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MULTIPLICATIVE GENERALIZED LIE n–DERIVATIONS ON COMPLETELY DISTRIBUTIVE COMMUTATIVE SUBSPACE LATTICE ALGEBRAS

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Abstract. Let $Alg\mathcal{L}$ be a completely distributive commutative subspace lattice algebra and let $\delta : Alg\mathcal{L} \to Alg\mathcal{L}$ be a nonlinear map. It is shown that δ is a multiplicative generalized Lie n-derivation on $Alg\mathcal{L}$ with an associated multiplicative generalized Lie n-derivation d if and only if $\delta(A) = \psi(A) + \xi(A)$ holds for every $A \in Alg\mathcal{L}$, where $\psi : Alg\mathcal{L} \to Alg\mathcal{L}$ is an additive generalized derivation and $\xi : Alg\mathcal{L} \to Z(Alg\mathcal{L})$ is a central-valued map vanishing on each $(n-1)$-th commutator $p_n(A_1, A_2, \cdots, A_n)$.

1. Introduction

Let $\mathcal{R}$ be an associative commutative unital ring and $\mathcal{A}$ be an algebra over $\mathcal{R}$. Recall that an $\mathcal{R}$-linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a *Jordan derivation* if $\delta(A^2) = \delta(A)A + A\delta(A)$ holds for all $A \in \mathcal{A}$; δ is called a *Lie derivation* if $\delta([A,B]) = [\delta(A),B] + [A,\delta(B)]$ holds for all $A,B \in \mathcal{A}$, where $[A,B] = AB - BA$ is the usual Lie product; δ is called a *Lie triple derivation* if $\delta([[A,B],C]) = [[\delta(A),B],C] + [[A,\delta(B)],C] + [[A,B],\delta(C)]$ holds for all $A,B,C \in \mathcal{A}$; δ is called a *generalized Lie derivation* if there exists a derivation d such that

$$\delta([A,B]) = \delta(A)B - \delta(B)A + Ad(B) - Bd(A) \quad \text{for all } A,B \in \mathcal{A}.$$

If there is no assumption of additivity for δ in the above definitions, then δ is said to be multiplicative (or nonlinear). We say a Lie derivation δ is *standard* if it can be decomposed as $\delta = \psi + \xi$, where ψ is an ordinary derivation and ξ is a linear mapping from $\mathcal{A}$ into the center of $\mathcal{A}$ vanishing on each commutator. Clearly, every (generalized) derivation is a (generalized) Lie derivation as well as a (generalized) Jordan derivation, and every (generalized) Lie (Jordan) derivation is a (generalized) Lie (Jordan) triple derivation. The converse is, in general, not true (see [4, 7, 16]). The standard problem is to find out whether (under some conditions) a Lie derivation is standard.

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In 1964, Martindale [18] introduced the notion of Lie derivations and proved that every Lie derivation on a primitive ring is standard. From then on, many mathematicians studied this problem and obtained abundant results(see [5, 19]). Hvala [9] studied generalized Lie derivations of a prime ring and observed that every generalized Lie derivation of a prime ring is standard, and Yu and Zhang [20] extended to consider nonlinear generalized Lie derivations of triangular algebras. With the development of research, many achievements about (nonlinear) Lie n-derivations have been obtained.

For $A,B \in \mathcal{A}$, let $[A,B] = AB - BA$ be the usual Lie product. Set $p_1(A) = A$, and for all integers $n \geqslant 2$,

$$p_n(A_1,A_2,\cdots,A_n) = [p_{n-1}(A_1,A_2,\cdots,A_{n-1}),A_n] = p_{n-1}([A_1,A_2],A_3,\cdots,A_n).$$

In [6, 12], they gave a definition about multiplicative generalized Lie n-derivations:

DEFINITION 1.1. [6, 12] Let $\mathcal{A}$ be an associated algebra. A map $\delta : \mathcal{A} \to \mathcal{A}$(not necessarily linear) is called a *multiplicative generalized Lie n-derivation* if there exists a multiplicative Lie n-derivation d on $\mathcal{A}$, such that

$$\delta(p_n(A_1,A_2,\cdots,A_n)) = p_n(\delta(A_1),A_2,\cdots,A_n) + \sum_{i=2}^{n} p_n(A_1,\cdots,d(A_i),\cdots,A_n), \quad (1.1)$$

for all $A_i \in \mathcal{A}$, and in this case, d is called *an associated multiplicative Lie n-derivation* of δ.

Clearly, (multiplicative generalized) Lie 2-derivations are (multiplicative generalized) Lie derivations, and (multiplicative generalized) Lie 3-derivations are (multiplicative generalized) Lie triple derivations. In this vein, there are indeed some interesting works. The concept of a Lie n-derivation was introduced by Abdullaev [1], where the form of Lie n-derivations of a certain von Neumann algebra was described. Benkovič and Eremita [2] showed that every multiplicative Lie n-derivation (under some conditions) on triangular rings has the standard form. Feng and Qi [6] extended Abdullaev's result to the case of multiplicative generalized Lie n-derivations on von Neumann algebra. Recently, Ma, Zhang and Liu [17] have obtained that multiplicative generalized Lie derivations on a reflexive algebra whose lattice is completely distributive and commutative is standard. More details can be seen in [3, 12] and its references.

Inspired by the works mentioned, it is reasonable to consider the multiplicative generalized Lie n-derivation of completely distributive commutative subspace lattice algebras in this work.

2. Mathematical preliminaries

Let us introduce the notations and the concepts. Let $\mathcal{H}$ be a Hilbert space over a real or complex field $\mathcal{F}$. A *subspace lattice* $\mathcal{L}$ of $\mathcal{H}$ is a strongly closed collection of projections on $\mathcal{H}$, if it is closed under the usual lattice operations $\bigvee$ and $\bigwedge$, and contains the zero operator 0 and the identity operator I. If each pair of projections in $\mathcal{L}$ commute, then $\mathcal{L}$ is called a *commutative subspace lattice(CSL)*, and the associated subspace lattice algebra $Alg\mathcal{L} = \{T \in B(\mathcal{H}) : T(L) \subseteq L, \forall L \in \mathcal{L}\}$ is called a *CSL* algebra. A totally ordered subspace lattice is called a *nest*. Recall that a subspace lattice is called *completely distributive* if $e = \bigvee\{N \in \mathcal{L} : N_- \not\supseteq e\}$ for every $0 \neq e \in \mathcal{L}$, where $N_- = \bigvee\{P \in \mathcal{L} : P \not\supseteq N\}$, and its associated subspace lattice algebra is called *completely distributive CSL algebra*(shortly written by *CDC* algebra). For standard definitions concerning completely distributive subspace lattice algebras see [10, 13].

