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SOME REFINEMENTS OF REAL POWER FORM INEQUALITIES FOR CONVEX FUNCTIONS VIA WEAK SUB–MAJORIZATION

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Abstract. The main goal of this article, is to develop a general method for improving some new real power inequalities for convex and log-convex functions, which extends and unifies two recent and important results due to M. Sababheh [Linear Algebra Appl. **506** (2016), 588–602] and D. Q. Huy et al. [Linear Algebra Appl. **656** (2023), 368–384]. Then by selecting some appropriate convex and log-convex functions, we obtain new mean inequalities for scalars and matrices, some new refinements and reverses of the Heinz and Hölder type inequalities for matrices. We get also some new and refined trace and numerical radius inequalities.

1. Introduction and preliminaries

Convex functions have played a key role in different fields, including mathematical inequalities, optimization, functional analysis, applied mathematics and mathematical physics. Recall that a function $f : I \to \mathbb{R}$ is said to be convex on the interval I if

$$f((1-\nu)a+\nu b) \leqslant (1-\nu)f(a)+\nu f(b), \tag{1}$$

for all $a,b \in I$ and $\nu \in (0,1)$. If this inequality is reversed, then f is said to be concave.

If $\log f$ is convex, then f is called log-convex. Accordingly, a log-convex function is a positive function satisfying

$$f((1-\nu)a+\nu b) \leqslant f(a)^{1-\nu} f(b)^{\nu}, \tag{2}$$

for the same parameters as in (1).

Recent studies of the topic have investigated possible refinements of the above inequalities, where adding a positive term to the left side becomes possible. This idea has been treated in [6, 17, 18, 19, 21, 22], where not only refinements have been investigated, but also reversed versions and much more have been discussed. Research related to the above discussion includes obtaining new forms that generalize these inequalities, getting refinements that minimizes the difference between the two sides of the inequalities and adding some acceptable terms that reverse such inequalities. For example, the

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inequality (1) was refined and reversed in [7]. Namely, the authors show the following two inequalities

$$f((1-\nu)a+\nu b) \leqslant (1-\nu)f(a)+\nu f(b)-2\mathrm{r}_0\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right), \quad (3)$$

and

$$(1-\nu)f(a)+\nu f(b) \leqslant f((1-\nu)a+\nu b)+2\mathrm{R}_0\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right), \quad (4)$$

where $a,b\in I$, $\mathrm{r}_0=\min\{\nu,1-\nu\}$ and $\mathrm{R}_0=\max\{\nu,1-\nu\}$.

A generalisation of the inequalities (3) and (4) is presented in [18] as follows.

THEOREM 1. *Let $f:[0,1]\to[0,\infty)$ is a convex function, $0<\nu\leqslant\tau<1$ and $\lambda\geqslant 1$. Then*

$$\left(\frac{\nu}{\tau}\right)^{\lambda} \leqslant \frac{((1-\nu)f(0)+\nu f(1))^{\lambda}-f^{\lambda}(\nu)}{((1-\tau)f(0)+\tau f(1))^{\lambda}-f^{\lambda}(\tau)} \leqslant \left(\frac{1-\nu}{1-\tau}\right)^{\lambda}. \quad (5)$$

The main objective of this paper is to provide a unified treatment of convex and log-convex functions. More precisely, we will present a general improvement of Theorem 1. As applications, by selecting some appropriate convex and log convex functions we will derive improved versions of some classical inequalities, that includes scalar and operator means, Heinz and Hölder type inequalities for matrices and the numerical radius.

Before ending this section, let us fix some notations and remind the reader to related concepts about matrices. Let $\mathbf{M}_n$ be the algebra of all complex matrices of order $n\times n$. A matrix $A\in\mathbf{M}_n$ is called Hermitian if $A=A^*$, where A^* is the adjoin of A. The notation $A\geqslant 0$ $(A>0)$ is used to mean that A is positive semi-definite (positive definite), if A and B are Hermitian and $A-B$ is positive semi-definite, then we write $A\geqslant B$. The set of all positive semi-definite matrices is denoted by $\mathbf{M}_n^+$ and the set of all definite matrices in $\mathbf{M}_n^+$ is denoted by $\mathbf{M}_n^{++}$. The singular values of a matrix $A\in\mathbf{M}_n$ are the eigenvalues of the positive semi-definite matrix $|A|:=(A^*A)^{1/2}$, denoted by $s_j(A)$ for $j=1,2,3,\ldots,n$; arranged in a non-increasing order. A unitarily invariant norm $|||\cdot|||$ on $\mathbf{M}_n$ is a matrix norm that satisfies the invariance property: $|||UAV|||=|||A|||$ for every $A\in\mathbf{M}_n$ and for all unitary matrices $U,V\in\mathbf{M}_n$. The trace norm is given by $||A||_1:=\mathrm{tr}|A|=\sum_{j=1}^n s_j(A)$, where tr is the usual trace. This norm is unitarily invariant. An important example of unitarily invariant norm is the Hilbert-Schmidt norm $||\cdot||_2$ defined by

