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A CONSTRUCTIVE PROOF OF A NONCOMMUTATIVE FEJÉR–RIESZ THEOREM

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Abstract. In this paper, we present a constructive proof of Popescu's Fejér-Riesz theorem for noncommuting polynomials representing nonnegative "multi-Toeplitz" operators.

1. Introduction

The classical Fejér-Riesz theorem states the following: if a trigonometric polynomial

$$w(e^{it}) = \sum_{j=-m}^{m} c_j e^{ijt}$$

is nonnegative for all real t, then it is expressible in the form

$$w(e^{it}) = |p(e^{it})|^2$$

for some analytic outer polynomial $p(z) = \sum_{j=0}^{m} a_j z^j$. (Concretely, a polynomial is *outer* if it has no zeroes in the open disk $|z| < 1$.)

For a proof, refer to Lemma 2.1 in [8]. There is also an operator version where the coefficients of w are matrices or operators ([10]), see also [2].

The Fejér-Riesz theorem can be reformulated as a statement about Toeplitz operators: the function w may be interpreted as the symbol of a *Toeplitz operator* T_w; in particular if S denotes the unilateral shift on $\ell^2(\mathbb{N})$ then T_w is the operator defined by

$$T_w = c_0 I + \sum_{k=1}^{m} c_k S^k + \sum_{k=1}^{m} c_{-k} S^{*k},$$

and then the factorization $w = |p|^2$ is equivalent to the factorization of operators

$$T_w = T_p^* T_p$$

where $T_p = \sum_{k=0}^{m} a_k S^k = p(S)$. The equivalence of the two formulations follows easily from the fact that S is an isometry.

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It turns out that this operator formulation admits a generalization in the noncommutative setting, to so-called *multi-Toeplitz* operators, where the single isometry S is replaced by a *row isometry*; for example, the d tuple of left shifts, $(L_1, L_2, \dots, L_d)$ or d tuple of right shifts, $(R_1, R_2, \dots, R_d)$. Precise definitions are given in the next section. Following an idea of Dritschel and Woerdeman [2], this paper develops a constructive proof of Popescu's Fejér-Riesz theorem in the noncommutative setting. We have a nonnegative multi-Toeplitz polynomial operator

$$T_Q := Q_0 \otimes I_{\ell^2(\mathcal{F}_d^+)} + \sum_{0<|v|\leqslant n} Q_v \otimes L_v + \sum_{0<|v|\leqslant n} Q_v^* \otimes L_v^*.$$

We then find a multi-Toeplitz operator factorization of this polynomial which is also outer,

$$T_Q := T_F^* T_F$$

where $T_F := \sum_{0\leqslant|v|\leqslant n} F_v \otimes L_v$ [This is a slight rewording of Theorem 1.6 in [9]]. There are several such results in the same spirit ([6], [4], [7]), depending on which operators were used to define the positivity condition; for example, in [6], McCullough has considered two different factorization theorems with positivity defined by testing on either unitary operators or self-adjoint operators.

2. Preliminaries

We begin by recalling some relevant definitions:

DEFINITION 2.1. Let $\mathcal{F}_d^+$ denote the free monoid on d letters. This is the set of all finite words in the letters $1, 2, \dots, d$, including the empty word. We write $|w|$ for the *length* of the word, that is, the total number of letters that appear. We write $\varnothing$ for the empty word; by convention $|\varnothing| = 0$. The *Fock space*, $\ell^2(\mathcal{F}_d^+)$ is the Hilbert space with orthonormal basis indexed by $\mathcal{F}_d^+$: we write ξ_w for the basis vector labeled by the word $w \in \mathcal{F}_d^+$. When $d = 1$, $\mathcal{F}_d^+$ is identified with the natural numbers $\mathbb{N}$.

DEFINITION 2.2. The *left shift operator* L_j is defined as

$$L_j \xi_w = \xi_{jw}$$

for $j = 1, \dots, d$ and extending linearly. Similarly, the *left backward shift operator* L_j^* is defined as

$$L_j^* \xi_w = \begin{cases} \xi_v, & w = jv \\ 0, & \text{otherwise} \end{cases}$$

for $j = 1, \dots, d$. Thus $\{L_j | j = 1, \dots, d\}$ forms a system of isometries with orthogonal ranges:

$$L_i^* L_j = \delta_{ij} I. \tag{1}$$

Analogously we can define the *right shift operator* R_j by

$$R_j \xi_w = \xi_{wj}$$

for $j = 1, \dots, d$ and extending linearly. Similarly, the *right backward shift operator* R_j^* is defined as

$$R_j^* \xi_w = \begin{cases} \xi_v, & w = vj \\ 0, & \text{otherwise} \end{cases}$$

for $j = 1, \dots, d$. Note that left shifts and right shifts commute with each other; that is, $L_i R_j = R_j L_i$ for all $i, j = 1, \dots, d$.