In [11], they proved that the collection of finite sums of rank-one operators in a *CDC* algebra is strongly dense. This result will be frequently used in studying *CDC* algebra. Set $\mathcal{U}(\mathcal{L}) = \{e \in \mathcal{L} : e \neq 0, e_- \neq H\}$.

LEMMA 2.1. [11] *Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$. Then the rank one operator $x \otimes y$ belongs to $Alg\mathcal{L}$ if and only if there is an element $E \in \mathcal{U}(L)$ such that $x \in E$ and $y \in E_-^{\perp}$. Here $x \otimes y$ is defined as $(x \otimes y)z = (z,y)x$ for $z \in \mathcal{H}$.*

Let $Alg\mathcal{L}$ be a *CDC* algebra. We say $e, e' \in \mathcal{U}(\mathcal{L})$ are connected if there exist finitely many projections $e_1, e_2, \ldots, e_m \in \mathcal{U}(\mathcal{L})$, such that e_i and e_{i+1} are comparable for each $i = 0, 1, \ldots, m$, where $e_0 = e, e_{m+1} = e'$. $\mathcal{C} \subseteq \mathcal{U}(\mathcal{L})$ is called a connected component if each pair in $\mathcal{C}$ is connected and any element in $\mathcal{U}(\mathcal{L}) \diagdown \mathcal{C}$ is not connected with any element in $\mathcal{C}$. Recall that a *CDC* algebra $Alg\mathcal{L}$ is *irreducible* if and only if the commutant is trivial, i.e. $(Alg\mathcal{L})' = \mathcal{F}I$, which is also equivalent to the condition that $\mathcal{L} \bigcap \mathcal{L}^{\perp} = \{0, I\}$, where $\mathcal{L}^{\perp} = \{e^{\perp} : e \in \mathcal{L}\}$. Clearly, Nest algebra is irreducible. In [8, 14], it turns out that any *CDC* algebra can be decomposed into the direct sum of irreducible *CDC* algebras.

LEMMA 2.2. [8, 14] *Let $Alg\mathcal{L}$ be a CDC algebra on a separable Hitbert space $\mathcal{H}$. Then there are no more than countably many connected components $\{\mathcal{C}_n : n \in \Lambda\}$ of $\mathcal{E}(\mathcal{L})$ such that $\mathcal{E}(\mathcal{L}) = \cup\{e : e \in \mathcal{C}_n, n \in \Lambda\}$. Let $e_m = \vee\{e : e \in \mathcal{C}_m, m \in \Lambda\}$. Then $\{e_m, m \in \Lambda\} \subseteq \mathcal{L} \cap \mathcal{L}^{\perp}$ is a subset of pairwise orthogonal projections, and the algebra $Alg\mathcal{L}$ can be written as a direct sum:*

$$Alg\mathcal{L} = \sum_{m \in \Lambda} \oplus (Alg\mathcal{L})e_m,$$

where each $(Alg\mathcal{L})e_m$ viewed as a subalgebra of operators acting on the range of e_m is an irreducible CDC algebra. Here, all convergence means strong convergence.

From the definition of e_n, we know that its linear span is a Hilbert space $\mathcal{H}$, and pairwise orthogonal projections. It follows that the identity and center of $Alg\mathcal{L}$ is $I = \sum_{m \in \Lambda} \oplus e_m$ and $\mathcal{Z}(Alg\mathcal{L}) = \sum_{m \in \Lambda} \oplus \lambda_m e_m$, respectively, where $\lambda_m \in \mathcal{F}$. In [14], they prove that each Jordan isomorphism between irreducible *CDC* algebras is the sum of an isomorphism and an anti-isomorphism.

LEMMA 2.3. [15] *Let $Alg\mathcal{L}$ be a non-trivially irreducible completely distributive commutative subspace lattice algebra on a complex Hilbert space $\mathcal{H}$. Then there exists a non-trivial projection $e \in \mathcal{L}$, such that $e(Alg\mathcal{L})e^{\perp}$ is faithful $Alg\mathcal{L}$ bimodule, i.e., for all $A \in Alg\mathcal{L}$, if $Ae(Alg\mathcal{L})e^{\perp} = \{0\}$, then $Ae = 0$ and if $e(Alg\mathcal{L})e^{\perp}A = \{0\}$, then $e^{\perp}A = 0$.*

Let I be the identity operator on $\mathcal{H}$. If $\mathcal{L}$ is non-trivial, by Lemma 2.3, there exists a non-trivial projection $e \in \mathcal{L}$, such that $e(Alg\mathcal{L})e^{\perp}$ is faithful $Alg\mathcal{L}$ bimodule. Set

$e_1 = e, e_2 = I - e_1$, then e_1, e_2 are projections of $Alg\mathcal{L}$. Moreover, by the definitions of p_n and e_i, we have following results.

LEMMA 2.4. [2] *Let $Alg\mathcal{L}$ be a non-trivially irreducible CDC algebra on a complex Hilbert space $\mathcal{H}$ and $e_1 \in Alg\mathcal{L}$ be an associated non-trivial projection, $e_2 = I - e_1$. Then, for all $A \in Alg\mathcal{L}$, and any positive integer $n \geqslant 2$, we have*

$$p_n(A, e_1, \cdots, e_1) = (-1)^{n-1} e_1 A e_2 \text{ and } p_n(A, e_2, \cdots, e_2) = e_1 A e_2.$$

LEMMA 2.5. *Let $Alg\mathcal{L}$ be a non-trivially irreducible CDC algebra on a complex Hilbert space $\mathcal{H}$ with non-trivial projections e_1, e_2, and $\delta : Alg\mathcal{L} \to Alg\mathcal{L}$ be a multiplicative generalized Lie n-derivation with an associated multiplicative Lie n-derivation d. Then there exists an inner derivation $d' : Alg\mathcal{L} \to Alg\mathcal{L}$ and a multiplicative generalized Lie n-derivation $\delta' : Alg\mathcal{L} \to Alg\mathcal{L}$, such that*

$$\delta = d' + \delta' \quad \text{and} \quad e_1 \delta'(e_2) e_2 = 0.$$

Proof. Define maps $d', \delta' : Alg\mathcal{L} \to Alg\mathcal{L}$ by

$$d'(A) = [\delta(e_2), A] \quad \text{and} \quad \delta'(A) = \delta(A) - d'(A)$$

for all $A \in Alg\mathcal{L}$. Clearly, d' is an inner derivation and δ' is a multiplicative generalized Lie n-derivation. Moreover, it follows from $\delta'(e_2) = \delta(e_2) - d'(e_2) = \delta(e_2) - [\delta(e_2), e_2]$ that $e_1 \delta'(e_2) e_2 = 0$. The proof is completed. □

REMARK 2.1. From Lemma 2.4, we can obtain

$$\begin{aligned} 0 &= \delta(p_n(e_2, e_2, \cdots, e_2)) = p_n(\delta(e_2), e_2, \cdots, e_2) + p_n(e_2, d(e_2), \cdots, e_2) \\ &= e_1 \delta(e_2) e_2 + e_1 d(e_2) e_2. \end{aligned}$$

It follows from Lemma 2.5 that $e_1 d(e_2) e_2 = 0$.

