa}\frac{|a_n|}{\sigma_n}(Nr)^{[\mathbf{n}]} + \sum_{[\mathbf{n}]=\kappa+1}^{+\infty}\frac{1}{2^{[\mathbf{n}]}} \\ &< \sum_{[\mathbf{n}]=0}^{\kappa}\frac{|a_n|}{\sigma_n}(Nr)^{[\mathbf{n}]} + 1, \text{ if } r > R_1. \end{aligned} \tag{4.22}$$

However,

$$\sum_{[\mathbf{n}]=0}^{\kappa}\frac{|a_n|}{\sigma_n}(Nr)^{[\mathbf{n}]} = \sum_{[\mathbf{n}]=0}^{\gamma}\frac{|a_n|}{\sigma_n}(Nr)^{[\mathbf{n}]} + \sum_{[\mathbf{n}]=\gamma+1}^{\kappa}\frac{|a_n|}{\sigma_n}(Nr)^{[\mathbf{n}]}$$
$$< r^{\gamma}\sum_{[\mathbf{n}]=0}^{\gamma}\frac{|a_n|}{\sigma_n}N^{[\mathbf{n}]} + (\kappa-\gamma)\cdot\max_{\gamma+1\leqslant[\mathbf{n}]\leqslant\kappa}\frac{|a_n|}{\sigma_n}(Nr)^{[\mathbf{n}]}. \tag{4.23}$$

Now

$$\max_{\gamma+1\leqslant[\mathbf{n}]\leqslant\kappa}\frac{|a_n|}{\sigma_n}(Nr)^{[\mathbf{n}]} \leqslant \max_{\gamma+1\leqslant[\mathbf{n}]}\frac{|a_n|}{\sigma_n}(Nr)^{[\mathbf{n}]}$$
$$< \max_{\gamma+1\leqslant[\mathbf{n}]}\left(\frac{(e)^{\frac{1}{\eta}}}{N\left(\exp\left(\beta^{-1}\left(\ln\left(\left(\frac{\exp(\alpha(\ln([\mathbf{n}])))}{\eta\xi}\right)^{\frac{1}{\eta}}\right)\right)\right)\right)}\right)^{[\mathbf{n}]}(Nr)^{[\mathbf{n}]}$$
$$\leqslant \max_{1\leqslant[\mathbf{n}]}\left(\frac{(e)^{\frac{1}{\eta}}}{\left(\exp\left(\beta^{-1}\left(\ln\left(\left(\frac{\exp(\alpha(\ln([\mathbf{n}])))}{\eta\xi}\right)^{\frac{1}{\eta}}\right)\right)\right)\right)}\right)^{[\mathbf{n}]} r^{[\mathbf{n}]}$$
$$\leqslant \max_{1\leqslant[\mathbf{n}]}\left(\frac{(e)^{\frac{1}{\eta}}r}{\left(\exp\left(\beta^{-1}\left(\ln\left(\left(\frac{\exp(\alpha(\ln([\mathbf{n}])))}{\eta\xi}\right)^{\frac{1}{\eta}}\right)\right)\right)\right)}\right)^{[\mathbf{n}]}.$$

The maximum is achieved for all these multi-indices with

$$[\mathbf{n}] = \exp(\alpha^{-1}(\ln(\eta\xi(\exp(\beta(\ln(r))))^{\eta})));$$

hence from above we get that

$$\max_{\gamma+1\leqslant[\mathbf{n}]\leqslant\kappa}\frac{|a_n|}{\sigma_n}(Nr)^{[\mathbf{n}]} < \exp\left(\frac{1}{\eta}\cdot\exp(\alpha^{-1}(\ln(\eta\xi(\exp(\beta(\ln(r))))^{\eta})))\right).$$

Therefore, if $r > R_1$, then from (4.22), (4.23) and above we obtain that

$$M[\mathcal{F};\overline{\mathbf{S}}_r] < r^{\gamma}\sum_{[\mathbf{n}]=0}^{\gamma}\frac{|a_n|}{\sigma_n}N^{[\mathbf{n}]} + (\kappa-\gamma)\cdot\exp\left(\frac{1}{\eta}\cdot\exp(\alpha^{-1}(\ln(\eta\xi(\exp(\beta(\ln(r))))^{\eta})))\right)+1$$

i.e.,

$$M[\mathcal{F};\overline{\mathbf{S}}_r] < r^{\gamma}\sum_{[\mathbf{n}]=0}^{\gamma}\frac{|a_n|}{\sigma_n}N^{[\mathbf{n}]} + (\exp(\alpha^{-1}(\ln(\eta\xi(\exp(\beta(\ln(2((e)^{\frac{1}{\eta}}r)))))^{\eta}))) - \gamma)$$
$$\cdot\exp\left(\frac{1}{\eta}\cdot\exp(\alpha^{-1}(\ln(\eta\xi(\exp(\beta(\ln(r))))^{\eta})))\right)+1$$

