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PRESERVERS OF CONDITION SPECTRA AND PSEUDO SPECTRA OF HERMITIAN MATRIX JORDAN PRODUCTS

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Abstract. Let $\mathscr{H}_n$ be the real space of $n\times n$ complex Hermitian matrices. Complete descriptions are given of the maps of $\mathscr{H}_n$ leaving invariant the pseudo spectral radius or the condition spectral radius of Jordan product of matrices. As application, maps on $\mathscr{H}_n$ that preserve the condition spectrum of Jordan product of matrices are classified.

1. Introduction

Throughout, $\mathscr{M}_n$ will be the algebra of all $n\times n$ matrices over the field of complex numbers $\mathbb{C}$ with identity matrix I. For $A\in\mathscr{M}_n$ we write A^* for its adjoint, A^{tr} for its transpose, $\sigma(A)$ for its spectrum, and $\|A\|$ the (spectral) norm of A. For $0<\varepsilon<1$, the ε-condition spectrum of A is defined by

$$\sigma_\varepsilon(A) := \{z\in\mathbb{C} : \|zI-A\|\|(zI-A)^{-1}\|\geqslant \varepsilon^{-1}\}$$

with the convention that $\|zI-A\|\|(zI-A)^{-1}\|=\infty$ when $zI-A$ is not invertible, and is a non empty set, compact, perfect set (no isolated points); see [14]. The ε-condition spectral radius of A is

$$r_\varepsilon(A) := \sup\{|z| : z\in\sigma_\varepsilon(A)\}.$$

For $\varepsilon>0$, the ε-pseudo spectrum of A is given by

$$\Lambda_\varepsilon(A) := \{z\in\mathbb{C} : \|(zI-A)^{-1}\|>\varepsilon^{-1}\}$$

with the convention that $\|(zI-A)^{-1}\|=\infty$ if $z\in\sigma(A)$, and coincides with the set $\cup\{\sigma(A+E) : E\in\mathscr{M}_n, \|E\|<\varepsilon\}$. The ε-pseudo spectral radius of A is

$$\delta_\varepsilon(A) := \sup\{|z| : z\in\sigma_\varepsilon(A)\}.$$

Unlike the spectrum, which is a purely algebraic concept, both the ε-condition spectrum and ε-pseudospectrum depend on the norm and contain the usual spectrum as a subset.

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Recently, general preserver problems with respect to various algebraic operations on $\mathcal{M}_n$ or on operator algebras, attracted a lot of attention of researchers in the fields; see for instance [2, 6, 7, 9, 10, 13, 15, 17] and the references therein. On the subject focused on the structures of nonlinear transformations on $\mathcal{M}_n$ that respect the condition spectra or the pseudo spectra of certain algebraic operations, we mention: [8] where the authors studied mappings on $\mathcal{M}_n$ that preserve the pseudo spectrum of different kind of binary operations on matrices, [11] concerned with the preservers of the pseudo spectra of the matrix Lie products, [5] treated the preservers of the condition spectra of the usual matrix products or the Jordan matrix triple products, and [3, 4] general preserver problems that to do with the corresponding pseudo spectra case are considered.

The aim of this paper is to study maps on $\mathcal{H}_n$, the real space of $n \times n$ complex Hermitian matrices, that preserve the condition spectra or the pseudo spectra of matrix Jordan products. Complex Hermitian matrices are used to describe different geometries on $\mathbb{C}^n$, and in particular, they are used to describe all possible unitary geometries on $\mathbb{C}^n$. More on different geometries can be found in a book by Artin [1]. In the next section, we characterize map Φ on $\mathcal{H}_n$ that preserves the ε-condition spectral radius of Jordan products in a sense that

$$r_\varepsilon(\Phi(A)\Phi(B)+\Phi(B)\Phi(A)) = r_\varepsilon(AB+BA) \quad (A,B \in \mathcal{H}_n).$$

It is shown that such a map Φ has a nice structure. Precisely, Φ has the form $A \mapsto \xi(A)UAU^*$ or $T \mapsto \xi(A)UA^{tr}U^*$ for some unitary matrix $U \in \mathcal{M}_n$ and a (general) functional ξ from $\mathcal{H}_n$ into $\{-1,1\}$. As a consequence, we describe mappings on $\mathcal{H}_n$ that preserve the ε-condition spectrum of Jordan product of matrices. We prove that such transformations are of standard forms up a square root of unity. While the last section is devoted to the preservers of ε-pseudo spectral radius of matrix Jordan products of Hermitian matrices. Particularly, the obtained result in Theorem 3 below leads to pseudo spectral radius version of [10, Theorem 5.1] which describes non linear maps on $\mathcal{H}_n$ that are ε-pseudo spectrum preserving.

