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A GENERALIZATION OF KLEINECKE–SHIROKOV THEOREM FOR MATRICES

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Dedicated to the memory of the first author.

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Abstract. For given square matrices A and B we denote by $Y = AB - BA$ and by $Z = AY - YA$. It is well known that if A and Y commute, i.e., if $Z = 0$, then Y is a nilpotent matrix. In this note we show that the same is true if $YZ = ZY$. We also generalize this result by using commutators of higher order.

The Kleinecke-Shirokov theorem [4, 6] (see also [1] and the references therein) asserts that if, for given bounded operators A and B on a Banach space, A and its commutator $AB - BA$ commute then $AB - BA$ is a quasinilpotent operator. In the finite dimensional case this result is also known as Jacobson's Lemma [3]. We will remain in finite dimensions and will replace the above condition with a weaker one.

All the matrices in the sequel are assumed to be complex or defined over an algebraically closed field. Let A and B be two square matrices and denote by $\delta_A(B) := AB - BA$ their commutator. Similarly, for a given matrix B of order $r \times s$ and matrices A_1 and A_2 of order $r \times r$ and $s \times s$, respectively, we denote

$$\delta_{A_1A_2}(B) := A_1B - BA_2$$

and, for $k = 2, 3, \ldots,$ we successively define

$$\delta^k_{A_1A_2}(B) := A_1\delta^{k-1}_{A_1A_2}(B) - \delta^{k-1}_{A_1A_2}(B)A_2.$$

Assume that square matrix A has a block diagonal matrix form with diagonal square blocks A_1, $A_2, \ldots,$ A_n (not necessarily of the same size) and matrix B is of the same order as A and with the same block partition,

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & B_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \cdots & B_{nn} \end{bmatrix}. \tag{1}$$

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Then, it is easy to verify that the commutator $\delta_A(B)$ has the following form

$$\delta_A(B) = \begin{bmatrix} \delta_{A_1}(B_{11}) & \delta_{A_1A_2}(B_{12}) & \cdots & \delta_{A_1A_n}(B_{1n}) \\ \delta_{A_2A_1}(B_{21}) & \delta_{A_2}(B_{22}) & \cdots & \delta_{A_2A_n}(B_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{A_nA_1}(B_{n1}) & \delta_{A_nA_2}(B_{n2}) & \cdots & \delta_{A_n}(B_{nn}) \end{bmatrix}. \tag{2}$$

For given square matrices A and B let us denote by $Y := \delta_A(B)$ and by $Z := \delta_A(Y)$. If we assume that A and Y commute, that is if $Z = 0$, then, by Kleinecke-Shirokov theorem, Y is a nilpotent matrix. We can ask ourselves if the same is true in the case when A and Y are *quasi-commutative* in the sense of McCoy [5], that is, if $\delta_A(Z) = 0$ and $\delta_Y(Z) = 0$. In fact, the second condition is sufficient.

THEOREM 1. *If for given square matrices A and B over algebraically closed field it holds*

$$YZ = ZY,$$

where $Y = \delta_A(B)$ and $Z = \delta_A(Y)$, then Y is a nilpotent matrix.

Proof. We may assume that Y is expressed in the Jordan canonical form as

$$Y = \begin{bmatrix} Y_1 & 0 & \cdots & 0 \\ 0 & Y_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Y_p \end{bmatrix}, \tag{3}$$

where we grouped the blocks corresponding to the same eigenvalue. So, for each $k \in \{1,2,\ldots,p\}$ the block Y_k has the form

$$Y_k = \begin{bmatrix} \lambda_k & t_1^{(k)} & 0 & \cdots & 0 \\ 0 & \lambda_k & t_2^{(k)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & t_{r_{k-1}}^{(k)} \\ 0 & 0 & \cdots & \cdots & \lambda_k \end{bmatrix}, \tag{4}$$

where r_k is the algebraic multiplicity of eigenvalue λ_k and numbers $t_1^{(k)}, t_2^{(k)}, \ldots, t_{r_{k-1}}^{(k)}$ equal either 0 or 1. Let us denote by

$$N_k := \begin{bmatrix} 0 & t_1^{(k)} & 0 & \cdots & 0 \\ 0 & 0 & t_2^{(k)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & t_{r_{k-1}}^{(k)} \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}. \tag{5}$$

Assume that matrices A and Z are in the same block partition form as Y. Fix any pair of indices i and j from $\{1,2,\ldots,p\}$ where $i < j$. Clearly, $Z = \delta_A(Y) = -\delta_Y(A)$.

Then, for the block Z_{ij} of Z and for the block A_{ij} of A, taking into the account the form of blocks given in (2), we obtain

$$-Z_{ij} = \delta_{Y_iY_j}(A_{ij}) = (\lambda_i I + N_i)A_{ij} - A_{ij}(\lambda_j I + N_j) = qA_{ij} + N_iA_{ij} - A_{ij}N_j,$$

where $q = \lambda_i - \lambda_j \neq 0$. Since Y_i and Y_j correspond to different eigenvalues and Y and Z commute, it follows that $Z_{ij} = 0$ (see [5, p. 329]). Let us denote simply $m = r_i$ and $n = r_j$. Then, we have the following matrix equation for the $m \times n$ block $X = A_{ij}$,

$$(qI_m + N_i)X - XN_j = 0, \tag{6}$$

where I_m denotes the identity matrix of order m. With the vectorization of this equation and using the Kronecker product (see [2, p. 257]), we obtain

$$S\,\mathrm{vec}(X) = 0$$

where $\mathrm{vec}(X)$ is the matrix of order $nm \times 1$ consisting of columns $c_1, c_2, \ldots, c_n$ of matrix X and S is of the form