Therefore, without loss of generality, we can assume that the multiplicative generalized Lie n-derivation δ and its associated multiplicative Lie n-derivation d of δ on non-trivially irreducible *CDC* algebra satisfies $e_1 \delta(e_2) e_2 = e_1 d(e_2) e_2 = 0$. Moreover, assume that all algebras in this paper are $(n-1)$-torsion free.

3. Multiplicative generalized Lie n-derivations on irreducible completely distributive commutative subspace lattice algebras

In this section, we begin with the irreducible case.

THEOREM 3.1. *Let $Alg\mathcal{L}$ be an irreducible completely distributive commutative subspace lattice algebra on a complex Hilbert space $\mathcal{H}$ and $\delta : Alg\mathcal{L} \to Alg\mathcal{L}$ be a nonlinear map. Then δ is a multiplicative generalized Lie $n(\geqslant 2)$-derivation if and only if for every $A \in Alg\mathcal{L}$, $\delta(A) = \psi(A) + \xi(A)$, where $\psi : Alg\mathcal{L} \to Alg\mathcal{L}$ is an* *additive generalized derivation and* $\xi : Alg\mathcal{L} \to Z(Alg\mathcal{L})$ *vanishes on each* $(n-1)$*-th commutator* $p_n(A_1, A_2, \cdots, A_n)$.

Proof. If $\delta(A) = \psi(A) + \xi(A)$, it is easy to check that δ is a multiplicative generalized Lie n-derivation. So we only need to show "only if" part.

Two cases arise:

Case 1. If $\mathcal{L}$ is trivial, then $Alg\mathcal{L}$ is a C*-algebra. It follows from the main Theorem of [6] that δ is standard.

Case 2. Assume that $\mathcal{L}$ is non-trivial, then there exists a non-trivial projection $e_1 \in \mathcal{L}$. Set $e_2 = I - e_1$. Then, for every A in $Alg\mathcal{L}$, A can be decomposed as: $A = e_1Ae_1 + e_1Ae_2 + e_2Ae_2$. Set $\mathcal{A}_{ij} = e_i(Alg\mathcal{L})e_j$, then, $Alg\mathcal{L}$ can be decomposed as

$$Alg\mathcal{L} = e_1(Alg\mathcal{L})e_1 \oplus e_1(Alg\mathcal{L})e_2 \oplus e_2(Alg\mathcal{L})e_2 = \mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{22}.$$

We divide the proof into several claims.

Claim 1. $\delta(\mathcal{A}_{ii}) \subseteq \mathcal{A}_{11} + \mathcal{A}_{22}$ and $\delta(\mathcal{A}_{12}) \subseteq \mathcal{A}_{12}$.

For every $A_{ii} \in \mathcal{A}_{ii}$, note that $[A_{ii}, e_2] = 0$, so we obtain

$$\begin{aligned} 0 &= \delta(e_1A_{ii}e_2) = \delta(p_n(A_{ii}, e_2, \cdots, e_2)) \\ &= p_{n-1}([\delta(A_{ii}), e_2], \cdots, e_2) + p_{n-1}([A_{ii}, d(e_2)], e_2, \cdots, e_2) \\ &\quad + \sum_{i=3}^{n} p_{n-1}([A_{ii}, e_2], \cdots, d(e_2), \cdots, e_2) \\ &= e_1\delta(A_{ii})e_2 + e_1A_{ii}d(e_2)e_2 - e_1d(e_2)A_{ii}e_2 \end{aligned}$$

by using Lemma 2.4 and the fact that $\delta(0) = 0$. Following from $e_1d(e_2)e_2 = 0$ and $e_1\delta(A_{ii})e_2 = 0$, $\delta(\mathcal{A}_{ii}) \subseteq \mathcal{A}_{11} + \mathcal{A}_{22}$.

For every $A_{12} \in \mathcal{A}_{12}$, by Lemma 2.4, one has

$$\begin{aligned} \delta(A_{12}) &= \delta(e_1A_{12}e_2) = \delta(p_n(A_{12}, e_2, \cdots, e_2)) \\ &= p_n(\delta(A_{12}), e_2, \cdots, e_2) + \sum_{i=2}^{n} p_n(A_{12}, e_2, \cdots, d(e_2), \cdots, e_2) \\ &= e_1\delta(A_{12})e_2 + (n-1)[A_{12}, d(e_2)]. \end{aligned}$$

Multiplying above equation left by e_1 and right by e_2, we obatin $(n-1)e_1[A_{12}, d(e_2)]e_2 = (n-1)[A_{12}, d(e_2)] = 0$. Following from the fact that $Alg\mathcal{L}$ is $(n-1)$-torsion free, then $[\mathcal{A}_{12}, d(e_2)] = 0$. Consequently, $\delta(A_{12}) = e_1\delta(A_{12})e_2 \in \mathcal{A}_{12}$.

Claim 2. $d(e_1), d(e_2) \in \mathcal{F}I$.

Since the center of each irreducible *CDC* algebra coincides with $\mathcal{F}I$, by using $[\mathcal{A}_{12}, d(e_2)] = 0$ and Lemma 2.3, we can obtain $d(e_2) \in \mathcal{F}I$. Then, for every $A_{12} \in \mathcal{A}_{12}$, since d is a multiplicative Lie n-derivation, and thus,

$$\begin{aligned} d(A_{12}) &= d((p_{n-1}(A_{12}, e_2, \cdots, e_2)) = d((p_n(e_1, A_{12}, e_2, \cdots, e_2)) \\ &= e_1[d(e_1), A_{12}]e_2 + e_1[e_1, d(A_{12})]e_2 = e_1[d(e_1), A_{12}]e_2 + e_1d(A_{12})e_2. \end{aligned}$$

From $d(\mathcal{A}_{12}) \in \mathcal{A}_{12}$ and $0 = e_1[d(e_1), A_{12}]e_2 = [d(e_1), A_{12}]$, we have $d(e_1) \in \mathcal{F}I$.