$$||A||_2:=\mathrm{tr}(AA^*)^{\frac{1}{2}}=\left(\sum_{i,j}|a_{i,j}|^2\right)^{\frac{1}{2}}, \qquad \big(A=(a_{i,j})\big).$$

Speaking of the eigenvalues of a Hermitian matrix A, we use the notation $\lambda_j(A)$ to mean the j-th eigenvalue of A, when written in a decreasing order.

The comparison between matrices has been done in different ways, among which the Löwener partial order $\leqslant$ is the strongest. More precisely, when A and B are Hermitian such that $A \leqslant B$, we infer that $\lambda_j(A) \leqslant \lambda_j(B), \forall j$, which is another perspective to compare between A and B. Notice that the relation $\lambda_j(A) \leqslant \lambda_j(B), \forall j$ implies $\sum_{j=1}^{k} \lambda_j(A) \leqslant \sum_{j=1}^{k} \lambda_j(B)$ for $1 \leqslant k \leqslant n$. This last comparison is what we call weak majorization, and is denoted by $\prec_w$. Thus, we have

$$A \leqslant B \Rightarrow \lambda_j(A) \leqslant \lambda_j(B), \forall j \Rightarrow A \prec_w B \Rightarrow |||A||| \leqslant |||B|||$$

where $|||.|||$ stands for unitarily invariant norms. For $A,B \in \mathbf{M}_n^{++}$, $\nu \in [0,1]$ and $p \in \mathbb{R} \setminus \{0\}$, the operators arithmetic, geometric, harmonic and power means are defined respectively, by $A\nabla_\nu B := (1-\nu)A + \nu B$, $A\sharp_\nu B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\nu} A^{1/2}$, $A!_\nu B := ((1-\nu)A^{-1} + \nu B^{-1})^{-1}$ and

$$A\sharp_{p,\nu} B := A^{1/2}\left((1-\nu)I + \nu(A^{-1/2}BA^{-1/2})^p\right)^{\frac{1}{p}} A^{1/2};\ p \in \mathbb{R} \setminus \{0\}.$$

Thus, the value $p \to 0$ gives the geometric mean, while the values $p = 1, -1$ give the arithmetic and harmonic means, respectively.

In the sequel, we will adopt the following notations for $a,b > 0$ and $\nu \in (0,1)$.

$$a\nabla_\nu b = (1-\nu)a + \nu b,\ \ a\sharp_\nu b = a^{1-\nu} b^{\nu} \text{ and } a!_\nu b = ((1-\nu)a^{-1} + \nu b^{-1})^{-1},$$

to denote the arithmetic, geometric and harmonic means respectively. The arithmetic means for scalars and operators can be rewritten by simplification as $a\nabla b$ and $A\nabla B$ for $\nu = \frac{1}{2}$.

This paper is organized as follows. After the forgoing section, we state and prove our main results concerning the refinement of Theorem 1 using the weak sub-majorization in the second section. In section 3, we present the application of our main results to scalar means. Section 4 is devoted to present some new refinement of matrix inequalities. In section 5, we discuss a new matrix norm inequalities via convexity and log-convexity and we finish this paper by given some new inequalities for the generalized numerical radius in section 6.

2. Some new inequalities for convex functions via weak sub-majorization

In this section, we give an improved version of Theorem 1. We begin with recalling the theory of weak sub-majorization. Throughout this section, we denote by $a^* = (a_1^*, \ldots, a_n^*)$ the vector obtained from the vector $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ by rearranging the components of it in decreasing order. Then, for two vectors $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ in $\mathbb{R}^n, a$ is said to be weakly sub-majorized by b, written $a \prec_w b$, if

$$\sum_{i=1}^{k} a_i^* \leqslant \sum_{i=1}^{k} b_i^*$$

for all $k = 1, \ldots, n$. An important feature of the theory of weak sub-majorization which will be used in proofs of our results is given by the following lemma.

