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ISOLATION AMONGST COMPOSITION OPERATORS ON $L^p(\mu)$–SPACES $(1 \leqslant p \leqslant \infty)$

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Abstract. In this paper we show that each composition operator is isolated in $comp(L^p)$ $(1 \leqslant p \leqslant \infty)$ under the norm topology. We also prove that while each composition operator is non-isolated in $comp(\ell^p)$ $(1 \leqslant p < \infty)$ under the strong operator topology, every composition operator is isolated in $comp(L^\infty)$ under the strong operator topology.

1. Introduction

The study of composition operators first arose in B. O. Koopman's formulations of classical mechanics in 1931 [12]. These operators were first studied on the space of analytic functions by J. V. Ryff [21]. However, the word "Composition Operator" appeared for the first time in the work of Nordgren in 1968 [18]. There are at least two distinct important settings in which the study of composition operator has been done extensively, namely: (i) composition operators defined on the space of holomorphic functions and (ii) composition operators defined on the $L^p(\mu)$ associated with a σ- finite measure space. The study of various properties of composition operators on spaces having analytic structure was done prominently by Ryff- Cowen [21, 6], Nordgren-Rosenthal-Wintrobe [20]. For more details on work done for the analytic setting we suggest the excellent monographs of Cowen-MacCluer [7] and Shapiro [22]. The work of Koopman [12] is the only early work on composition operators which falls in the second category (measure-theoretic rather than holomorphic). Later authors who have studied composition operators in the measure theoretic setting include Banach [1], Lamperti [14], Nordgren [19] (the one place where both holomorphic and measure-theoretic settings are discussed together), Singh [26, 27, 24], Lo [15, 16], Takagi-Yokouchi [28, 30], Flytzanis-Kanakis [9], Jabbarzadeh-Pourreza [10, 11], Kumar [13] among others. A good reference of work done on composition operators in measure theoretic framework is [25].

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Composition operator. Let X be a non-empty set and V(X) be a linear space of complex valued functions on X under point wise addition and scalar multiplication. If φ is a self map on X into itself such that composition $f \circ \varphi$ belongs to V(X) for each $f \in V(X)$, then φ induces a linear transformation C_φ on V(X), defined as

$$C_\varphi(f) = f \circ \varphi \; \forall \; f \in V(X).$$

The transformation C_φ is known as composition transformation. If V(X) is a Banach or Hilbert space and C_φ is a bounded linear operator on V(X), then C_φ is called composition operator.

DEFINITION 1. Let φ be a self map on the set of natural numbers $\mathbb{N}$. Then φ induces a linear transformation C_φ on ℓ^p, defined by

$$C_\varphi\Big(\sum_{n=1}^{\infty} x_n \chi_n\Big) = \sum_{n=1}^{\infty} x_n \chi_{\varphi^{-1}(n)}$$

where χ_n stands for the characteristic function of the set $\{n\}$.

Let $(X, \mathfrak{B}, \mu)$ be a σ-finite measure space, $L^p(X, \mathfrak{B}, \mu)$ be the Banach space of all p-summable complex-valued measurable functions on X and $L^\infty(X, \mathfrak{B}, \mu)$ denote the Banach space of all essentially bounded complex-valued measurable functions on X. Let $\varphi : X \to X$ be a measurable transformation. Now we state a necessary and sufficient condition on φ so that C_φ become a composition operator on $L^p(X, \mathfrak{B}, \mu)$.

The following results can be found in [19].

THEOREM 1. *Necessary and sufficient condition for a measurable transformation φ to induce a bounded operator on $L^p(\mu)$ $(1 \leqslant p < \infty)$ defined by $C_\varphi(f) = f \circ \varphi$ are $\mu\varphi^{-1} \ll \mu$ and $\frac{d\mu\varphi^{-1}}{d\mu}$ is bounded. In this case*

$$\|C_\varphi\| = \left\| \frac{d\mu\varphi^{-1}}{d\mu} \right\|_\infty^{\frac{1}{p}}$$

where $\| \; \|_\infty$ indicates the essential norm.

In case when $X = \mathbb{N}$ and μ is the counting measure on X, then $L^p(\mathbb{N}, \mu) = \ell^p$. In this setting theorem 1 gets the following form.