Claim 3. For every $A_{ii} \in \mathcal{A}_{ii}$, $\delta(A_{ii}) \in \mathcal{A}_{ii} + \mathcal{F}e_j$ ($i, j = 1,2$ and $i \neq j$).

Take any $A_{ij} \in \mathcal{A}_{ij}$. If $n > 3$,

$$p_n(A_{11}, A_{22}, A_{12}, e_1 \cdots, e_1) = p_n(A_{11}, A_{22}, A_{12}, e_2 \cdots, e_2) = 0. \tag{3.1}$$

Using Claim 2 and noting that d is a multiplicative Lie n-derivation, we have

$$0 = d(p_n(A_{11}, A_{22}, A_{12}, e_2 \cdots, e_2)) = [[d(A_{11}), A_{22}] + [A_{11}, d(A_{22})], A_{12}].$$

Noting that $[d(A_{11}), A_{22}] \in \mathcal{A}_{22}$ and $[A_{11}, d(A_{22})] \in \mathcal{A}_{11}$, and combining Lemma 2.3, we have

$$[d(A_{11}), A_{22}],\ [A_{11}, d(A_{22})] \in \mathcal{F}I. \tag{3.2}$$

Also from Eq. (3.1), by Claim 2 and $[A_{11}, A_{22}] = 0$, one can obtain

$$\begin{aligned} 0 &= \delta(p_n(A_{11}, A_{22}, A_{12}, e_1 \cdots, e_1)) \\ &= p_{n-1}([\delta(A_{11}), A_{22}], A_{12}, \cdots, e_1) + p_{n-1}([A_{11}, d(A_{22})], A_{12}, \cdots, e_1) \\ &\quad + p_{n-1}([A_{11}, A_{22}], d(A_{12}), \cdots, e_1) + \sum_{i=4}^{n-1} p_{n-1}([A_{11}, A_{22}], A_{12}, \cdots, d(e_1), \cdots, e_1) \\ &= p_{n-1}([\delta(A_{11}), A_{22}], A_{12}, e_1, \cdots, e_1) + p_{n-1}([A_{11}, d(A_{22})], A_{12}, e_1, \cdots, e_1) \end{aligned}$$

It follows from Eq. (3.2) and Lemma 2.4 that when $n > 3$, we have

$$0 = p_{n-1}([\delta(A_{11}), A_{22}], A_{12}, \cdots, e_1) = (-1)^{n-3} e_1[[\delta(A_{11}), A_{22}], A_{12}]e_2. \tag{3.3}$$

From Claim 1, we can assume that there exists $B_{ii} \in \mathcal{A}_{ii}$ such that $\delta(A_{11}) = B_{11} + B_{22}$. By using this in Eq. (3.3), we get for all $A_{ij} \in \mathcal{A}_{ij}$, $A_{12}[\delta(A_{11}), A_{22}] = A_{12}[B_{22}, A_{22}] = 0$, which implies $\mathcal{A}_{12}[B_{22}, A_{22}] = 0$. Hence, by Lemma 2.3, we obtain $[B_{22}, A_{22}] = 0$ for all $n > 3$. It means that $B_{22} \in \mathcal{F}e_2$.

When $n = 2$, $p_2(A_{11}, A_{22}) = 0$, and when $n = 3$, $p_3(A_{11}, A_{22}, A_{12}) = 0$, as we have seen in the proof of above, just a special case. And hence, for every $A_{11} \in \mathcal{A}_{11}$, $\delta(A_{11}) = B_{11} + B_{22} \in \mathcal{A}_{11} + \mathcal{F}e_2$.

Similarly, we have $\delta(A_{22}) \in \mathcal{A}_{22} + \mathcal{F}e_1$.

Next, from Claim 3, we define two maps $\theta : Alg\mathcal{L} \to \mathcal{F}I$ and $F : Alg\mathcal{L} \to Alg\mathcal{L}$, respectively, by

$$\theta(A) = e_2\delta(e_1Ae_1)e_2 + e_1\delta(e_2Ae_2)e_1 \text{ and } F(A) = \delta(A) - \theta(A)$$

for every $A \in Alg\mathcal{L}$. It follows from Claim 2 and 3 that for all $A_{ij} \in \mathcal{A}_{ij}$,

$$F(A_{12}) = \delta(A_{12}) \ \text{ and } \ F(A_{ii}) - \delta(A_{ii}) \in \mathcal{F}I. \tag{3.4}$$

In addition, since d is a multiplicative Lie n-derivation, by a similar argument to that of [19], one can obtain that there exists an additive derivation $f : Alg\mathcal{L} \to Alg\mathcal{L}$ and a central-valued map $\gamma : Alg\mathcal{L} \to \mathcal{F}I$ annihilating each $(n-1)$th commutator, such that $d(A) = f(A) + \gamma(A)$. Moreover, by Claim 1 and Claim 3, the following claim is true.

Claim 4. For every $A_{ij} \in \mathcal{A}_{ij}$, $F(A_{ij}) \in \mathcal{A}_{ij}$.

Claim 5. For every $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$, we have
(i) $F(A_{11}A_{12}) = F(A_{11})A_{12} + A_{11}f(A_{12})$ and $F(A_{12}A_{22}) = F(A_{12})A_{22} + A_{12}f(A_{22})$;
(ii) $F(A_{ii}B_{ii}) = F(A_{ii})B_{ii} + A_{ii}f(B_{ii})$.