From the above definition we get that $(L_1, L_2, \dots, L_d)$ and $(R_1, R_2, \dots, R_d)$ are row-isometries (this just means that equation (1) holds).

For any $w \in \mathcal{F}_d^+$, $w = i_1 i_2 \cdots i_n$ we denote $L_w = L_{i_1} L_{i_2} \cdots L_{i_n}$. So for any $w = i_1 i_2 \cdots i_n$ and $v = j_1 j_2 \cdots j_m$ in $\mathcal{F}_d^+$, $L_w^* L_v = L_{i_n}^* \cdots L_{i_2}^* L_{i_1}^* L_{j_1} \cdots L_{j_m}$. Thus

$$L_w^* L_v = \begin{cases} L_x, & \text{if } v = wx \\ L_y^*, & \text{if } w = vy \\ 0, & \text{otherwise} \end{cases}.$$

DEFINITION 2.3. In the classical setting, T is said to be a *Toeplitz operator* if $S^* T S = T$ where S is a unilateral shift. An operator T is *L-multi-Toeplitz* if $L_i^* T L_j = \delta_{ij} T$ where L_j is a left shift operator. Similarly, T is called *R-multi-Toeplitz* if $R_i^* T R_j = \delta_{ij} T$ where R_j is a right shift operator.

EXAMPLE 1. Any left shift operator L_w is R-multi-Toeplitz. Indeed, since L_i and R_j commute with each other, R_j commutes with L_w for all w and thus we have

$$\begin{aligned} R_i^* L_w R_j &= R_i^* R_j L_w \\ &= \delta_{ij} L_w. \end{aligned}$$

Similarly, L_v^* is R-multi-Toeplitz for any word v. Therefore for any noncommutative polynomials f, g, we have that $f(L)^* + g(L)$ is R-multi-Toeplitz. Moreover, a multi-Toeplitz polynomial of the form $g(L)$ is called *analytic*.

Next let us consider an R-multi-Toeplitz polynomial operator with scalar coefficients, say,

$$T := \sum_{0 \leqslant |v| \leqslant n} q_v L_v + \sum_{0 < |v| \leqslant n} q_v^* L_v^*. \tag{2}$$

Then corresponding to the Fock space basis $\{\xi_v\}_{v \in \mathcal{F}_d^+}$ we get its matrix representation which is a multi-Toeplitz matrix:

$$\left[\begin{array}{c|ccc|c|ccccccc|cccc}
q_0 & q_1^* & \cdots & q_d^* & \cdots & q_{11\cdots1}^* & \cdots & q_{1d\cdots d}^* & \cdots & q_{d1\cdots1}^* & \cdots & q_{dd\cdots d}^* & 0 & \cdots & 0 & \cdots \\
\hline
q_1 & q_0 & \cdots & 0 & \cdots & q_{1\cdots1}^* & \cdots & 0 & \cdots & q_{d\cdots d}^* & \cdots & 0 & q_{11\cdots1}^* & \cdots & 0 & \cdots \\
\vdots & \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots \\
q_d & 0 & \cdots & q_0 & \cdots & 0 & \cdots & q_{1\cdots1}^* & \cdots & 0 & \cdots & q_{d\cdots d}^* & 0 & \cdots & q_{dd\cdots d}^* & \ddots \\
\hline
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\
\hline
q_{11\cdots1} & q_{1\cdots1} & \cdots & 0 & \cdots & q_0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & q_1^* & \cdots & 0 & \ddots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & & \vdots & \vdots & \ddots & \vdots & \ddots \\
q_{1d\cdots d} & 0 & \cdots & q_{1\cdots1} & \cdots & 0 & \cdots & q_0 & \cdots & 0 & \cdots & 0 & 0 & & 0 & \\
\vdots & & \ddots & & \ddots & \vdots & & & \ddots & & & \vdots & \vdots & \ddots & \vdots & \ddots \\
q_{d1\cdots1} & q_{d\cdots d} & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & q_0 & \cdots & 0 & 0 & & 0 & \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & & & & & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots \\
q_{dd\cdots d} & 0 & \cdots & q_{d\cdots d} & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & q_0 & 0 & \cdots & q_d^* & \\
\hline
0 & q_{11\cdots1} & \cdots & 0 & \cdots & q_1 & \cdots & 0 & \cdots & 0 & \cdots & 0 & q_0 & \cdots & 0 & \\
\vdots & & \ddots & & \ddots & \vdots & \ddots & & \ddots & & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots \\
0 & 0 & \cdots & q_{dd\cdots d} & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & q_d & 0 & \cdots & q_0 & \\
\hline
\vdots & & \ddots & & \ddots & & \ddots & & \ddots & & \ddots & & & \ddots & & \ddots
\end{array}\right].$$

Here we have used lexicographic ordering for ordering the elements of the word set, $\mathcal{F}_d^+$. Also, it is convenient to view the basis $\{\xi_w\}$ as partitioned into blocks according to the length of the word w, which induces a corresponding block structure in the above matrix.