i.e.,

$$
\begin{aligned}
M[\mathcal{F};\overline{\mathbf{S}}_r] < \exp\Big(\frac{1}{\eta}\cdot\exp(\alpha^{-1}(\ln(\eta\xi(\exp(\beta(\ln(r))))^{\eta})))\Big) \\
\cdot\Big\{\exp(\alpha^{-1}(\ln(\eta\xi(\exp(\beta(\ln(2((e)^{\frac{1}{\eta}}r)))))^{\eta})))-\gamma \\
+\exp\Big(-\frac{1}{\eta}\cdot\exp(\alpha^{-1}(\ln(\eta\xi(\exp(\beta(\ln(r))))^{\eta})))\Big) \\
+r^{\gamma}\exp\Big(-\frac{1}{\eta}\cdot\exp(\alpha^{-1}(\ln(\eta\xi(\exp(\beta(\ln(r))))^{\eta})))\Big)\sum_{[n]=0}^{\gamma}\frac{|a_{\mathbf{n}}|}{\sigma_{\mathbf{n}}}N^{[n]}\Big\}
\end{aligned}
$$

Given any $\varepsilon>0$, there is a number $R=R(\varepsilon)>R_1$ such that the expression in the right side of the above inequality is less than $\exp\Big(\frac{1}{\eta}\cdot\exp(\alpha^{-1}(\ln(\eta\varepsilon(\exp(\beta(\ln(r))))^{\eta})))\Big)$ provided that $r>R$. Hence,

$$
M[\mathcal{F};\overline{\mathbf{S}}_r] < \exp\Big(\frac{1}{\eta}\cdot\exp(\alpha^{-1}(\ln(\eta(\xi+\varepsilon)(\exp(\beta(\ln(r))))^{\eta})))\Big) \quad \text{for all } r>R. \quad \square
$$

THEOREM 5. *If $\mathcal{F}(\mathbf{X})$ is an entire matrix function of finite generalized growth order $\rho(\mathcal{F})$ $(0<\rho(\mathcal{F})<+\infty)$ and growth type $\lambda(\mathcal{F})$, then*

$$
\lambda(\mathcal{F})=\limsup_{[\mathbf{n}]\to+\infty}\frac{\exp(\alpha(\ln([\mathbf{n}])))}{\left(\exp\left(\beta\left(\ln\left(\frac{1}{\frac{N}{e}\left\{\frac{|a_n|}{\sigma_n}\right\}^{\frac{1}{[\mathbf{n}]}}}\right)\right)\right)\right)^{\rho(\mathcal{F})}}. \tag{4.24}
$$

Proof. Suppose $\lambda(\mathcal{F})$ is finite. Then, for given any $\pi>\lambda(\mathcal{F})$, there is a number $R=R(\pi)>0$ such that

$$
M[\mathcal{F};\overline{\mathbf{S}}_r] < \exp\Big(\frac{1}{\rho(\mathcal{F})}\cdot\exp(\alpha^{-1}(\ln(\rho(\mathcal{F})\pi(\exp(\beta(\ln(r))))^{\rho(\mathcal{F})})))\Big) \quad \text{for all } r>R.
$$

According to Cauchy's inequality in (3.2), we obtain for all $r>R$ that

$$
\begin{aligned}
N^{[\mathbf{n}]}\left|\frac{a_n}{\sigma_n}\right| &\leqslant \frac{NM[\mathcal{F};\overline{\mathbf{S}}_r]}{r^{[\mathbf{n}]}} \\
&< \frac{N\exp\Big(\frac{1}{\rho(\mathcal{F})}\cdot\exp(\alpha^{-1}(\ln(\rho(\mathcal{F})\pi(\exp(\beta(\ln(r))))^{\rho(\mathcal{F})})))\Big)}{r^{[\mathbf{n}]}}.
\end{aligned} \tag{4.25}
$$

The minimum value of $\frac{\exp\Big(\frac{1}{\rho(\mathcal{F})}\cdot\exp(\alpha^{-1}(\ln(\rho(\mathcal{F})\pi(\exp(\beta(\ln(r))))^{\rho(\mathcal{F})})))\Big)}{r^{[\mathbf{n}]}}$ occurs for

$$
r=\exp\Big(\beta^{-1}\Big(\ln\Big(\Big(\frac{\exp(\alpha(\ln([\mathbf{n}])))}{\rho(\mathcal{F})\pi}\Big)^{\frac{1}{\rho(\mathcal{F})}}\Big)\Big)\Big);
$$

thus from (4.25), we get that

$$N^{[\mathbf{n}]}\left|\frac{a_n}{\sigma_n}\right| < N\frac{\exp\left(\frac{[\mathbf{n}]}{\rho(\mathcal{F})}\right)}{\left(\exp\left(\beta^{-1}\left(\ln\left(\left(\frac{\exp(\alpha(\ln([\mathbf{n}])))}{\rho(\mathcal{F})\pi}\right)^{\frac{1}{\rho(\mathcal{F})}}\right)\right)\right)\right)^{[\mathbf{n}]}}, \tag{4.26}$$

for all these multi-indices with $[\mathbf{n}] > \gamma$ and

$$r = \exp\left(\beta^{-1}\left(\ln\left(\left(\frac{\exp(\alpha(\ln([\mathbf{n}])))}{\rho(\mathcal{F})\pi}\right)^{\frac{1}{\rho(\mathcal{F})}}\right)\right)\right) > R(\pi).$$