2. Preservers of condition spectra

In this section, we give the structure of non linear maps on $\mathcal{H}_n$ that preserve the condition spectral radius or the condition spectrum of Jordan product of Hermitian matrices. First, we fix some notation. The inner product on $\mathbb{C}^n$ will be denoted by $\langle .,. \rangle$. For $x, f \in \mathbb{C}^n$, as usual we denote by $x \otimes f$ the rank at most one matrix in $\mathcal{M}_n$ given by $x \otimes fz = \langle z, f \rangle x$ for every column vector $z \in \mathbb{C}^n$. We know that all at most rank one matrices in $\mathcal{M}_n$ can written into this form.

Before stating the main results of this section, we collect some lemmas needed in what follows. The first one summarizes some properties of the condition spectrum; see [14].

LEMMA 1. *For $0 < \varepsilon < 1$ and $A \in \mathcal{M}_n$, the following statements hold.*

(i) $\sigma_\varepsilon(A) = \sigma(A)$ *if and only if A is a scalar multiple of the identity.*

(ii) $\sigma_\varepsilon(\alpha+\beta A)=\alpha+\beta\sigma_\varepsilon(A)$ *for all* $\alpha,\beta\in\mathbb{C}$.

(iii) $\sigma_\varepsilon(A^{tr})=\sigma_\varepsilon(A)$ *and* $\sigma_\varepsilon(UAU^*)=\sigma_\varepsilon(A)$ *for every unitary matrix* $U\in\mathscr{M}_n$.

In the sequel, for $r\geqslant 0$ and $a\in\mathbb{C}$ we will denote by $\overline{D}(a,r)$ the closed disc of $\mathbb{C}$ centered at a and of radius r. The second lemma, quoted from [5], describe the condition spectra of Hermitian matrices in terms of their usual spectrums.

LEMMA 2. *Let* $0<\varepsilon<1$ *and* $A\in\mathscr{M}_n$ *be a Hermitian matrix. Then*

$$\sigma_\varepsilon(A)=\bigcup_{\alpha,\beta\in\sigma(A)}\overline{D}\left(\frac{\alpha-\beta\varepsilon^2}{1-\varepsilon^2},\frac{\varepsilon|\alpha-\beta|}{1-\varepsilon^2}\right)$$

and

$$r_\varepsilon(A)=\max\left(\frac{|i(A)-\varepsilon s(A)|}{1-\varepsilon},\frac{|s(A)-\varepsilon i(A)|}{1-\varepsilon}\right)$$

where $i(A)$ *and* $s(A)$ *denote the infimum and the supremum of* $\sigma(A)$, *respectively.*

The third one, established in [12], determines the structure of mappings on $\mathscr{H}_n$ that preserve zero-Jordan product of matrices when restricted to the set of rank one matrices.

LEMMA 3. *Let* $n\geqslant 3$ *and let* ϕ *be a map from* $\mathscr{H}_n$ *into itself. If* ϕ *satisfies*

$$\phi(A)\phi(B)+\phi(B)\phi(A)=0\Longleftrightarrow AB+BA=0\quad(A,B\in\mathscr{H}_n),$$

then there is a functional $h:\mathscr{H}_n\to\mathbb{R}\setminus\{0\}$ *and a unitary matrix* $U\in\mathscr{M}_n$ *such that either*

(i) $\phi(A)=h(A)UAU^*$ *for every rank one matrix* $A\in\mathscr{H}_n$, *or*

(ii) $\phi(A)=h(A)UA^{tr}U^*$ *for every rank one matrix* $A\in\mathscr{H}_n$.

The next two lemmas may be of independent interest.