THEOREM 2. [25] *A necessary and sufficient condition on φ to induce a composition operator on $\ell^p (1 \leqslant p < \infty)$ is that the set $\{|\varphi^{-1}(n)| : n \in \mathbb{N}\}$ is bounded where $|\varphi^{-1}(n)|$ denotes the cardinality of the fiber $\varphi^{-1}(n)$. In this case $\|C_\varphi\| =$* $\max |\varphi^{-1}(n)|^{\frac{1}{p}} \quad n \in \mathbb{N}$.

Isolation problem.

DEFINITION 2. Let Y be a Banach space and $\mathscr{B}(Y)$ denote the Banach algebra of all bounded linear operators on Y. Let $T \in A \subseteq \mathscr{B}(Y)$. Then T is said to be isolated in A under the norm topology if there exists $\varepsilon > 0$ such that $B_\varepsilon(T) \cap A = \{T\}$, where $B_\varepsilon(T) = \{S \in \mathscr{B}(Y) : \|T - S\| < \varepsilon\}$.

DEFINITION 3. An operator T is said to be isolated in A under strong operator topology, if there exists $\varepsilon > 0$ and $y \in Y$ such that $B(T, y, \varepsilon) \cap A = \{T\}$, where $B(T, y, \varepsilon) = \{S \in \mathscr{B}(Y); \|(S - T)y\| < \varepsilon\}$.

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic function, $\tilde{\varphi}$ denote the radial limit function induced by φ on the unit circle and σ denote the normalized Lebesgue measure on the unit circle.

The study of "Isolation problem" originates from the following result of Berkson.

Berkson Isolation Theorem [2]. Let φ be an analytic map of $\mathbb{D}$ into $\mathbb{D}$ such that $\sigma(A) > 0$, where A=$\tilde{\varphi}^{-1}(K)$. Let ψ be an analytic map of $\mathbb{D}$ into itself and let C_φ and C_ψ be the corresponding composition operators on $H^p(\mathbb{D})$ $(1 \leqslant p < \infty)$. If $\|C_\psi - C_\varphi\| < \left(\frac{\sigma(A)}{2}\right)^{\frac{1}{p}}$, then $\varphi = \psi$. In the above theorem K denotes the unit circle in the complex plane.

Berkson's theorem, which is a topological statement about the $comp(H^p)$, the space of all composition operators on $H^p(\mathbb{D})$ endowed with norm topology, says that whenever the radial limit function $\tilde{\varphi}$ of φ has modulus one on a subset of unit circle having positive Lebesgue measure, then C_φ is isolated. Later, there was further sharpening and elaborations of the work of Berkson by MacCluer, Shapiro- Sundberg [23], Chandra [4], Bourdon [3], Hammond-MacClure [17] and most recently, by Cheng and Dai [5]. Though there has been substantial work on composition operators over $L^p(\mu)$ spaces beginning with Banach's book [1] (see the discussion in the opening paragraph of the introduction), there does not appear to be any work on this category of composition operators over $L^p(\mu)$ spaces connected with the Isolation Problem.

In this paper we study the Isolation Problem for composition operators of category (ii) namely, composition operators on $L^p(\mu)$, where μ is an arbitrary σ-finite measure, and then discuss the Isolation problem for composition operators on the elementary sequence spaces ℓ^p $(1 \leqslant p \leqslant \infty)$.

Notation and Terminology. Let $\mathbb{N}$ denote the set of all positive integers and μ be the counting measure on X. For $1 \leqslant p < \infty$, $L^p(\mathbb{N}, \mu) = \ell^p$ denotes the Banach space of all p-summable real or complex sequences and $L^\infty(\mathbb{N}, \mu) = \ell^\infty$ denotes the Banach space of all bounded sequences of complex numbers. For a subset A of a non-empty set X, the characteristic function of A, denoted by χ_A, is defined as

$$\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & \text{otherwise} \end{cases}$$

Also $|A|$ denotes the cardinality of the set A. Let $\mathbb{C}$ denote the set of all complex numbers and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk in the complex plane $\mathbb{C}$. Let $comp(\ell^p)$ $(1 \leqslant p \leqslant \infty)$ denote the set of all composition operators on the space ℓ^p and $comp(L^p)$ $(1 \leqslant p \leqslant \infty)$ denote the set of all composition operators on $L^p(X, \mathfrak{B}, \mu)$.

The paper is arranged as follows.

In the second section we show that every composition operator C_φ is isolated in $comp(L^p)$ $(1 \leqslant p < \infty)$ under the norm operator topology. We also prove that a composition operator is non-isolated in $comp(\ell^p)$ $(1 \leqslant p < \infty)$ under the strong operator topology.