For every $A_{ij} \in \mathcal{A}_{ij}$, since $p_n(A_{11}, A_{12}, e_2, e_2 \cdots, e_2) = A_{11}A_{12} \in \mathcal{A}_{12}$, by Claim 3, 4 and Lemma 2.4, it yields that

$$\begin{aligned}
F(A_{11}A_{12}) &= \delta(A_{11}A_{12}) = \delta(p_n(A_{11}, A_{12}, e_2, e_2 \cdots, e_2)) \\
&= p_{n-1}([\delta(A_{11}), A_{22}], e_2, \cdots, e_2) + p_{n-1}([A_{11}, d(A_{12})], e_2, \cdots, e_2) \\
&= p_{n-1}([\delta(A_{11}) - \theta(A_{11}), A_{12}], e_2, \cdots, e_2) \\
&\quad + p_{n-1}([A_{11}, f(A_{12}) + \gamma(A_{12})], e_2, \cdots, e_2) \\
&= p_{n-1}([F(A_{11}), A_{12}], e_2, \cdots, e_2) + p_{n-1}([A_{11}, f(A_{12})], e_2, \cdots, e_2) \\
&= F(A_{11})A_{12} + A_{11}f(A_{12}).
\end{aligned}$$

Analogously, one can show that for all $A_{ij} \in \mathcal{A}_{ij}$, $F(A_{12}A_{22}) = F(A_{12})A_{22} + A_{12}f(A_{22})$. Thus, (i) holds true. It remains to prove (ii). Let $A_{ij}, B_{ij} \in \mathcal{A}_{ij} (i, j = 1, 2)$, by (i), we have

$$F(A_{11}B_{11}A_{12}) = F(A_{11}B_{11})A_{12} + A_{11}B_{11}f(A_{12}),$$

and

$$\begin{aligned}
F(A_{11}B_{11}A_{12}) &= F(A_{11})B_{11}A_{12} + A_{11}f(B_{11}A_{12}) \\
&= F(A_{11})B_{11}A_{12} + A_{11}f(B_{11})A_{12} + A_{11}B_{11}f(A_{12}).
\end{aligned}$$

Now, together with above two equalities, it implies that $(F(A_{11}B_{11}) - F(A_{11})B_{11} - A_{11}f(B_{11}))A_{12} = 0$. By using Lemma 2.3, we obtain $F(A_{11}B_{11}) = F(A_{11})B_{11} + A_{11}f(B_{11})$.

Analogously, we can prove that $F(A_{22}B_{22}) = F(A_{22})B_{22} + A_{22}f(B_{22})$.

Claim 6. For any $A_{ij} \in \mathcal{A}_{ij}$, we have
(i) $F(A_{11} + A_{12}) - F(A_{11}) - F(A_{12}) \in \mathcal{F}I$;
(ii) $F(A_{12} + A_{22}) - F(A_{12}) - F(A_{22}) \in \mathcal{F}I$.

For every $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$, if $n \geqslant 3$, by the fact that $A_{11}A_{12} = [A_{11} + B_{12}, A_{12}]$, and considering Lemma 2.4, Claim 3 and Eq. (3.4), we have

$$\begin{aligned}
F(A_{11}A_{12}) &= \delta(A_{11}A_{12}) = \delta(p_{n-1}([A_{11} + B_{12}, A_{12}], e_2, \cdots, e_2)) \\
&= p_n(\delta(A_{11} + B_{12}), A_{12}, e_2, \cdots, e_2) + p_n(A_{11} + B_{12}, d(A_{12}), e_2, \cdots, e_2) \\
&= p_{n-1}([\delta(A_{11} + B_{12}), A_{12}], e_2, \cdots, e_2) + p_{n-1}([A_{11} + B_{12}, d(A_{12}], e_2, \cdots, e_2) \\
&= e_1[\delta(A_{11} + B_{12}) - \theta(A_{11} + A_{12}), A_{12}]e_2 + e_1[A_{11} + B_{12}, f(A_{12}) + \gamma(A_{12})]e_2 \\
&= e_1[F(A_{11} + B_{12}), A_{12}]e_2 + A_{11}f(A_{12}).
\end{aligned}$$

On the other hand, by Claim 5 (i) and Eq. (3.4), one can obtain

$$F(A_{11}A_{12}) = F(A_{11})A_{12} + A_{11}f(A_{12}) = [F(A_{11}), A_{12}] + A_{11}f(A_{12}).$$

Comparing the above two equalities gives that for all $A_{12} \in \mathcal{A}_{12}$, $e_1[F(A_{11} + B_{12}) - F(A_{11}), A_{12}]e_2 = 0$, and then,

$$e_1(F(A_{11} + B_{12}) - F(A_{11}))e_1 + e_2(F(A_{11} + B_{12}) - F(A_{11}))e_2 \in \mathcal{F}I. \tag{3.5}$$

Also by Lemma 2.4, Claim 3 and Eq. (3.4), one has

$$\begin{aligned} F(B_{12}) &= \delta(B_{12}) = \delta(p_n(A_{11} + B_{12}, e_2 \cdots, e_2)) \\ &= p_n(\delta(A_{11} + B_{12}), e_2, \cdots, e_2) = p_n(F(A_{11} + B_{12}), e_2, \cdots, e_2) \\ &= e_1 F(A_{11} + B_{12})e_2. \end{aligned}$$

This implies that $e_1(F(A_{11} + B_{12}) - F(B_{12}))e_2 = 0$. Then, it follows from Eq. (3.5) that $F(A_{11} + A_{12}) - F(A_{11}) - f(A_{12}) \in \mathcal{F}I$. Note that when $n = 2$, using the fact that $A_{11}A_{12} = [A_{11} + B_{12}, A_{12}]$, as we have seen in the proof of above, just a special case and hence (i) holds true.

Analogously, one can prove that $F(A_{12} + A_{22}) - F(A_{12}) - f(A_{22}) \in \mathcal{F}I$.

Claim 7. For any $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$ and $1 \leqslant i \leqslant j = 2$, $F(A_{ij} + B_{ij}) = F(A_{ij}) + F(B_{ij})$.

Take any $A_{12}, B_{12} \in \mathcal{A}_{12}$. Noting that $A_{12} + B_{12} = p_n(e_1 + A_{12}, B_{12} + e_2, e_2, \cdots, e_2)$, by Eq. (3.4) and Claim 6, one can obtain

$$\begin{aligned} &F(A_{12} + B_{12}) \\ &= \delta(A_{12} + B_{12}) = \delta(p_n(e_1 + A_{12}, B_{12} + e_2, e_2, \cdots, e_2)) \\ &= p_n(\delta(e_1 + A_{12}), B_{12} + e_2, e_2, \cdots, e_2) + p_n(e_1 + A_{12}, d(B_{12} + e_2), e_2, \cdots, e_2) \\ &= p_n(F(e_1 + A_{12}), B_{12} + e_2, e_2, \cdots, e_2) + p_n(e_1 + A_{12}, f(B_{12} + e_2), e_2, \cdots, e_2) \\ &= p_n(F(e_1) + F(A_{12}), B_{12} + e_2, e_2, \cdots, e_2) + p_n(e_1 + A_{12}, f(B_{12}), e_2, \cdots, e_2) \\ &= e_1[F(e_1) + F(A_{12}), B_{12} + e_2]e_2 + e_1[e_1 + A_{12}, f(B_{12})]e_2 \\ &= F(e_1)B_{12} + F(A_{12}) + f(B_{12}) = F(A_{12}) + F(B_{12}). \end{aligned} \tag{3.6}$$

When $n = 2$, using the fact $A_{12} + B_{12} = [e_1 + A_{12}, B_{12} + e_2]$, Eq. (3.6) still holds true.