LEMMA 4. *Let* $A,B\in\mathscr{M}_n$ *be Hermitian matrices and* $\lambda>0$. *If*

$$|\langle Ax,x\rangle|+\lambda\,\|Ax\|=|\langle Bx,x\rangle|+\lambda\,\|Bx\| \tag{1}$$

holds for all unit vector $x\in\mathbb{C}^n$, *then* $A=\pm B$.

Proof. By the equality (1),

$$\begin{aligned}(1+\lambda)\,|\langle Ax,x\rangle| &\leqslant |\langle Ax,x\rangle|+\lambda\,\|Ax\|\\ &=|\langle Bx,x\rangle|+\lambda\,\|Bx\|\\ &\leqslant(1+\lambda)\,\|Bx\|\end{aligned}$$

for every unit vector $x \in \mathbb{C}^n$. This together with the fact that A is a Hermitian matrix, we see that $\|A\| \leqslant \|B\|$, thus we must have $\|A\| = \|B\|$.

Now, by the assumption on A, there is a orthonormal basis $f_1, \ldots, f_n$ of eigenvectors of A; thus, $A = \sum_{k=1}^n \mu_k f_k \otimes f_k$ and $Af_k = \mu_k f_k$ for $k = 1, 2, \ldots, n$, where $\mu_1, \ldots, \mu_n$ are the corresponding eigenvalues with $|\mu_1| \geqslant |\mu_2| \geqslant \ldots \geqslant |\mu_n|$ whenever $i < j$. We have

$$|\langle Af_1, f_1 \rangle| = \|Af_1\| = |\mu_1| = \|A\| = \|B\|,$$

and so by taking into account the equality (1), we have

$$\begin{aligned}(1+\lambda)\|B\| &= |\mu_1| + \lambda |\mu_1| \\ &= |\langle Af_1, f_1 \rangle| + \lambda \|Af_1\| \\ &= |\langle Bf_1, f_1 \rangle| + \lambda \|Bf_1\| \\ &\leqslant |\langle Bf_1, f_1 \rangle| + \lambda \|B\|.\end{aligned}$$

So $\|B\| \leqslant |\langle Bf_1, f_1 \rangle|$, and thus we must have $\|B\| = |\langle Bf_1, f_1 \rangle|$; which implies that there exists a scalar η_1 with $|\eta_1| = \|B\|$ such that $Bf_1 = \eta_1 f_1$. Since $|\eta_1| = |\mu_1|$, one gets $Bf_1 = \xi_1 \mu_1 f_1$ where $\xi_1 = \pm 1$. Hence $[f_1]$, the linear span of f_1, is invariant under B and $B = \xi_1 \mu_1 I_1 \oplus B_1$, where I_1 is the identity on $[f_1]$ and B_1 is the restriction of B on $\oplus_{k=2}^n [f_k]$. Similarly, considering B_1 and $A_1 = \sum_{k=2}^n \mu_k f_k \otimes f_k$ on $\oplus_{k=2}^n [f_k]$, one gets that $Bf_2 = \xi_2 \mu_2 f_2$ with $\xi_2 = \pm 1$. Continuing this process, and by induction, we see that, for every $1 \leqslant k \leqslant n$, $Bf_k = \xi_k \mu_k f_k$ with $\xi_k = \pm 1$. We claim that $\xi_k = \xi_1$ for all $1 \leqslant k \leqslant n$. Assume, to the contrary, that $\mu_1 \neq 0 \neq \mu_k$ and $\xi_1 = 1$ but $\xi_k = -1$ for some $1 \leqslant k \leqslant n$. Take $x = \dfrac{f_1 + f_k}{\sqrt{2}}$, and note that x is a unit vector satisfying $\|Ax\| = \|Bx\|$, but

$$|\langle Ax, x \rangle| = \frac{|\mu_1 + \mu_k|}{2} \neq \frac{|\mu_1 - \mu_k|}{2} = |\langle Bx, x \rangle|;$$

which contradicts the equality (1). This proves the claim and shows that $A = B$ or $A = -B$, as desired. □

LEMMA 5. *Let $0 < \varepsilon < 1$ and $A, B \in \mathscr{M}_n$ be Hermitian matrices. If*

$$r_\varepsilon(Ax \otimes x + x \otimes xA) = r_\varepsilon(Bx \otimes x + x \otimes xB) \tag{2}$$

holds for all unit vector $x \in \mathbb{C}^n$, then $A = \pm B$.