Now we do some relabeling of the indexes here and define for $d = 1, \ldots, n$:

$$q_k := col(q_w)_{w\in\mathcal{F}_d^+;|w|=k} \text{ and } q_{-k} := row(q_w^*)_{w\in\mathcal{F}_d^+;|w|=k},$$

and also identifying

$$q_1 \otimes I_d := \begin{pmatrix} q_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & q_1 \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ q_d & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & q_d \end{pmatrix}$$

and so on. Thus we get the following compact form of T, which makes it easier to see the multi-Toeplitz form of the matrix:

$$T = \begin{bmatrix} q_0 & q_{-1} & q_{-2} & \cdots & q_{-n} & 0 & 0 & \cdots \\ q_1 & q_0 \otimes I_d & q_{-1} \otimes I_d & \cdots & q_{-(n-1)} \otimes I_d & q_{-n} \otimes I_d & 0 & \cdots \\ q_2 & q_1 \otimes I_d & q_0 \otimes I_d \otimes I_d & \cdots & q_{-(n-2)} \otimes I_d \otimes I_d & q_{-(n-1)} \otimes I_d \otimes I_d & q_{-n} \otimes I_d^{\otimes 2} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\ q_n & q_{n-1} \otimes I_d & q_{n-2} \otimes I_d \otimes I_d & \cdots & q_0 \otimes I_d \otimes \cdots \otimes I_d & q_{-1} \otimes I_d^{\otimes n} & q_{-2} \otimes I_d^{\otimes n} & \ddots \\ 0 & q_n \otimes I_d & q_{n-1} \otimes I_d \otimes I_d & \cdots & q_1 \otimes I_d^{\otimes n} & q_0 \otimes I_d^{\otimes (n+1)} & q_{-1} \otimes I_d^{\otimes (n+1)} & \ddots \\ 0 & 0 & q_n \otimes I_d \otimes I_d & \cdots & q_2 \otimes I_d^{\otimes n} & q_1 \otimes I_d^{\otimes (n+1)} & q_0 \otimes I_d^{\otimes (n+2)} & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

We are going to use a Schur complement technique from Dritschel and Woerdeman [2] in the proof of the main theorem. So let us define the following:

DEFINITION 2.4. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and

$$M = \begin{pmatrix} A & B^* \\ B & C \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2$$

is a positive semidefinite operator, then there exists a unique contraction $G : \overline{ran}(C) \to \overline{ran}(A)$ such that $B = A^{1/2} G C^{1/2}$. The *Schur complement of M supported on $\mathcal{H}_1$* is defined to be positive semidefinite operator $A^{1/2}(1 - GG^*)A^{1/2}$.

An alternative way to define the Schur complement of M supported on $\mathcal{H}_1$ is via

$$\langle Sf, f \rangle = \inf \left\{ \left\langle \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle : g \in \mathcal{H}_2 \right\};$$

that is, $S : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ is the largest positive semidefinite operator which may be subtracted from A in M such that the resulting operator matrix remains positive semidefinite.

REMARK. Consider any positive semidefinite operator matrix, M, say

$$M = \begin{pmatrix} A & B^* \\ B & C \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2$$

and let S_M be the Schur complement of M supported on $\mathcal{H}_1$. Then for the positive semidefinite matrix $M \otimes I_d$, the Schur complement supported on $\mathcal{H}_1 \otimes \mathbb{C}^d$ is $S_{M \otimes I_d} = S_M \otimes I_d$.

We conclude this section with some notation: if Q is an operator from $\mathcal{H}$ to $\mathcal{H}$ for some Hilbert space $\mathcal{H}$, then we understand $Q \otimes I_d$ to be an operator from $\mathcal{H} \otimes \mathbb{C}^d$ to $\mathcal{H} \otimes \mathbb{C}^d$. We will write $\mathcal{H}_i := \mathcal{H} \otimes (\mathbb{C}^d)^{\otimes i}$ for $i \geqslant 0$.