In section three we show that every composition operator in comp(L^∞) is isolated both under the norm topology and under the strong operator topology. In the last section we show that norm of sum of two composition operators on $L^\infty(\mu)$ is the sum of their norms. We also generalize this result for sum of finite number of composition operators. These results form what we call as "an extremal property" satisfied by composition operators on $L^\infty(\mu)$.

2. Isolation amongst composition operators on $L^p(X, \mu)$ $(1 \leqslant p < \infty)$

In this section we show that every composition operator C_φ is isolated in $comp(L^p)$ $(1 \leqslant p < \infty)$ under the norm operator topology. We also prove that every composition operator on ℓ^p $(1 \leqslant p < \infty)$ is non-isolated in $comp(\ell^p)$ $(1 \leqslant p < \infty)$ under the strong operator topology.

THEOREM 3. *Let $(X, \mathfrak{B}, \mu)$ be a σ-finite measure space and $\varphi : X \to X$ be a measurable transformation which induces a composition operator $C_\varphi : L^p(\mu) \to L^P(\mu)$. Then C_φ is isolated in $comp(L^p)$ $(1 \leqslant p < \infty)$ under the norm topology.*

Proof. Let C_ψ be another composition operator on $L^p(\mu)$ such that $C_\varphi \neq C_\psi$. Then $\varphi \neq \psi$. Hence there exists a measurable set E such that $0 < \mu(E) < \infty$ and $\mu(\varphi(E) \cap \psi(E))$=0. Let $f = \chi_{\varphi(E)}$. Then $\|f\| = \mu(\varphi(E))^{\frac{1}{p}}$.

Now

$$(C_\varphi - C_\psi)f = \chi_{\varphi^{-1}\varphi(E)} - \chi_{\psi^{-1}\varphi(E)}.$$

Further, $\mu(\varphi(E) \cap \psi(E)) = 0$ implies $\mu(\psi^{-1}(\varphi(E)) \cap E) = 0$, $E \subset \varphi^{-1}\varphi(E)$ and $\mu(E \cap \psi^{-1}(\varphi(E))) = 0$.

Therefore

$$\|(C_\varphi - C_\psi)f\| \geqslant (\mu(E))^{\frac{1}{p}}.$$

Thus

$$\|C_\varphi - C_\psi\| \geqslant \frac{(\mu(E))^{\frac{1}{p}}}{\mu(\varphi(E))^{\frac{1}{p}}}$$

This shows that C_φ is isolated in $comp(L^p)$ $(1 \leqslant p < \infty)$ under the norm topology. $\square$

The following corollary is the immediate consequence of the above theorem.

COROLLARY 1. *Let C_φ be a composition operator on ℓ^p $(1 \leqslant p < \infty)$. If C_ψ is another composition operator on ℓ^p different from C_φ then $\|C_\varphi - C_\psi\| \geqslant 1$. Hence C_φ is isolated in comp(ℓ^p) with norm topology.*

REMARK 1. Let $\varphi : X \to X$ be a map and $\varphi_1 = \varphi$ and, more generally, $\varphi_n = \varphi \circ \varphi_{n-1}$. Then for each $n \geqslant 1$, the composition operator $C_\varphi^n = C_{\varphi_n}$. Hence by Theorem 3, for each non-negative integer n, C_φ^n is isolated in comp(L^p) $(1 \leqslant p < \infty)$ under the norm topology.

The next theorem shows that a composition operator on ℓ^p $(1 \leqslant p < \infty)$ is non-isolated under the strong operator topology.

THEOREM 4. *Every composition operator is non-isolated in comp(ℓ^p) $(1 \leqslant p < \infty)$ endowed with the strong operator topology.*

Proof. Let $\varepsilon > 0$ and $f = \sum_{n=1}^{\infty} f_n \chi_n \in \ell^p$. Since $f \in \ell^p$ there exists $N_1 \in \mathbb{N}$ such that

$$\sum_{n=k}^{\infty} |f_n|^p < \left(\frac{\varepsilon}{2}\right)^p \quad \forall\, k \geqslant N_1.$$

Also $C_\varphi f = \sum_{n=1}^{\infty} f_n \chi_{\varphi^{-1}(n)} \in \ell^p$, hence there exists $N_2 \in \mathbb{N}$ such that

$$\sum_{n=k}^{\infty} |f_n|^p |\varphi^{-1}(n)| < \left(\frac{\varepsilon}{2}\right)^p \forall\, k \geqslant N_2.$$