$$S = (I_n \otimes (qI_m + N_i)) - N_j^T \otimes I_m.$$

It is easy to see that S consists of n^2 blocks of size $m \times m$, all its diagonal blocks are equal to

$$Q = \begin{bmatrix} q & t_1^{(i)} & 0 & \cdots & 0 \\ 0 & q & t_2^{(i)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & q & t_{m-1}^{(i)} \\ 0 & 0 & 0 & \cdots & q \end{bmatrix},$$

all its subdiagonal blocks are successively equal to: $-t_1^{(j)}I_m$, $-t_2^{(j)}I_m, \ldots, -t_{n-1}^{(j)}I_m$, while all the other blocks of S are zero. Since S is of block lower triangular form, we have $\det(S) = \det(Q)^n = q^{mn} \neq 0$. Hence, S is nonsingular, consequently $\mathrm{vec}(X) = 0$, that is $X = 0$. Thus, $A_{ij} = 0$.

In the same way we obtain the following equation for the block A_{ji} of A,

$$-Z_{ji} = \delta_{Y_jY_i}(A_{ji}) = (\lambda_j I + N_j)A_{ji} - A_{ji}(\lambda_i I + N_i) = -qA_{ji} + N_jA_{ji} - A_{ji}N_i.$$

Further, with the vectorization as above, we obtain for the $n \times m$ block $X = A_{ji}$:

$$T\,\mathrm{vec}(X) = 0,$$

where

$$T = (I_m \otimes (-qI_n + N_j)) - N_i^T \otimes I_n.$$

Now, T consists of m^2 blocks of order $n \times n$, all its diagonal blocks are equal to

$$P = \begin{bmatrix} -q & t_1^{(j)} & 0 & \cdots & 0 \\ 0 & -q & t_2^{(j)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -q & t_{n-1}^{(j)} \\ 0 & 0 & 0 & \cdots & -q \end{bmatrix},$$

all its subdiagonal blocks are successively equal to: $-t_1^{(i)}I_n$, $-t_2^{(i)}I_n$, $\cdots$, $-t_{m-1}^{(i)}I_n$, and all the other blocks equal zero. Matrix T is of lower block triangular form, consequently, $\det(T) = \det(P)^m = (-q)^{mn} \neq 0$, hence $\mathrm{vec}(X) = 0$. Thus, $A_{ji} = 0$. Since i and j were arbitrary indices, matrix A has block diagonal form as in (1) where $n = p$.

Finally, we take into account that $Y = \delta_A(B)$ where B has the block form (1) with $n = p$. Thus, for each $k \in \{1,2,\dots,p\}$ we have $Y_k = \delta_{A_k}(B_{kk})$ and, consequently, $\mathrm{tr}(Y_k) = 0$, hence $\lambda_k = 0$. So, Y is a nilpotent matrix. □

As a consequence we obtain Kleinecke-Shirokov theorem (or Jacobson's lemma) for matrices.

COROLLARY 1. *If matrices A and $Y = \delta_A(B)$ commute then Y is nilpotent.*

Moreover, we have a generalization to the quasi-commutativity condition.

COROLLARY 2. *If matrices A and $Y = \delta_A(B)$ quasi-commute (i.e., if $\delta_A(Y)$ commutes with both A and Y) then Y is nilpotent.*

The assumption of the above theorem was that Y commutes with $\delta_A(Y)$, which is equivalent to the assumption that Y commutes with $\delta_Y(A)$. This can be further generalized in assuming that Y commutes with some higher order commutator $\delta_Y^{k_0}(A)$ instead of $\delta_Y(A)$. Let $\mathbb{N} = \{1,2,\dots\}$ denote the set of natural numbers.

THEOREM 2. *If for given square matrices A, B over algebraically closed field and $Y = \delta_A(B)$ it holds*

$$Y\delta_Y^{k_0}(A) = \delta_Y^{k_0}(A)Y,$$

for some $k_0 \in \mathbb{N}$, then Y is a nilpotent matrix.

Proof. With the same notations as above, where Y is of the form (3), we have for fixed $i, j \in \{1,2,\dots,p\}$, $i < j$, the following successive relations for the block $X = A_{ij}$ of the matrix A:

$$\begin{aligned}
X^{(1)} &:= \delta_{Y_iY_j}(X) = qX + N_iX - XN_j,\\
X^{(2)} &:= \delta_{Y_iY_j}(X^{(1)}) = qX^{(1)} + N_iX^{(1)} - X^{(1)}N_j,\\
&\ \vdots\\
X^{(k_0+1)} &:= \delta_{Y_iY_j}(X^{(k_0)}) = qX^{(k_0)} + N_iX^{(k_0)} - X^{(k_0)}N_j.
\end{aligned}$$

Since $X^{(k_0+1)} = \delta_{Y_iY_j}^{k_0+1}(X) = 0$ by assumption, we have for $X = X^{(k_0)}$ the matrix equation (6) and, as in the proof of the Theorem 1, we obtain $X^{(k_0)} = 0$. Following the above successive relations we obtain in the same way $X^{(k_0-1)} = \cdots = X^{(1)} = 0$ and, finally, $X = 0$, hence $A_{ij} = 0$. In the similar way we can prove that also $A_{ji} = 0$. This means, as above, that A is of the block diagonal form (1). Since $Y = \delta_A(B)$, for each $k \in \{1,2,\dots,p\}$ we again obtain that $Y_k = \delta_{A_k}(B_{kk})$ and, consequently $\mathrm{tr}(Y_k) = 0$ and $\lambda_k = 0$. It follows that Y is nilpotent. □

QUESTION 1. Can Theorems 1 and 2 be generalized for bounded operators on infinite dimensional spaces?

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