Proof. Let x be an arbitrary unit vector $x \in \mathbb{C}^n$, and set $R_A = Ax \otimes x + x \otimes xA$. Write $Ax = \alpha x + \beta y$ for some scalars $\alpha, \beta \in \mathbb{C}$ and a unit vector $y \in \mathbb{C}^n$ with $\langle x, y \rangle = 0$, and note that $\alpha = \langle Ax, x \rangle$ is a real number since A is Hermitian. Then, with respect to a suitable orthonormal basis in $\mathbb{C}^n = [x, y] \oplus [x, y]^\perp$,

$$R_A = \begin{pmatrix} 2\alpha & \bar{\beta} \\ \beta & 0 \end{pmatrix} \oplus 0.$$

Clearly, R_A is an Hermitian matrix, and so by taking into account Lemma 2, we have

$$r_\varepsilon(R_A) = \max\left\{\frac{|i(R_A)-\varepsilon s(R_A)|}{1-\varepsilon}, \frac{|s(R_A)-\varepsilon i(R_A)|}{1-\varepsilon}\right\}$$

where $i(R_A)$ and $s(R_A)$ denote the infimum and the supremum of $\sigma(R_A)$, respectively. Straightforward computations give that

$$s(R_A) = \alpha + \sqrt{\alpha^2 + |\beta|^2} = \langle Ax, x\rangle + \|Ax\| \geqslant 0$$

and

$$i(R_A) = \alpha - \sqrt{\alpha^2 + |\beta|^2} = \langle Ax, x\rangle - \|Ax\| \leqslant 0,$$

implying that

$$|i(R_A) - \varepsilon s(R_A)| = |(1-\varepsilon)\langle Ax, x\rangle - (1+\varepsilon)\|Ax\||$$

and

$$|s(R_A) - \varepsilon i(R_A)| = |(1-\varepsilon)\langle Ax, x\rangle + (1+\varepsilon)\|Ax\||.$$

From this together the fact that

$$\max(|a-b|, |a+b|) = b + |a|$$

for any real numbers a, b satisfying $|a| \leqslant b$, we infer that

$$r_\varepsilon(R_A) = \frac{1+\varepsilon}{1-\varepsilon}\|Ax\| + |\langle Ax, x\rangle|.$$

Similarly, considering $R_B = Bx \otimes x + x \otimes xB$, one gets

$$r_\varepsilon(R_B) = \frac{1+\varepsilon}{1-\varepsilon}\|B(x)\| + |\langle B(x), x\rangle|.$$

Thus, by the equality (2), we have

$$\frac{1+\varepsilon}{1-\varepsilon}\|A(x)\| + |\langle A(x), x\rangle| = \frac{1+\varepsilon}{1-\varepsilon}\|B(x)\| + |\langle B(x), x\rangle|$$

for all unit vector $x \in \mathbb{C}^n$. Lemma 4 tells us that $A = \pm B$, and the proof is therefore complete. $\square$

We now have collected all the necessary ingredients and are therefore in a position to state and prove the main results of this section. The following theorem is one of the purposes of this paper. It characterizes nonlinear maps on $\mathscr{H}_n$ that preserve the condition spectral radius of Jordan product of matrices.

THEOREM 1. *Let $n \geqslant 3$ and $0 < \varepsilon < 1$. A map $\Phi : \mathscr{H}_n \to \mathscr{H}_n$ satisfies*

$$r_\varepsilon(\Phi(A)\Phi(B) + \Phi(B)\Phi(A)) = r_\varepsilon(AB + BA) \qquad (A, B \in \mathscr{H}_n) \tag{3}$$

if and only if there is a functional $\xi : \mathscr{H}_n \to \{-1, 1\}$ and a unitary matrix $U \in \mathscr{M}_n$ such that Φ has the form