Let $N = \max\{N_1, N_2\}$. For each $n \geqslant 1$ define $\varphi_n : \mathbb{N} \to \mathbb{N}$ as

$$\varphi_n(k) = \begin{cases} \varphi(m), & \text{when } k \in \varphi^{-1}(m),\ 1 \leqslant m \leqslant n \\ k+n, & \text{otherwise} \end{cases}$$

Then

$$\|C_\varphi f - C_{\varphi_n} f\|_p = \left\| \sum_{m=1}^{\infty} f_m \chi_{\varphi^{-1}(m)} - \sum_{m=1}^{\infty} f_m \chi_{\varphi_n^{-1}(m)} \right\|_p.$$

Since $\varphi_n^{-1}(m) = \varphi^{-1}(m)$ for each $1 \leqslant m \leqslant n$, and $|\varphi_n^{-1}(m)| = 1$ for each $m \geqslant n+1$, hence

$$\|C_\varphi f - C_{\varphi_n} f\|_p = \left\| \sum_{m=n+1}^{\infty} f_m \chi_{\varphi^{-1}(m)} - \sum_{m=n+1}^{\infty} f_m \chi_{\varphi_n^{-1}(m)} \right\|_p.$$

This implies that

$$\begin{aligned}\|C_\varphi f - C_{\varphi_n} f\|_p &\leqslant \Big\| \sum_{m=n+1}^{\infty} f_m \chi_{\varphi^{-1}(m)} \Big\| + \Big\| \sum_{m=n+1}^{\infty} f_m \chi_m \Big\| \\ &= \Big(\sum_{m=n+1}^{\infty} |f_m|^p |\varphi^{-1}(m)| \Big)^{\frac{1}{p}} + \Big(\sum_{m=n+1}^{\infty} |f_m|^p \Big)^{\frac{1}{p}} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \ \ \forall\, n \geqslant N.\end{aligned}$$

Thus C_φ is non-isolated in $comp(\ell^p)$ under the strong operator topology. □

REMARK 2. From the Theorem 4 it follows that every composition operator is non-isolated in $comp(\ell^p)$ $(1 \leqslant p < \infty)$ under the weak operator topology.

REMARK 3. We do not know whether the Theorem 4 holds for composition operators on a more general setting of $L^p(\mu)$ $(1 \leqslant p < \infty)$.

3. Isolation amongst composition operators on $L^\infty(X,\mu)$

In this section we show that every composition operator on $L^\infty(X,\mu)$ is isolated in $comp(L^\infty)$ under the strong operator topology.

THEOREM 5. *Let $(X,\mathfrak{B},\mu)$ be a σ-finite measure space and $\varphi : X \to X$ be a measurable transformation which induces a composition operator $C_\varphi : L^\infty(\mu) \to L^\infty(\mu)$. Then C_φ is isolated in $comp(L^\infty)$ under the strong topology.*

Proof. Let C_ψ be another composition operator on $L^p(\mu)$. Then there exists a measurable set $E \in \mathfrak{B}$ with $0 < \mu(E) < \infty$ and $\mu(\varphi(E) \cap \psi(E)) = 0$. Since $\mu\varphi^{-1} \ll \mu$, it follows that $\mu(\varphi(E)) > 0$. Let $f_0 = \chi_{\varphi(E)} - \chi_{\psi(E)}$. Then $\|f_0\| = 1$.

Now

$$(C_\varphi - C_\psi) f_0 = \chi_{\varphi^{-1}(\varphi(E))} - \chi_{\psi^{-1}(\varphi(E))} - \chi_{\varphi^{-1}(\psi(E))} + \chi_{\psi^{-1}(\psi(E))}.$$

Since $E \subset \varphi^{-1}(\varphi(E)) \cap \psi^{-1}(\psi(E))$, hence

$$\|(C_\varphi - C_\psi) f_0\| = 2.$$

Hence C_φ is isolated in $comp(L^\infty)$ under the strong operator topology. □

The following corollaries follow from the above theorem.

COROLLARY 2. *Every composition operators on $L^\infty(\mu)$ is isolated in $comp(L^\infty)$ under the norm topology.*

COROLLARY 3. *Every composition operator is isolated in comp*(ℓ^∞) *under the strong operator as well as norm operator topology.*

REMARK 4. We do not know whether a composition operator on $L^\infty(\mu)$ is isolated in $comp(L^\infty)$ under the weak operator topology.

The following theorem shows that every composition operator on $L^\infty(\mu)$ has norm one.

THEOREM 6. *Let* C_φ *be a composition operator on* $L^\infty(\mu)$. *Then* $\|C_\varphi\| = 1$.