Taking any $A_{11}, B_{11} \in \mathcal{A}_{11}$ and any $A_{12} \in \mathcal{A}_{12}$, by Eq. (3.6) and Claim 5(i), one has

$$F((A_{11} + B_{11})A_{12}) = F(A_{11} + B_{11})A_{12} + (A_{11} + B_{11})F(A_{12})$$

and

$$F((A_{11} + B_{11})A_{12}) = F(A_{11})A_{12} + F(B_{11})A_{12} + A_{11}F(A_{12}) + B_{11}F(A_{12})$$

Comparing the above relations, one can obtain that for all $A_{12} \in \mathcal{A}_{12}$,

$$(F(A_{11} + B_{11}) - F(A_{11}) - F(B_{11}))A_{12} = 0,$$

it follows from Lemma 2.3 that $F(A_{11}+B_{11})=F(A_{11})+F(B_{11})$. Similarly, one can obtain $F(A_{22}+B_{22})=F(A_{22})+F(B_{22})$.

Claim 8. For any $A_{ij}\in\mathcal{A}_{ij}$, we have $F(A_{11}+A_{12}+A_{22})-F(A_{11})-F(A_{12})-F(A_{22})\in\mathcal{FI}$.

Take any $A_{ij}\in\mathcal{A}_{ij}$. If $n\geqslant 3$, note that

$$\begin{aligned} p_n(A_{11}+A_{12}+A_{22},B_{12},e_2,\cdots,e_2) &= p_n(A_{11}+A_{22},B_{12},e_2,\cdots,e_2)\\ &= A_{11}B_{12}-B_{12}A_{22}\in\mathcal{A}_{12}, \end{aligned}$$

then, by Lemma 2.4, Claim 5, 6 and Eq. (3.4), we have

$$\begin{aligned} &F(A_{11}B_{12}-B_{12}A_{22})=\delta(p_n(A_{11}+A_{12}+A_{22},B_{12},e_2,\cdots,e_2))\\ &=p_n(\delta(A_{11}+A_{12}+A_{22}),B_{12},e_2,\cdots,e_2)+p_n(A_{11}+A_{12}+A_{22},d(B_{12}),e_2,\cdots,e_2)\\ &=p_n(F(A_{11}+A_{12}+A_{22}),B_{12},e_2,\cdots,e_2)+p_n(A_{11}+A_{12}+A_{22},f(B_{12}),e_2,\cdots,e_2)\\ &=e_1[F(A_{11}+A_{12}+A_{22}),B_{12}]e_2+A_{11}f(B_{12})-f(B_{12})A_{22}. \end{aligned}$$

and

$$\begin{aligned} &F(A_{11}B_{12}-B_{12}A_{22})=\delta(p_n(A_{11}+A_{22},B_{12},e_2,\cdots,e_2))\\ &=\delta(p_n(A_{11},B_{12},e_2,\cdots,e_2)+p_n(A_{22},B_{12},e_2,\cdots,e_2))\\ &=p_n(F(A_{11}),B_{12},e_2,\cdots,e_2)+p_n(F(A_{22}),B_{12},e_2,\cdots,e_2)\\ &\quad+p_n(A_{11},f(B_{12}),e_2,\cdots,e_2)+p_n(A_{22},f(B_{12}),e_2,\cdots,e_2)\\ &=e_1[F(A_{11})+F(A_{22}),B_{12}]e_2+A_{11}f(B_{12})-f(B_{12})A_{22}. \end{aligned}$$

Comparing the above relations, one can obtain that for all $B_{12}\in\mathcal{A}_{12}$,

$$e_1[F(A_{11}+A_{12}+A_{22})-F(A_{11})-F(A_{22}),B_{12}]e_2=0. \tag{3.7}$$

When $n=2$, using the fact $[A_{11}+A_{12}+A_{22},B_{12}]=[A_{11}+A_{22},B_{12}]$, one can obtain that Eq. (3.7) still holds true. Then, one has

$$\begin{aligned} &e_1(F(A_{11}+A_{12}+A_{22})-F(A_{11})-F(A_{22}))e_1\\ &+e_2(F(A_{11}+A_{12}+A_{22})-F(A_{11})-F(A_{22}))e_2\in\mathcal{FI}. \end{aligned}$$

Similar argument to that of Claim 6, notice that

$$\begin{aligned} &e_1(F(A_{11}+A_{12}+A_{22})-F(A_{11})-F(A_{22}))e_2\\ &=p_n(F(A_{11}+A_{12}+A_{22})-F(A_{11})-F(A_{22},e_2,\cdots,e_2)\\ &=p_n(\delta(A_{11}+A_{12}+A_{22}),e_2,\cdots,e_2)-p_n(\delta(A_{11}),e_2,\cdots,e_2)-p_n(\delta(A_{22}),e_2,\cdots,e_2)\\ &=\delta(p_n(A_{11}+A_{12}+A_{22},e_2,\cdots,e_2)-p_n(A_{11},e_2,\cdots,e_2)-p_n(A_{22},e_2,\cdots,e_2))\\ &=\delta(p_n(A_{12},e_2,\cdots,e_2)=\delta(A_{12})=F(A_{12}). \end{aligned}$$

Thus, the claim is true.

Now, from Claim 8, we define two maps: $h: Alg\mathcal{L} \to \mathcal{F}I$ and $\psi: Alg\mathcal{L} \to Alg\mathcal{L}$ respectively, by

$$h(A) = F(A_{11}+A_{12}+A_{22}) - F(A_{11}) - F(A_{12}) - F(A_{22}) \text{ and } \psi(A) = F(A) - h(A) \tag{3.8}$$

for all $A = A_{11}+A_{12}+A_{22} \in Alg\mathcal{L}$. It is easy to see that for all $A_{ij} \in \mathcal{A}_{ij}$, $h(A_{ij}) = 0$. Hence from Claim 7, we get that for all $A_{ij} \in \mathcal{A}_{ij}$,

$$\psi(A_{11}+A_{12}+A_{22}) = \psi(A_{11}) + \psi(A_{12}) + \psi(A_{22}). \tag{3.9}$$

Thus, for all $A \in Alg\mathcal{L}$, one has

$$\delta(A) = F(A) + \theta(A) = \psi(A) + h(A) + \theta(A) = \psi(A) + \xi(A), \tag{3.10}$$

where $\xi \equiv h + \theta$ is a central-valued map on an irreducible completely distributive commutative subspace lattice algebra $Alg\mathcal{L}$.