$$A \mapsto \xi(A)UAU^* \text{ or } A \mapsto \xi(A)UA^{tr}U^*.$$

Proof. Checking the 'if' part is straightforward, so we will only deal with the 'only if' part. So assume that Φ satisfices (3). From the first statement of Lemma 1, the map Φ preserves zero Jordan product of matrices in both directions, i.e., for any $A,B \in \mathscr{H}_n$, $\Phi(A)\Phi(B)+\Phi(B)\Phi(A) = 0$ if and only if $AB+BA = 0$. Therefore, by Lemma 3, there is a functional $h : \mathscr{H}_n \to \mathbb{R}\setminus\{0\}$ and a unitary matrix $U \in \mathscr{M}_n$ such that either

$$\Phi(A) = h(A)UAU^* \text{ or } \Phi(A) = h(A)UA^{tr}U^*$$

for all rank one matrix $A \in \mathscr{H}_n$. Assume firstly that $\Phi(A) = h(A)UAU^*$ for every rank one matrix $A \in \mathscr{H}_n$. We assert that the map h can be chosen so that

$$h(A) = \pm 1$$

for every $A \in \mathscr{H}_n$. Indeed, every rank one Hermitian matrix A can be written as $A = x\otimes x$ for some nonzero column vector $x \in \mathbb{C}^n$, and so

$$2\|x\|^2 r_\varepsilon(x\otimes x) = r_\varepsilon(2(x\otimes x)^2) = r_\varepsilon(2\Phi(x\otimes x)^2) = 2h(x\otimes x)^2\|x\|^2 r_\varepsilon(x\otimes x)$$

implying that $|h(x\otimes x)| = 1$ for every rank one matrix $x\otimes x \in \mathscr{H}_n$. Redefine h, if necessary, by letting $h(A) = 1$ when A is not of rank one, we get a functional $h : \mathscr{H}_n \to \{-1,1\}$ as asserted.

Set

$$\Psi(A) = h(A)U^*\Phi(A)U$$

for every $A \in \mathscr{H}_n$, and note that

$$\Psi(x\otimes x) = x\otimes x \tag{4}$$

for every rank one matrix $x\otimes x \in \mathscr{H}_n$.

Now, let us prove that, for every $A \in \mathscr{H}_n$, $\Psi(A) = \ell(A)A$ with $\ell(A) = \pm 1$. To do so, let A be an arbitrary Hermitian matrix in $\mathscr{H}_n$. Since the map Ψ satisfies (3) and (4), one gets

$$r_\varepsilon(\Psi(A)x\otimes x + x\otimes x\Psi(A)) = r_\varepsilon(Ax\otimes x + x\otimes xA)$$

for all unit vector x in $\mathbb{C}^n$. As the matrices A and $\Psi(A)$ are Hermitian, Lemma 5 tells us that $\Psi(A) = \pm A$ as required. By letting $\xi(A) = h(A)\ell(A)$, we get a functional $\xi : \mathscr{H}_n \to \{-1,1\}$ for which the map Φ has the form $\Phi : A \mapsto \xi(A)UAU^*$, as desired.

In the remainder case when $\Phi(A) = h(A)UA^{tr}U^*$ for all rank one matrix $A \in \mathscr{H}_n$, set $\chi(A) := \Phi(A^{tr})$ and $h'(A) := h(A^{tr})$ so that $\chi(A) = h'(A)UAU^*$ for every $A \in \mathscr{H}_n$. From Lemma 1, the map χ satisfies (3). Thus, by what has been shown above, the map h' can be chosen so that $h'(A) = \pm 1$ for every $A \in \mathscr{H}_n$; which yields the desired conclusion in this case too. The proof of the theorem is therefore complete. $\square$

As a consequence, the following result describes maps on $\mathscr{H}_n$ that preserve the condition spectrum of Jordan product of matrices.