Rewriting (4.26), we have

$$N^{\frac{\rho(\mathcal{F})}{[\mathbf{n}]}} > \frac{N^{\rho(\mathcal{F})}}{e}\cdot\left(\exp\left(\beta^{-1}\left(\ln\left(\left(\frac{\exp(\alpha(\ln([\mathbf{n}])))}{\rho(\mathcal{F})\pi}\right)^{\frac{1}{\rho(\mathcal{F})}}\right)\right)\right)\right)^{\rho(\mathcal{F})}\cdot\left\{\left|\frac{a_n}{\sigma_n}\right|\right\}^{\frac{\rho(\mathcal{F})}{[\mathbf{n}]}}. \tag{4.27}$$

Hence it follows from (4.27) that

$$\pi \geqslant \frac{1}{\rho(\mathcal{F})}\limsup_{[\mathbf{n}]\to+\infty}\frac{\exp(\alpha(\ln([\mathbf{n}])))}{\left(\exp\left(\beta\left(\ln\left(\left(\frac{1}{\frac{N^{\rho(\mathcal{F})}}{e}\left\{\left|\frac{a_n}{\sigma_n}\right|\right\}^{\frac{\rho(\mathcal{F})}{[\mathbf{n}]}}}\right)^{\frac{1}{\rho(\mathcal{F})}}\right)\right)\right)\right)^{\rho(\mathcal{F})}}.$$

Since π is an arbitrary number exceeding $\lambda(\mathcal{F})$, so we get from above that

$$\lambda(\mathcal{F}) \geqslant \frac{1}{\rho(\mathcal{F})}\limsup_{[\mathbf{n}]\to+\infty}\frac{\exp(\alpha(\ln([\mathbf{n}])))}{\left(\exp\left(\beta\left(\ln\left(\left(\frac{1}{\frac{N^{\rho(\mathcal{F})}}{e}\left\{\left|\frac{a_n}{\sigma_n}\right|\right\}^{\frac{\rho(\mathcal{F})}{[\mathbf{n}]}}}\right)^{\frac{1}{\rho(\mathcal{F})}}\right)\right)\right)\right)^{\rho(\mathcal{F})}},$$

where the right-hand side is clearly finite. Now let π_1 be any number exceeding the right-hand side of (4.24). Then, there is a number $\gamma = \gamma(\pi_1) > 0$ such that

$$N^{[\mathbf{n}]}\left|\frac{a_n}{\sigma_n}\right| < \frac{\exp\left(\frac{[\mathbf{n}]}{\rho(\mathcal{F})}\right)}{\left(\exp\left(\beta^{-1}\left(\ln\left(\left(\frac{\exp(\alpha(\ln([\mathbf{n}])))}{\rho(\mathcal{F})\pi}\right)^{\frac{1}{\rho(\mathcal{F})}}\right)\right)\right)\right)^{[\mathbf{n}]}},\quad \text{for all } [\mathbf{n}] > \gamma.$$

Applying Lemma 1 with $\xi = \pi_1$ and $\eta = \rho(\mathcal{F})$, given such that

$$M[\mathcal{F};\overline{\mathbf{S}}_r] < \exp\left(\frac{1}{\rho(\mathcal{F})}\cdot\exp(\alpha^{-1}(\ln(\rho(\mathcal{F})(\pi_1+\varepsilon)(\exp(\beta(\ln(r))))^{\rho(\mathcal{F})})))\right)$$

$$\text{for all } r > R.$$

Therefore, $\lambda(\mathcal{F}) \leqslant \pi_1$ and because of the choice of π_1,

$$\lambda(\mathcal{F}) \leqslant \frac{1}{\rho(\mathcal{F})} \limsup_{[\mathbf{n}]\to+\infty} \frac{\exp(\alpha(\ln([\mathbf{n}])))}{\left(\exp\left(\beta\left(\ln\left(\left(\frac{1}{\frac{N^{\rho(\mathcal{F})}}{e}\{|\frac{a_n}{\sigma_n}|\}^{\frac{\rho(\mathcal{F})}{[\mathbf{n}]}}}\right)^{\frac{1}{\rho(\mathcal{F})}}\right)\right)\right)\right)^{\rho(\mathcal{F})}}.$$

Thus, the result is derived. Also, if the right-hand side of (4.24) is finite so is $\lambda(\mathcal{F})$, and if $\lambda(\mathcal{F})$ is infinite, so is the right-hand side of (4.24). $\square$

5. Concluding remarks

In this paper, we investigate certain properties of generalized order and the generalized type the entire function of several complex matrices variables in hyperspherical region which considerably extend some recent works of Abul-Ez et al. [1]. Accordingly, it is interesting to study about the similar properties of linear substitution for entire function of several complex matrices variables. In fact some results in this direction have also been explored by Abul-Ez et al. [1]. Those outcomes may also be extended by using the concepts of generalized order and the generalized type which are left to the interested readers or the involved authors for future study in this research subject.

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