In addition, we make use of the following notation from [2]: Typically, we will index rows and columns of an $n \times n$ matrix with $0, \dots, n-1$. For $\Lambda \subseteq \{0, \dots, n -$

1} and an $n \times n$ matrix M, we write $S(M;\Lambda)$, or $S(\Lambda)$ when there is no chance of confusion, for the Schur complement supported on the rows and columns labeled by elements of Λ. It is usual to view $S(\Lambda)$ as an $m \times m$ matrix, where $m = \text{card}\Lambda$, however it is often useful to take $S(\Lambda)$ as an $n \times n$ matrix by padding the rest of the entries in this $n \times n$ matrix with zeros. For notational convenience we have used $S(m)$ for the Schur complement supported on rows and columns labeled by $\{0, \dots, m\}$.

3. Main Theorem

Theorem 3.1 and Corollary 3.2 below are multi-Toeplitz versions of Proposition 3.1 and Corollary 3.2 from [2].

THEOREM 3.1. *Consider the positive semidefinite multi-Toeplitz operator matrix*

$$T_Q = \begin{bmatrix} Q_0 & Q_{-1} & Q_{-2} & \cdots\cdots \\ Q_1 & Q_0 \otimes I_d & Q_{-1} \otimes I_d & \cdots\cdots \\ Q_2 & Q_1 \otimes I_d & Q_0 \otimes I_d \otimes I_d & \cdots\cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

acting on $\mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots$. *Then*

1. *The Schur complements of* T_Q *satisfy the reccurrence relation:*

$$S(m) = \begin{bmatrix} A & B^* \\ B & S(m-1) \otimes I_d \end{bmatrix} \qquad (*)$$

for appropriate choice of $A : \mathcal{H}_0 \to \mathcal{H}_0$ *and* $B^* : \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \to \mathcal{H}_0$.

2. *Suppose in addition that* $Q_j = 0$ *for* $j \geqslant m+1$ *(for some fixed positive integer* m*). Then the recursion formula (*), will hold with* $A = Q_0$ *and* $B = col(Q_i)_{i=1}^m$.

REMARK. Given

$$T_Q = \begin{bmatrix} Q_0 & Q_{-1} & Q_{-2} & \cdots\cdots \\ Q_1 & Q_0 \otimes I_d & Q_{-1} \otimes I_d & \cdots\cdots \\ Q_2 & Q_1 \otimes I_d & Q_0 \otimes I_d \otimes I_d & \cdots\cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

we observe that T_Q can be identified with

$$T_Q = \begin{bmatrix} Q_0 & row(Q_{-j})_{j\geqslant 1} \\ col(Q_j)_{j\geqslant 1} & T_Q \otimes I_d \end{bmatrix}.$$

Proof of Theorem 3.1. Let us write

$$S(m) = \begin{bmatrix} A & B^* \\ B & C \end{bmatrix} : \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_m \to \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_m.$$

By definition of Schur complement, we have that

$$T_Q - \begin{bmatrix} S(m) & 0 \\ 0 & 0 \end{bmatrix} \geqslant 0.$$

That is, we have that

$$\left[\begin{array}{c|c} Q_0 & row(Q_{-j})_{j\geqslant 1} \\ \hline col(Q_j)_{j\geqslant 1} & T_Q \otimes I_d \end{array}\right] - \left[\begin{array}{c|cc} A & B^* & 0 \\ \hline B & C & 0 \\ 0 & 0 & 0 \end{array}\right] \geqslant 0. \tag{3}$$

Then leaving out 0th row and column in (3) we get,

$$T_Q \otimes I_d - \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} \geqslant 0.$$

So we have again from the definition of Schur complement that $C \leqslant S(m-1) \otimes I_d$. Now leaving out the rows and columns $1,\dots,m$ in (3), we get

$$\begin{bmatrix} Q_0 - A & row(Q_j^*)_{j\geqslant m+1} \\ col(Q_j)_{j\geqslant m+1} & T_Q \otimes I_d^{\otimes(m+1)} \end{bmatrix} \geqslant 0.$$

That is,

$$A \leqslant S\left(\begin{bmatrix} Q_0 & row(Q_j^*)_{j\geqslant m+1} \\ col(Q_j)_{j\geqslant m+1} & T_Q \otimes I_d^{\otimes(m+1)} \end{bmatrix} ; 0 \right) := \tilde{A}.$$

Note that when $Q_j = 0$ for all $j \geqslant m+1$, then $\tilde{A} = Q_0$.
Next consider the following operator matrix:

$$\begin{bmatrix} Q_0 - \tilde{A} & X & row(Q_j^*)_{j\geqslant m+1} \\ X^* & (Q_{i-j}\otimes I_d)_{i=1,j=1}^{m} - S(m-1)\otimes I_d & (Q_{i-j}\otimes I_d)_{i=1,j=m+1}^{m+1,\infty} \\ col(Q_j)_{j\geqslant m+1} & (Q_{i-j}\otimes I_d)_{i=m+1,j=1}^{\infty,m} & T_Q \otimes I_d^{\otimes(m+1)} \end{bmatrix}. \tag{4}$$