Claim 9. ψ is an additive generalized derivation with associated derivation f.

Take any $A = A_{11}+A_{12}+A_{22}, B = B_{11}+B_{12}+B_{22} \in Alg\mathcal{L}$ $(A_{ij}, B_{ij} \in \mathcal{A}_{ij})$. By Claim 7 and Eqs. (3.8), (3.9), one obtains

$$\begin{aligned}
\psi(A+B) &= \psi(A_{11}+A_{12}+A_{22}+B_{11}+B_{12}+B_{22}) \\
&= \psi(A_{11}+B_{11}) + \psi(A_{12}+B_{12}) + \psi(A_{22}+B_{22}) \\
&= \psi(A_{11}) + \psi(B_{11}) + \psi(A_{12}) + \psi(B_{12}) + \psi(A_{22}) + \psi(B_{22}) \\
&= \psi(A_{11}+A_{12}+A_{22}) + \psi(B_{11}+B_{12}+B_{22}) = \psi(A) + \psi(B).
\end{aligned}$$

From Claims 4 and 5, Eqs. (3.4), (3.8) and (3.9), we have

$$\begin{aligned}
\psi(AB) &= \psi(A_{11}B_{11}+A_{11}B_{12}+A_{12}B_{22}+A_{22}B_{22}) \\
&= \psi(A_{11}B_{11}) + \psi(A_{11}B_{12}) + \psi(A_{12}B_{22}) + \psi(A_{22}B_{22}) \\
&= F(A_{11}B_{11}) + F(A_{11}B_{12}) + F(A_{12}B_{22}) + F(A_{22}B_{22}) \\
&= F(A_{11})B_{11} + A_{11}f(B_{11}) + F(A_{11})B_{12} + A_{11}f(B_{12}) \\
&\quad + F(A_{12})B_{22} + A_{12}f(B_{22}) + F(A_{22})B_{22} + A_{22}f(B_{22}) \\
&= (\psi(A_{11}) + \psi(A_{12}) + \psi(A_{22}))(B_{11}+B_{12}+B_{22}) \\
&\quad + (A_{11}+A_{12}+A_{22})(f(B_{11}) + f(B_{12}) + f(B_{22})) \\
&= \psi(A)B + Af(B).
\end{aligned}$$

This shows that G is an additive generalized derivation with associated derivation f. At last, following from Eq. (3.10), we only need prove that ξ sends each $(n-1)$-th commutator $p_n(A_1, A_2, \cdots, A_n)$ to zero.

Claim 10. $\xi(p_n(A_1, A_2, \cdots, A_n)) = 0$ holds for all $A_i \in Alg\mathcal{L}$.

Take any $A_i \in Alg\mathcal{L}$. By the above all claims, one has

$$\begin{aligned}\xi(p_n(A_1,A_2,\cdots,A_n)) &= \delta(p_n(A_1,A_2,\cdots,A_n)) - \psi(p_n(A_1,A_2,\cdots,A_n))\\ &= p_{n-1}([\delta(A_1),A_2],\cdots,A_n) + \sum_{i=2}^{n} p_n(A_1,\cdots,d(A_i),\cdots,A_n)\\ &\quad -p_{n-1}([\psi(A_1),A_2],\cdots,A_n) - \sum_{i=2}^{n} p_n(A_1,\cdots,f(A_i),\cdots,A_n)\\ &= p_{n-1}([F(A_1),A_2],\cdots,A_n) + \sum_{i=2}^{n} p_n(A_1,\cdots,f(A_i),\cdots,A_n)\\ &\quad -p_{n-1}([\psi(A_1),A_2],\cdots,A_n) - \sum_{i=2}^{n} p_n(A_1,\cdots,f(A_i),\cdots,A_n)\\ &= p_{n-1}([F(A_1)-\psi(A_1),A_2],\cdots,A_n) = 0\end{aligned}$$

The proof is completed. □

4. Main results

In this section, we study multiplicative generalized Lie n-derivations on completely distributive commutative subspace lattice algebras. The main result reads as follows.

THEOREM 4.1. *Let $Alg\mathcal{L}$ be an associated completely distributive commutative subspace lattice algebra on a complex Hilbert space $\mathcal{H}$ and $\delta : Alg\mathcal{L} \to Alg\mathcal{L}$ be a nonlinear map. Then δ is a multiplicative generalized Lie $n(\geqslant 2)$-derivation if and only if for every $A \in Alg\mathcal{L}$, $\delta(A) = \psi(A) + \xi(A)$, where $\psi : Alg\mathcal{L} \to Alg\mathcal{L}$ is an additive generalized derivation and $\xi : Alg\mathcal{L} \to Z(Alg\mathcal{L})$ vanishes on each $(n-1)$th commutator $p_n(A_1,A_2,\cdots,A_n)$.*

Proof. From the proof of Theorem 3.1, we know that only need to check the case that $\mathcal{L}$ is non-trivial.

Let $e_m = \vee\{e : e \in \mathcal{C}_m, m \in \Lambda\}$ be the projections of $\mathcal{L}$ as in Lemma 2.2. Following from Lemma 2.2, $Alg\mathcal{L} = \sum_{m\in\Lambda} \oplus (Alg\mathcal{L})e_m$ is the irreducible decomposition of $Alg\mathcal{L}$. Fixing an index m, we know that e_m is also a Hilbert space and

$$(Alg\mathcal{L})e_m = e_m(Alg\mathcal{L})e_m = Alg(e_m\mathcal{L}).$$

Then for each m, $Alg(e_m\mathcal{L})$ is an irreducible CDC algebra on a Hilbert space e_m. Let δ be a multiplicative generalized Lie $n(\geqslant 2)$-derivation on $Alg\mathcal{L}$. It follows from Theorem 3.1 that one can define two maps $\delta_m, d_m : Alg(e_m\mathcal{L}) \to Alg(e_m\mathcal{L})$ by

$$\delta(A) = \delta_m(A) = \psi_m(A) + \xi_m(A) \text{ and } d(A) = d_m(A) = f_m(A) + \gamma_m(A). \tag{4.1}$$

for all $A \in Alg(e_m\mathcal{L})$, where $\psi_m : Alg(e_m\mathcal{L}) \to Alg(e_m\mathcal{L})$ is an additive generalized derivation with associated derivation f_m, and $\xi_m : Alg(e_m\mathcal{L}) \to Z(Alg(e_m\mathcal{L}))$ is a central-valued map annihilating all $(n-1)$-th commutators $p_n(A_1,A_2,\cdots,A_n)$.