THEOREM 2. *Let $n \geqslant 3$ and $0 < \varepsilon < 1$. A map $\Phi : \mathscr{H}_n \to \mathscr{H}_n$ satisfies*

$$\sigma_\varepsilon(\Phi(A)\Phi(B)+\Phi(B)\Phi(A)) = \sigma_\varepsilon(AB+BA) \qquad (A,B \in \mathscr{H}_n) \tag{5}$$

if and only if there is a scalar $c \in \{-1,1\}$ *and a unitary matrix* $U \in \mathscr{M}_n$ *such that* Φ *has the form*

$$A \mapsto cUAU^* \quad or \quad A \mapsto cUA^{tr}U^*.$$

Proof. The sufficiency condition can be readily checked. To prove the necessity, assume that Φ satisfies (5), and note that the map Φ preserves the ε-condition spectral radius of Jordan product of matrices. By Theorem 1, there exist a functional $\xi : \mathscr{H}_n \to \{-1,1\}$ and a unitary matrix $U \in \mathscr{M}_n$ such that either $\phi(T) = \xi(T)UTU^*$ or $\phi(T) = \xi(T)UT^*U^*$ for all $T \in \mathscr{H}_n$.

Assume firstly that $\Phi(T) = \xi(T)UTU^*$ for all $T \in \mathscr{H}_n$. We claim that

$$\xi(x\otimes x) = \xi(I) \tag{6}$$

for all unit vector $x \in \mathbb{C}^n$. Fix an arbitrary unit vector $x \in \mathbb{C}^n$, and note that, by the equality (5),

$$\begin{aligned} 2\sigma_\varepsilon(x\otimes x) &= \sigma_\varepsilon((x\otimes x)I + I(x\otimes x)) \\ &= \sigma_\varepsilon(\Phi(x\otimes x)\Phi(I) + \Phi(I)\Phi(x\otimes x)) \\ &= \sigma_\varepsilon(2\xi(I)\xi(x\otimes x)x\otimes x) \\ &= 2\xi(I)\xi(x\otimes x)\sigma_\varepsilon(x\otimes x). \end{aligned}$$

Using Lemma 2, one gets

$$\sigma_\varepsilon(x\otimes x) = \overline{D}\left(\frac{1}{1-\varepsilon^2}, \frac{\varepsilon}{1-\varepsilon^2}\right) \cup \overline{D}\left(\frac{-\varepsilon^2}{1-\varepsilon^2}, \frac{\varepsilon}{1-\varepsilon^2}\right),$$

and consequently $\xi(x\otimes x)\xi(I) = 1$. This yields that $\xi(x\otimes x) = \xi(I)$ since $\xi(I)^2 = 1$, and proves the claim.

Next, let us prove that

$$\xi(A) = \xi(x\otimes x) \tag{7}$$

for all $A \in \mathscr{H}_n$ and all $x \in \mathbb{C}^n$. Let $A \in \mathscr{H}_n$ be an arbitrary matrix, and let x be a fixed unit vector in $\mathbb{C}^n$. From the proof of Lemma 5, we have

$$\sigma(Ax\otimes x + x\otimes xA) = \{0, \lambda_1, \lambda_2\},$$

where $\lambda_1 = \langle Ax, x\rangle + \|Ax\| \geqslant 0$ and $\lambda_2 = \langle Ax, x\rangle - \|Ax\| \leqslant 0$. Thus, it follows from Lemma 2 that

$$\sigma_\varepsilon(Ax\otimes x + x\otimes xA) = \bigcup_{\alpha,\beta\in\{0,\lambda_1,\lambda_2\}} \overline{D}\left(\frac{\alpha-\beta\varepsilon^2}{1-\varepsilon^2}, \frac{\varepsilon|\alpha-\beta|}{1-\varepsilon^2}\right). \tag{8}$$

On the other hand, the third statement of Lemma 1 together with (5) yields that

$$\xi(A)\xi(x\otimes x)\sigma_\varepsilon(Ax\otimes x + x\otimes xA)) = \sigma_\varepsilon(Ax\otimes x + x\otimes xA).$$

From this together with the above equality (8), we infer that $\xi(A)\xi(x\otimes x)=1$, and consequently $\xi(A)=\xi(x\otimes x)$ since $\xi(x\otimes x)^2=1$. By combining (6) and (7), we obtain $\xi(A)=\xi(I)$ for all $A\in\mathscr{H}_n$, and thus, by letting $c=\xi(I)$, the desired conclusion in the theorem follows in the case when Φ has the first form.