Proof. Given that $C_\varphi : L^\infty(\mu) \to L^\infty(\mu)$, then $(C_\varphi f)(x) = f(\varphi(x))\ \forall\ x \in X$. Therefore

$$|(C_\varphi f)(x)| = |f(\varphi(x))|\ \ \forall x \in X.$$
$$\|C_\varphi f\| = \sup_{x\in X} |(f \circ \varphi)(x)|\ \ \forall x \in X.$$

This implies that

$$\begin{aligned}\|C_\varphi f\| &= \sup_{x\in X} |f(\varphi(x))|\ \ \forall x \in X\\ &\leqslant \sup_{x\in X} |f(x)|\ \ \forall x \in X\\ &= \|f\|.\end{aligned}$$

Hence

$$\|C_\varphi f\| \leqslant \|f\|. \tag{1}$$

Conversely, let E be a measurable set with $0 < \mu(E) < \infty$ and let $f_0 = \chi_{\varphi(E)}$ then $\|f_0\| = 1$ and $\|C_\varphi f_0\| = 1$. Hence $\|C_\varphi\| = 1$. □

REMARK 5. Every self map φ of $\mathbb{N}$ induces a composition operator on ℓ^∞. However, we note that this statement is not true in the setting of an arbitrary σ-finite measure space.

4. Composition operators on $L^\infty(\mu)$ satisfying an extremal property

In this section we show that norm of finite sum of composition operators on $L^\infty(\mu)$ is equal to the sum of their norms. This property can also be seen as a generalization of Daugavet equations [8] for composition operators.

DEFINITION 4. We say that two composition operators C_φ and C_ψ satisfy an extremal property if

$$\|C_\varphi + C_\psi\| = \|C_\varphi\| + \|C_\psi\|.$$

THEOREM 7. *Let C_φ and C_ψ be two composition operators on $L^\infty(\mu)$. Then $\|C_\varphi + C_\psi\| = \|C_\varphi\| + \|C_\psi\| = 2$.*

Proof. Let C_ψ be another composition operator on $L^\infty(\mu)$. As $\varphi \neq \psi$, there exists a measurable set $E \in \mathfrak{B}$ with $0 < \mu(E) < \infty$ and $\mu(\varphi(E) \cap \psi(E)) = 0$. Since $\mu\varphi^{-1} \ll \mu$, it follows that $\mu(\varphi(E)) > 0$ similarly $\mu(\psi(E)) > 0$. Let $f_0 = \chi_{\varphi(E)} + \chi_{\psi(E)}$. Then $\|f_0\| = 1$.

Now

$$(C_\varphi + C_\psi)f_0 = \chi_{\varphi^{-1}(\varphi(E))} + \chi_{\psi^{-1}(\varphi(E))} + \chi_{\varphi^{-1}(\psi(E))} + \chi_{\psi^{-1}(\psi(E))}.$$

Since $E \subset \varphi^{-1}(\varphi(E)) \cap \psi^{-1}(\psi(E))$, hence

$$\|(C_\varphi + C_\psi)f_0\| = 2.$$

This implies that

$$\|C_\varphi + C_\psi\| = \|C_\varphi\| + \|C_\psi\| = 2. \quad \square$$

The following example shows that the above theorem fails when $1 \leqslant p < \infty$.

EXAMPLE 1. Define $\varphi, \psi : \mathbb{N} \to \mathbb{N}$ such that

$$\varphi(2n-1) = \varphi(2n) = 2n-1 \ \ \forall n \geqslant 1$$

$$\psi(2n-1) = \psi(2n) = 2n \ \ \forall n \geqslant 1$$

Let C_φ and C_ψ be two composition operators induced on ℓ^p $(1 \leqslant p < 1)$. Then $\|C_\varphi\| = \|C_\psi\| = 2^{\frac{1}{p}}$. For $f = \sum_{n=1}^\infty f_n \chi_n \in \ell^p$

$$\begin{aligned}
(C_\varphi + C_\psi)f &= C_\varphi f + C_\psi f \\
&= \sum_{n=1}^\infty f_n \chi_{\varphi^{-1}(n)} + \sum_{n=1}^\infty f_n \chi_{\psi^{-1}(n)} \\
&= \sum_{n=1}^\infty f_{2n-1} \chi_{\{2n-1,2n\}} + \sum_{n=1}^\infty f_{2n} \chi_{\{2n-1,2n\}} \\
&= \sum_{n=1}^\infty (f_{2n-1} + f_{2n}) \chi_{\{2n-1,2n\}} \\
\|(C_\varphi + C_\psi)f\|^p &\leqslant \sum_{n=1}^\infty 2|f_{2n-1} + f_{2n}|^p \\
&\leqslant 2^p \Big(\sum_{n=1}^\infty |f_{2n-1}|^p + |f_{2n}|^p \Big)
\end{aligned}$$