In [14], it turns out that for a CDC algebra, the algebra generated by all rank-one operators in $Alg\mathcal{L}$ is ultraweakly dense. Choosing a set $E\in\mathcal{U}(L)$, for any $x\in E$ and $y\in E_{-}^{\perp}$, by Lemma 2.1, one can obtain that $x\otimes y\in Alg\mathcal{L}$ is a rank-one operator. For every $u\otimes v\in Alg(e_m\mathcal{L})$ and $A\in Alg(e_m\mathcal{L})$, it follows from Theorem 3.1 that

$$\psi_m((u\otimes v)A(x\otimes y))=\psi_m(u\otimes v)A(x\otimes y)+(u\otimes v)f_m(A)(x\otimes y)+(u\otimes v)Af_m(x\otimes y). \tag{4.2}$$

Assuming that $\{A_k\}$ strongly converges to A, where $\{A_k\},A\in Alg(e_m\mathcal{L})$, it follows from (4.2) that

$$\begin{aligned}
&(u\otimes v)f_m(A_k)(x\otimes y)\\
=\ &\psi_m((u\otimes v)A_k(x\otimes y))-\psi_m(u\otimes v)A_k(x\otimes y)-(u\otimes v)A_kf_m(x\otimes y)\\
\to\ &\psi_m((u\otimes v)A(x\otimes y))-\psi_m(u\otimes v)A(x\otimes y)-(u\otimes v)Af_m(x\otimes y)\\
=\ &(u\otimes v)f_m(A)(x\otimes y).
\end{aligned}$$

This shows that f_m is strongly convergent, and then ψ_m is strongly convergent.

For $A^1,A^2,\cdots,A^n\in Alg\mathcal{L}$, we assume that $\{A_k^i\}$ strongly converges to A^i, respectively. Since $Alg\mathcal{L}=\sum_{m\in\Lambda}\oplus(Alg\mathcal{L})e_m$, and $\{e_m\}$ are pairwise orthogonal projections, then for every e_m, $\{A_k^ie_m\}$ strongly converges to A^ie_m, respectively, and

$$A_k^iA_k^j=(\sum_{m\in\Lambda}\oplus A_k^ie_m)(\sum_{i\in\Lambda}\oplus A_k^je_m)=\sum_{m\in\Lambda}\oplus A_k^iA_k^je_m.$$

Then, for every x in Hilbert space $\mathcal{H}$, it follows from the proof of Theorem 3.1 and Eq. (4.1) that

$$\begin{aligned}
&\delta(p_n(A_k^1,A_k^2,\cdots,A_k^n))x\\
=\ &\delta(p_n(\sum_{m\in\Lambda}\oplus A_k^1e_m,\sum_{m\in\Lambda}\oplus A_k^2e_m,\cdots,\sum_{m\in\Lambda}\oplus A_k^ne_m))x\\
=\ &(p_n(\delta(\sum_{m\in\Lambda}\oplus A_k^1e_m),\sum_{m\in\Lambda}\oplus A_k^2e_m,\cdots,\sum_{m\in\Lambda}\oplus A_k^ne_m)\\
&+\sum_{i=2}^{n}p_n(\sum_{m\in\Lambda}\oplus A_k^1e_m,\cdots,d(\sum_{m\in\Lambda}\oplus A_k^ie_m),\cdots,\sum_{m\in\Lambda}\oplus A_k^ne_m))x\\
=\ &(p_n(\sum_{m\in\Lambda}\oplus\psi_m(A_k^1)e_m,\sum_{m\in\Lambda}\oplus A_k^2e_m,\cdots,\sum_{m\in\Lambda}\oplus A_k^ne_m)\\
&+\sum_{i=2}^{n}p_n(\sum_{m\in\Lambda}\oplus A_k^1e_m,\cdots,\sum_{m\in\Lambda}\oplus f_m(A_k^i)e_m,\cdots,\sum_{m\in\Lambda}\oplus A_k^ne_m))x\\
=\ &(\sum_{m\in\Lambda}\oplus(p_n(\psi_m(A_k^1)e_m,A_k^2e_m,\cdots,A_k^ne_m)+\sum_{i=2}^{n}p_n(A_k^1e_m,\cdots,f_m(A_k^i)e_m,\cdots,A_k^ne_m))x\\
\to\ &\sum_{m\in\Lambda}\oplus(p_n(\psi_m(A^1)e_m,A^2e_m,\cdots,A^ne_m)+\sum_{i=2}^{n}p_n(A^1e_m,\cdots,f_m(A^i)e_m,\cdots,A^ne_m))x\\
=\ &\sum_{m\in\Lambda}\oplus(p_n(\delta_m(A^1)e_m,A^2e_m,\cdots,A^ne_m)+\sum_{i=2}^{n}p_n(A^1e_m,\cdots,d_m(A^i)e_m,\cdots,A^ne_m))x\\
=\ &\sum_{m\in\Lambda}\oplus\delta_m(p_n(A^1e_m,A^2e_m,\cdots,A^ne_m))x=\delta(p_n(A^1,A^2,\cdots,A^n))x.
\end{aligned}$$

It means that δ is strongly convergent on CDC algebra $Alg\mathcal{L}$. Thus, for every $A \in Alg\mathcal{L}$, we obtain that

$$\delta(A) = \sum_{m\in\Lambda} \oplus \delta_m(Ae_m) = \sum_{m\in\Lambda} \oplus(\psi_m(Ae_m) + \xi_m(Ae_m)).$$

Write $\psi(A) = \sum_{m\in\Lambda} \oplus \psi_m(Ae_m)$ and $\xi(A) = \sum_{m\in\Lambda} \oplus \xi_m(Ae_m)$, then we have $\delta = \psi + \xi$. The proof is completed. $\square$

5. Conclusions

In this paper, we use decomposition of algebraic structure and the properties of completely distributive commutative subspace lattice algebras to study the multiplicative generalized Lie n-derivation on certain CSL algebra. We proved that every multiplicative generalized Lie n-derivation on completely distributive commutative subspace lattice algebras is standard. Moreover, the purpose of this modification is to answer the classic problem of preserving mappings of some certain CSL algebra.

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