In the remainder case when Φ has the form $\Phi: A\mapsto \xi(A)UA^{tr}U^*$, set $\chi(A):=\Phi(A^{tr})$ and $\xi'(A):=\xi(A^{tr})$ so that $\chi(A)=\xi'(A)UAU^*$ for every $A\in\mathscr{H}_n$, and note that the map χ satisfies (5). Thus, by what has been shown above, $\xi'(A)=\xi'(I)$ for every $A\in\mathscr{H}_n$. This chows that Φ has the asserted form described in the theorem in this case too, and achieves the proof. $\square$

3. Preservers of pseudo spectra

In this section, we characterize pseudo spectral radius preservers of Jordan product of any pair of Hermitian matrices. The following theorem is one of the purposes of this paper. It extends [10, Theorem 5.1].

THEOREM 3. *Let $n\geqslant 3$ and $\varepsilon>0$. A map $\Phi:\mathscr{H}_n\to\mathscr{H}_n$ satisfies*

$$\delta_\varepsilon(\Phi(A)\Phi(B)+\Phi(B)\Phi(A))=\delta_\varepsilon(AB+BA)\qquad (A,B\in\mathscr{H}_n)\tag{9}$$

if and only if there is a functional $\xi:\mathscr{H}_n\to\{-1,1\}$ and a unitary matrix $U\in\mathscr{M}_n$ such that Φ has the form

$$A\mapsto \xi(A)UAU^*\ \ or\ \ A\mapsto\xi(A)UA^{tr}U^*.$$

Before proving this result, we provide some lemmas needed in the sequel. The first one gives some pseudo spectral properties of matrices; see [16].

LEMMA 6. *For $\varepsilon>0$ and $A\in\mathscr{M}_n$, the following statements hold.*

(i) $\Lambda_\varepsilon(A^{tr})=\Lambda_\varepsilon(A)$ *and* $\Lambda_\varepsilon(UAU^*)=\Lambda_\varepsilon(A)$ *for every unitary matrix* $U\in\mathscr{M}_n$.

(ii) If A is Hermitian, then $\Lambda_\varepsilon(A)=\sigma(A)+D(0,\varepsilon)$ and $\delta_\varepsilon(A)=\|A\|+\varepsilon$.

The second one, quoted from [10, Proposition 2.5], identifies the ε-pseudo spectral radius of rank one matrices.

LEMMA 7. *Let $\varepsilon>0$ and $x,f\in\mathbb{C}^n$ be arbitrary. Then*

$$\delta_\varepsilon(x\otimes f)=\frac{1}{2}\left(\sqrt{|\langle x,f\rangle|^2+4\varepsilon^2+4\varepsilon\|x\|\|f\|}+|\langle x,f\rangle|\right).$$

The third lemma, proved in [3], characterizes zero matrix through their pseudo spectral properties.

LEMMA 8. *Let $A\in\mathscr{M}_n$ be a matrix. Then $\delta_\varepsilon(A)=\varepsilon$ if and only if $A=0$.*

The next one gives the pseudo spectral radius version of Lemma 5.

LEMMA 9. *Let $\varepsilon > 0$ and $A, B \in \mathscr{M}_n$ be Hermitian matrices. If*

$$\delta_\varepsilon(Ax \otimes x + x \otimes xA) = \delta_\varepsilon(Bx \otimes x + x \otimes xB) \tag{10}$$

holds for all unit vector $x \in \mathbb{C}^n$, then $A = \pm B$.

Proof. Let x be an arbitrary unit vector $x \in \mathbb{C}^n$. Similar argument as the one given in the beginning of the proof of Lemma 5, allows to get $\sigma(Ax \otimes x + x \otimes xA) = \{0, \lambda_1, \lambda_2\}$, where $\lambda_1 = \langle Ax, x\rangle + \|Ax\| \geqslant 0$ and $\lambda_2 = \langle Ax, x\rangle - \|Ax\| \leqslant 0$. Since $Ax \otimes x + x \otimes xA$ is a Hermitian matrix, the second statement of Lemma 6 tells us that

$$\sigma_\varepsilon(Ax \otimes x + x \otimes xA) = D(0, \varepsilon) \cup D(\lambda_1, \varepsilon) \cup D(\lambda_2, \varepsilon);$$