The existence of an operator X making this into a positive semidefinite matrix is a variant of a standard operator matrix completion problem (see Theorem XVI.3.1 in [3]), so there always exists such an X. So, we fix such an X. (Note that when $\tilde{A} = Q_0$ we have necessarily that $X = 0$.) Now since (4) is positive semidefinite, we obtain that

$$\begin{bmatrix} \tilde{A} & row(Q_j^*)_{j=1}^{m} - X \\ col(Q_j)_{j=1}^{m} - X & S(m-1)\otimes I_d \end{bmatrix} \leqslant S(m) = \begin{bmatrix} A & B^* \\ B & C \end{bmatrix}.$$

This implies that $\tilde{A} \leqslant A$ and $S(m-1)\otimes I_d \leqslant C$. From the first part of the proof we also have $A \leqslant \tilde{A}$ and $C \leqslant S(m-1)\otimes I_d$, and thus the equalities $A = \tilde{A}$ and $C = S(m-1)\otimes I_d$ follow. Moreover, if $Q_j = 0$ for $j \geqslant m+1$, we have that $\tilde{A} = Q_0$ and $X = 0$, and thus $B = col(Q_i)_{i=1}^{m}$. □

COROLLARY 3.2. *Consider the positive semidefinite multi-Toeplitz operator matrix*

$$T_Q = \begin{bmatrix} Q_0 & Q_{-1} & Q_{-2} & \cdots\cdots \\ Q_1 & Q_0\otimes I_d & Q_{-1}\otimes I_d & \cdots\cdots \\ Q_2 & Q_1\otimes I_d & Q_0\otimes I_d\otimes I_d & \cdots\cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

acting on $\mathcal{H}_0\oplus\mathcal{H}_1\oplus\mathcal{H}_2\oplus\cdots$. *Then for each* $m\geqslant 0$, *there exist operators* $F_0,\ F_1,\ \ldots$ *with*

$$F_{m-k}\otimes I_d^{\otimes k} : \mathcal{H}_k \longrightarrow \overline{ran}F_0\otimes\mathbb{C}^m \subseteq \mathcal{H}\otimes\mathbb{C}^m$$

for $0\leqslant k\leqslant (m-1)$, *so that the Schur complements* $S(m)$ *of* T_Q *satisfy*

$$S(m) = \begin{bmatrix} F_0^* & F_1^* & F_2^* & \cdots & F_m^* \\ & F_0^*\otimes I_d & F_1^*\otimes I_d & \cdots & F_{m-1}^*\otimes I_d \\ & & F_0^*\otimes I_d^{\otimes 2} & \cdots & F_{m-2}^*\otimes I_d^{\otimes 2} \\ & & & \ddots & \vdots \\ & & & & F_0^*\otimes I_d^{\otimes m} \end{bmatrix} \begin{bmatrix} F_0 & & & & \\ F_1 & F_0\otimes I_d & & & \\ F_2 & F_1\otimes I_d & F_0\otimes I_d^{\otimes 2} & & \\ \vdots & \vdots & \vdots & \ddots & \\ F_m & F_{m-1}\otimes I_d & F_{m-2}\otimes I_d^{\otimes 2} & \cdots & F_0\otimes I_d^{\otimes m} \end{bmatrix}.$$

Proof. We will prove this by induction on m.

Base step: $S(0)$ being a positive semidefinite operator, we can write $S(0) = F_0^*F_0$ where $F_0 = (S(0))^{1/2}$.

Induction hypothesis: Let us assume that the result holds for $S(m-1)$.

By [2], Proposition 3.1, we have that $(S(m))_{m,m} = (S(m-1))_{m-1,m-1}\otimes I_d = F_0^*F_0\otimes I_d^{\otimes(m-1)}\otimes I_d = F_0^*F_0\otimes I_d^{\otimes m}$.

From [2] Corollary 2.3, we have that $S(m-1) = S(S(m);m-1)$. Thus applying Lemma 2.1 from [2] to

$$P = \begin{bmatrix} F_0 & & & & \\ F_1 & F_0\otimes I_d & & & \\ F_2 & F_1\otimes I_d & F_0\otimes I_d^{\otimes 2} & & \\ \vdots & \vdots & \vdots & \ddots & \\ F_{m-1} & F_{m-2}\otimes I_d & F_{m-3}\otimes I_d^{\otimes 2} & \cdots & F_0\otimes I_d^{\otimes(m-1)} \end{bmatrix}, \quad R = F_0\otimes I_d^{\otimes m}, \tag{5}$$