Therefore

$$\|(C_\varphi + C_\psi)f\|^p \leqslant 2^p \sum_{n=1}^{\infty} \|f\|^p$$

Hence $\|C_\varphi + C_\psi\| \leqslant 2 < \|C_\varphi\| + \|C_\psi\| = 2^{1+\frac{1}{p}}$.

Now we generalize the Theorem 7 as follows.

THEOREM 8. *Let $C_{\varphi_1}, C_{\varphi_2}, C_{\varphi_3}, \ldots, C_{\varphi_n}$ be any n composition operators on $L^\infty(\mu)$. Then $\left\|\sum_{i=1}^n C_{\varphi_i}\right\| = \sum_{i=1}^n \|C_{\varphi_i}\| = n$.*

Proof. *Case* 1. Suppose there exists a measurable set E with $0 < \mu(E) < \infty$ and $\mu(\bigcap_{i=1}^n \varphi_i(E)) > 0$. Let $E_0 = \bigcap_{i=1}^n \varphi_i(E)$ and $f_0 = \chi_{E_0}$. Then $\|f_0\| = 1$, Further, it is easy to see that

$$\begin{aligned}
\|(C_{\varphi_1} + C_{\varphi_2} + C_{\varphi_3} + \ldots + C_{\varphi_n})f_0\| &= \|C_{\varphi_1}\chi_{E_0} + C_{\varphi_2}\chi_{E_0} + C_{\varphi_3}\chi_{E_0} + \ldots + C_{\varphi_n}\chi_{E_0}\| \\
&= \|\chi_{\varphi_1^{-1}(E_0)} + \chi_{\varphi_2^{-1}(E_0)} + \chi_{\varphi_3^{-1}(E_0)} + \ldots + \chi_{\varphi_n^{-1}(E_0)}\| \\
&= n.
\end{aligned}$$

Hence, in this case

$$\left\|\sum_{i=1}^n C_{\varphi_i}\right\| = \sum_{i=1}^n \|C_{\varphi_i}\|.$$

Case 2. Suppose that $\mu(\bigcap_{i=1}^n \varphi_i(E)) = 0$ for each measurable set E. Let A$= \{1, 2, \ldots, n\}$. Choose a maximal subset A_1 of A for which there exists a measurable set E_1 with $0 < \mu(E_1) < \infty$ and $\mu(\bigcap_{i\in A_1} \varphi_i(E_1)) > 0$. Let $E_0 = \bigcap_{i\in A_1} \varphi_i(E_1)$. Now Let A_2 is a maximal subset of A$\setminus A_1$ for which there exists a measurable set E_2 of finite positive measure such that $E_0 \subseteq \bigcap_{i\in A_2} \varphi_i(E_2)$. Continuing this process, we get a partition $\{A_1, A_2, \ldots, A_k\}$ of A and measurable sets $\{E_1, E_2, \ldots, E_k\}$ such that

$$E_0 \subseteq \Big(\bigcap_{i\in A_1} \varphi_i(E_1)\Big) \cap \Big(\bigcap_{i\in A_2} \varphi_i(E_2)\Big) \cap \ldots \cap \Big(\bigcap_{i\in A_k} \varphi_i^{-1}(E_k)\Big).$$

Now let $f_0 = \chi_{E_0}$. Then $\|f_0\| = 1$. Also

$$\Big(\sum_{i=1}^n C_{\varphi_i}\Big)(f) = \sum_{i\in A_1} \chi_{\varphi_i^{-1}(E_0)} + \sum_{i\in A_2} \chi_{\varphi_i^{-1}(E_0)} + \ldots + \sum_{i\in A_k} \chi_{\varphi_i^{-1}(E_0)}.$$

Hence

$$\left\|\sum_{i=1}^n C_{\varphi_i}(f_0)\right\| = |A_1| + |A_2| + A_3| + \ldots + |A_k| = n.$$

Thus

$$\left\|\sum_{i=1}^n C_{\varphi_i}\right\| = \sum_{i=1}^n \|C_{\varphi_i}\| = n. \quad \square$$

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