which implies that

$$\begin{aligned} \delta_\varepsilon(Ax \otimes x + x \otimes xA) &= \max(|\lambda_1|, |\lambda_2|) + \varepsilon \\ &= \begin{cases} \|Ax\| + \langle Ax, x\rangle + \varepsilon, & \text{if } \langle Ax, x\rangle \geqslant 0 \\ \|Ax\| - \langle Ax, x\rangle + \varepsilon, & \text{if } \langle Ax, x\rangle \leqslant 0 \end{cases} \\ &= \|Ax\| + |\langle Ax, x\rangle| + \varepsilon. \end{aligned}$$

Similarly, $\delta_\varepsilon(Bx \otimes x + x \otimes xB) = \|Ax\| + |\langle Bx, x\rangle| + \varepsilon$. Thus, by (10),

$$\|Ax\| + |\langle Ax, x\rangle| = \|Bx\| + |\langle Bx, x\rangle|$$

for all unit vector $x \in \mathbb{C}^n$, and so, from Lemma 4, $A = \pm B$ as desired. $\square$

We are now ready to prove the main result of this section.

Proof of Theorem 3. We only need to check the necessity condition. So assume that Φ satisfies (9), and note that, by Lemma 8, the map Φ preserves zero Jordan product of matrices in both directions. Therefore, by Lemma 3, there is a functional $h : \mathscr{H}_n \to \mathbb{R} \setminus \{0\}$ an a unitary matrix $U \in \mathscr{M}_n$ such that either $\Phi(A) = h(A)UAU^*$ or $\Phi(A) = h(A)UA^{tr}U^*$ for all rank one matrix $A \in \mathscr{H}_n$.

Assume firstly that $\Phi(A) = h(A)UAU^*$ for every rank one matrix $A \in \mathscr{H}_n$. We claim that the map h can be chosen so that $h(A) = \pm 1$ for all $A \in \mathscr{H}_n$. Observe that, for every $x \in \mathbb{C}^n$, we have, by (9),

$$\delta_\varepsilon(2\|x\|^2 x \otimes x) = \delta_\varepsilon(2(x \otimes x)^2) = \delta_\varepsilon(2\Phi(x \otimes x)^2) = \delta_\varepsilon(2\xi(x \otimes x)^2 \|x\|^2 x \otimes x).$$

Using Lemma 7, one gets

$$2\|x\|^4 + \varepsilon = 2|\xi(x \otimes x)|^2 \|x\|^4 + \varepsilon;$$

which shows that $|h(x \otimes x)| = 1$ for every rank one matrix $x \otimes x \in \mathscr{H}_n$. Redefine h, if necessary, by letting $h(A) = 1$ when A is not of rank one, we get a functional $h : \mathscr{H}_n \to \{-1, 1\}$ as claimed.

Set $\Psi(A) := h(A)U^*\Phi(A)U$ for every $A \in \mathscr{H}_n$, and note that the map Ψ satisfies (9) and $\Psi(x\otimes x) = x\otimes x$ for every $x \in \mathbb{C}_n$. Therefore, for every $A \in \mathscr{H}_n$,

$$\delta_\varepsilon(\Psi(A)x\otimes x + x\otimes x\Psi(A)) = \delta_\varepsilon(Ax\otimes x + x\otimes xA)$$

for all unit vector in $\mathbb{C}^n$, and consequently, by Lemma 9, $\Psi(A) = \pm A$ for all $A \in \mathscr{H}_n$. Now, by inspecting the end of the proof of Theorem 1, with no extra efforts, one can see that same approach used there allows to get that the map Φ has the desired form; which achieves the proof. $\square$

4. Concluding remarks

We end this papper by the following remarks.

(a) We believe that our main results remains true in the case of 2×2 matrices. It would be nice to prove or disprove our conjecture.

(b) Our approach in this paper works for the space of complex Hermitian matrices setting. The following natural problem, concerning the general C^*-algebra setting, suggests itself.

PROBLEM. Can the real space $\mathscr{H}_n$ be replaced by the set of self-adjoint element of a general C^*-algebra in Theorems 1, 2 and 3 of this paper?

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