we get, there exist $(G_m\cdots G_1)$ so that

$$S(m) = \begin{bmatrix} F_0^* & F_1^* & F_2^* & \cdots & G_m^* \\ & F_0^*\otimes I_d & F_1^*\otimes I_d & \cdots & G_{m-1}^* \\ & & F_0^*\otimes I_d^{\otimes 2} & \cdots & G_{m-2}^* \\ & & & \ddots & \vdots \\ & & & & F_0^*\otimes I_d^{\otimes m} \end{bmatrix} \begin{bmatrix} F_0 & & & & \\ F_1 & F_0\otimes I_d & & & \\ F_2 & F_1\otimes I_d & F_0\otimes I_d^{\otimes 2} & & \\ \vdots & \vdots & \vdots & \ddots & \\ G_m & G_{m-1} & G_{m-2} & \cdots & F_0\otimes I_d^{\otimes m} \end{bmatrix},$$

and $ran(G_m \cdots G_1) \subseteq \overline{ran} F_0 \otimes (\mathbb{C}^d)^{\otimes m}$. Comparing with $S(m) = \begin{bmatrix} A & B^* \\ B & S(m-1)\otimes I_d \end{bmatrix}$ along with the induction hypothesis, we deduce that $S(m-1)\otimes I_d$ factors into

$$\begin{bmatrix} F_0^*\otimes I_d & F_1^*\otimes I_d & \cdots & F_{m-2}^*\otimes I_d & G_{m-1}^* \\ & F_0^*\otimes I_d^{\otimes 2} & \cdots & F_{m-3}^*\otimes I_d^{\otimes 2} & G_{m-2}^* \\ & & \ddots & \vdots & \vdots \\ & & & F_0^*\otimes I_d^{\otimes(m-1)} & G_1^* \\ & & & & F_0^*\otimes I_d^m \end{bmatrix} \begin{bmatrix} F_0\otimes I_d & & & & \\ F_1\otimes I_d & F_0\otimes I_d^{\otimes 2} & & & \\ F_2\otimes I_d & F_1\otimes I_d^{\otimes 2} & F_0\otimes I_d^{\otimes 3} & & \\ \vdots & \vdots & \vdots & \ddots & \\ G_{m-1} & G_{m-2} & G_{m-3} & \cdots & F_0\otimes I_d^{\otimes m} \end{bmatrix}$$

$$= \begin{bmatrix} F_0^*\otimes I_d & F_1^*\otimes I_d & \cdots & F_{m-2}^*\otimes I_d & F_{m-1}^*\otimes I_d \\ & F_0^*\otimes I_d^{\otimes 2} & \cdots & F_{m-3}^*\otimes I_d^{\otimes 2} & F_{m-2}^*\otimes I_d^{\otimes 2} \\ & & \ddots & \vdots & \vdots \\ & & & F_0^*\otimes I_d^{\otimes(m-1)} & F_1^*\otimes I_d^{\otimes(m-1)} \\ & & & & F_0^*\otimes I_d^{\otimes m} \end{bmatrix}$$

$$\times \begin{bmatrix} F_0\otimes I_d & & & & \\ F_1\otimes I_d & F_0\otimes I_d^{\otimes 2} & & & \\ F_2\otimes I_d & F_1\otimes I_d^{\otimes 2} & F_0\otimes I_d^{\otimes 3} & & \\ \vdots & \vdots & \vdots & \ddots & \\ F_{m-1}\otimes I_d & F_{m-2}\otimes I_d^{\otimes 2} & F_{m-3}\otimes I_d^{\otimes 3} & \cdots & F_0\otimes I_d^{\otimes m} \end{bmatrix},$$

and thus, equating the last rows of each of these products, we have

$$F_0^*\otimes I_d^{\otimes m}(G_{m-1} \quad G_{m-2} \quad \cdots \quad G_1) = F_0^*\otimes I_d^{\otimes m}(F_{m-1}\otimes I_d \quad F_{m-2}\otimes I_d^{\otimes 2} \cdots F_1\otimes I_d^{\otimes(m-1)}).$$

As

$$ran(G_{m-1} \quad G_{m-2} \quad \cdots \quad G_1) \subseteq \overline{ran} F_0 \otimes (\mathbb{C}^d)^{\otimes m}$$

and

$$ran(F_{m-1}\otimes I_d \quad F_{m-2}\otimes I_d^{\otimes 2} \quad F_{m-3}\otimes I_d^{\otimes 3} \cdots F_1\otimes I_d^{\otimes(m-1)}) \subseteq \overline{ran} F_0 \otimes (\mathbb{C}^d)^{\otimes m},$$

it follows that $G_j = F_j \otimes I_d^{\otimes(m-j)}$ for $j = 1,\cdots,m-1$. By setting $F_m := G_m$, we have our result. □

THEOREM 3.3. *Let* $(L_1, L_2, \ldots, L_d)$ *be the left* d*-shift, let* $\mathcal{H}$ *be a Hilbert space, and let* $\{Q_w\}_{|w|\leqslant n}$ *be operators* $Q_w : \mathcal{H} \to \mathcal{H}$ *such that the operator* $T_Q : \mathcal{H}\otimes \ell^2(\mathcal{F}_d^+) \to \mathcal{H}\otimes \ell^2(\mathcal{F}_d^+)$ *given by*

$$T_Q := Q_0 \otimes I_{\ell^2(\mathcal{F}_d^+)} + \sum_{0<|v|\leqslant n} Q_v \otimes L_v + \sum_{0<|v|\leqslant n} Q_v^* \otimes L_v^*$$

is positive semidefinite. Then there exist operators $F_0, \ldots, F_w : \mathcal{H} \to \mathcal{H}$, $|w| \leqslant n$, *such that for* $T_F := \sum_{0\leqslant|v|\leqslant n} F_v \otimes L_v$ *we have* $T_Q = T_F^* T_F$.

Proof. Note that this operator T_Q is an R-multi-Toeplitz operator. Let us consider the matrix representation T_Q in the basis $\beta = \{e_i \otimes \xi^v\}_{i,v}$, where $\{e_i\}_i$ is some fixed orthonormal basis for $\mathcal{H}$ and $v \in \mathcal{F}_d^+$. It is a multi-Toeplitz operator matrix, just like the one corresponding to the polynomial (2), but with operator entries Q_j instead. Then as done before, after relabeling the indexes, we get the multi-Toeplitz matrix:

$$T_Q = \begin{bmatrix} Q_0 & Q_{-1} & Q_{-2} & \cdots & Q_{-n} & 0 & 0 & \cdots \\ Q_1 & Q_0 \otimes I_d & Q_{-1} \otimes I_d & \cdots & Q_{-(n-1)} \otimes I_d & Q_{-n} \otimes I_d & 0 & \cdots \\ Q_2 & Q_1 \otimes I_d & Q_0 \otimes I_d \otimes I_d & \cdots & Q_{-(n-2)} \otimes I_d \otimes I_d & Q_{-(n-1)} \otimes I_d \otimes I_d & Q_{-n} \otimes I_d^{\otimes 2} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\ Q_n & Q_{n-1} \otimes I_d & Q_{n-2} \otimes I_d \otimes I_d & \cdots & Q_0 \otimes I_d \otimes \cdots \otimes I_d & Q_{-1} \otimes I_d^{\otimes n} & Q_{-2} \otimes I_d^{\otimes n} & \ddots \\ 0 & Q_n \otimes I_d & Q_{n-1} \otimes I_d \otimes I_d & \cdots & Q_1 \otimes I_d^{\otimes n} & Q_0 \otimes I_d^{\otimes (n+1)} & Q_{-1} \otimes I_d^{\otimes (n+1)} & \ddots \\ 0 & 0 & Q_n \otimes I_d \otimes I_d & \cdots & Q_2 \otimes I_d^{\otimes n} & Q_1 \otimes I_d^{\otimes (n+1)} & Q_0 \otimes I_d^{\otimes (n+2)} & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

In this case, we have that $Q_j = 0$ for $|j| \geqslant n+1$.

Now consider the Schur complement, $S(n) = \begin{bmatrix} A & B^* \\ B & C \end{bmatrix}$ (that is, supported on first $n+1$ rows and columns of T_Q). Then from Theorem 3.1 we get that

$$A = Q_0 \text{ and } B = col(Q_i)_{i=1}^n.$$

Comparing the first row of $S(n)$ above with the first row of the product in Corollary 3.2 factorization we get that:

$$Q_0 = \sum_{j=0}^n F_j^* F_j;$$

$$Q_{-j} = \sum_{k=j}^n F_k^* (F_{k-j} \otimes I_d^{\otimes j}) \text{ for } 1 \leqslant j \leqslant n.$$

(These F_i for $i = 0, \dots, n$ are the ones provided by Corollary 3.2.) From the multi-Toeplitz structure and self-adjointness of the matrix T_Q, the above equations give us the following factorization:

$$T_Q = T_F^* T_F$$

where

$$T_F = \begin{pmatrix} F_0 & & & & & & & \\ F_1 & F_0 \otimes I_d & & & & & & \\ F_2 & F_1 \otimes I_d & F_0 \otimes I_d^{\otimes 2} & & & & & \\ \vdots & \vdots & \vdots & \ddots & & & & \\ F_n & F_{n-1} \otimes I_d & F_{n-2} \otimes I_d^{\otimes 2} & \cdots & F_0 \otimes I_d^{\otimes n} & & & \\ 0 & F_n \otimes I_d & F_{n-1} \otimes I_d^{\otimes 2} & \cdots & F_1 \otimes I_d^{\otimes n} & F_0 \otimes I_d^{\otimes (n+1)} & & \\ 0 & 0 & F_n \otimes I_d^{\otimes 2} & \cdots & F_2 \otimes I_d^{\otimes n} & F_1 \otimes I_d^{\otimes (n+1)} & F_0 \otimes I_d^{\otimes (n+2)} & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

In this compact representation of T_F, recall $F_i = col(F_v)_{|v|=i}$. Thus the operator corresponding to the above matrix is,

$$T_F = \sum_{0 \leqslant |v| \leqslant n} F_v \otimes L_v.$$

The theorem is proved. □

The last step is to prove that the above factorization is outer. Let us first define what an outer function is:

DEFINITION 3.1. Let F be analytic; that is, $F(\xi) = \sum_{\substack{v \in \mathcal{F}_d^+ \\ |v| \leqslant n}} F_v \xi_v$, where n is some positive integer, $F_w \in B(\mathcal{H})$ for each $w \in \mathcal{F}_d^+$ and suppose that $T_F : \mathcal{H} \otimes \ell^2(\mathcal{F}_d^+) \longrightarrow \mathcal{H} \otimes \ell^2(\mathcal{F}_d^+)$ is defined as

$$T_F = \sum_{0 \leqslant |v| \leqslant n} F_v \otimes L_v.$$

We say F is *outer* if there exists a closed subspace, $\mathcal{M} \subset \mathcal{H}$ such that

$$\overline{ran}(T_F) = \mathcal{M} \otimes \ell^2(\mathcal{F}_d^+).$$

This above definition is equivalent to Popescu's definition in [9]. Let us also remark that in the case of scalar coefficients, F is outer if and only if $F(\xi)$ has no zeroes in the *row ball*. This means that whenever $(X_1, \ldots, X_d)$ are any $n \times n$ matrices with $\|\sum X_j X_j^*\| < 1$, then the matrix $F(X)$ is nonsingular. See [5] and [1].

PROPOSITION 3.4. *Under the same hypothesis as in Theorem 3.3, the F obtained from this theorem is outer.*

Proof. The proof is a multi-Toeplitz version of (i)-(iv) from Theorem 3.3 in [2]. From our Corollary 3.2, we get that

$$S(T_F^* T_F; 0) = S(T_Q; 0) = S(S(n); 0) = F_0^* F_0.$$

Then applying Lemma 2.1 from [2] we get that, $ran(col(F_j)_{j \geqslant 1}) \subset \overline{ran}(T_F \otimes I_d)$. If $h \in \mathcal{H}$, then $col(F_j h)_{j \geqslant 1} \in \overline{ran}(T_Q \otimes I_d)$. Thus there exists a sequence of vectors $(g_n)_n$ so that $\lim_{n \to \infty} (T_F \otimes I_d) g_n = col(F_j h)_{j \geqslant 1}$. But then

$$\lim_{n \to \infty} \begin{pmatrix} F_0 & 0 \\ col(F_j)_{j \geqslant 1} & (T_F \otimes I_d) \end{pmatrix} \begin{pmatrix} h \\ -g_n \end{pmatrix} = \begin{pmatrix} F_0 h \\ 0 \end{pmatrix}.$$

Thus $F_0 h \in \overline{ran} T_F$. Hence $\overline{ran}(F_0) \otimes \ell^2(\mathcal{F}_d^+) \subset \overline{ran} T_F$.

Also $ran(col(F_j)_{j \geqslant 1}) \subset \overline{ran}(T_F \otimes I_d)$ implies that

$$ran(F_j) \subset \overline{ran}(F_{j-1} \otimes I_d \quad F_{j-2} \otimes I_d^{\otimes 2} \quad F_{j-3} \otimes I_d^{\otimes 3} \cdots F_0 \otimes I_d^{\otimes j}).$$

This gives us $ran(F_j) \subset \overline{ran}(F_0 \otimes I_d^{\otimes j})$ for each $j \geqslant 1$. Recall that $F_j = col(F_w)_{|w|=j}$ and so we get that for all $w \in \mathcal{F}_d^+$, $ran(F_w) \subset \overline{ran}(F_0)$. Therefore, $\overline{ran}(F_0) \otimes \ell^2(\mathcal{F}_d^+) = \overline{ran} T_F$. Finally, from the Definition 3.1 of an outer function, it follows that F is outer. $\square$

Analogously, we have similar results for polynomial operators in right shifts.

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