LEMMA 1. [14, pp. 13] *Let* $a=(a_i)_{i=1}^n, b=(b_i)_{i=1}^n \in \mathbb{R}^n$ *and* $J \subset \mathbb{R}$ *be an interval containing the components of* a *and* b*. If* $a \prec_w b$ *and* $\phi : J \to \mathbb{R}$ *is a continuous increasing convex function, then*

$$\sum_{i=1}^n \phi(a_i) \leqslant \sum_{i=1}^n \phi(b_i).$$

In order to accomplish our results, we need some preliminary results as follows.

LEMMA 2. *Let* $f:[0,1] \to [0,+\infty)$ *be a convex function and let* $0 < \nu \leqslant \tau < 1$*. Then we have*

$$f(0)\nabla_\nu f(1) \geqslant f(\nu) + \frac{\nu}{\tau}\left(f(0)\nabla_\tau f(1) - f(\tau)\right) + 2\mathrm{r}_0\left(f(0)\nabla f(\tau) - f\left(\frac{\tau}{2}\right)\right),$$

where $\mathrm{r}_0 = \min\{\frac{\nu}{\tau}, 1-\frac{\nu}{\tau}\}$.

Proof. By using the inequality (3), we have

$$\begin{aligned} f(0)\nabla_\nu f(1) \;-\; &\frac{\nu}{\tau}\left(f(0)\nabla_\tau f(1) - f(\tau)\right) \\ &= \left(1-\frac{\nu}{\tau}\right) f(0) + \frac{\nu}{\tau} f(\tau) \\ &\geqslant f\left(\left(1-\frac{\nu}{\tau}\right)0 + \frac{\nu}{\tau}\tau\right) + 2\mathrm{r}_0\left(f(0)\nabla f(\tau) - f\left(\frac{\tau}{2}\right)\right) \\ &= f(\nu) + 2\mathrm{r}_0\left(f(0)\nabla f(\tau) - f\left(\frac{\tau}{2}\right)\right). \quad \square \end{aligned}$$

REMARK 1. Notice that Lemma 2 presents one refinement term of the first inequality in Theorem 1, for $\lambda = 1$.

LEMMA 3. *Let* f *be a convex function on* $[0,1]$, $0 < \nu \leqslant \tau < 1$ *and* $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3) \in \mathbb{R}^3$ *be two vectors with components*

$$a_1 = f(\nu), \;\; a_2 = \frac{\nu}{\tau}\left(f(0)\nabla_\tau f(1)\right), \;\; a_3 = 2r_0\left(f(0)\nabla f(\tau)\right),$$

and

$$b_1 = f(0)\nabla_\nu f(1), \;\; b_2 = \frac{\nu}{\tau} f(\tau), \;\; b_3 = 2r_0 f\left(\frac{\tau}{2}\right),$$

where $\mathrm{r}_0 = \min\{\frac{\nu}{\tau}, 1-\frac{\nu}{\tau}\}$*. Then, we have* $a \prec_w b$*, namely, the vectors* a^* *and* b^* *have components satisfying that*

$$a_1^* \leqslant b_1^*, \tag{6}$$

$$a_1^* + a_2^* \leqslant b_1^* + b_2^*, \tag{7}$$

$$a_1^* + a_2^* + a_3^* \leqslant b_1^* + b_2^* + b_3^*. \tag{8}$$

Proof. In order to prove (6) remark that b_1^* is exactly b_1. Indeed, on one hand we have $b_1 - a_2 = \left(1-\frac{\nu}{\tau}\right)f(0) \geqslant 0$. This shows that $b_1 \geqslant a_2$. Since $a_2 \geqslant b_2$, then we get that $b_1 \geqslant b_2$. On the other hand, we have

$$b_1 - a_3 = (1-\nu-\mathrm{r}_0)f(0) + \nu f(1) - \mathrm{r}_0 f(\tau). \tag{9}$$

First of all remark that

$$\left\{(\nu,\tau)\in[0,1]^2 \,:\, 0<\nu\leqslant\tau<1\right\} = \left\{(\nu,\tau)\in[0,1]^2 \,:\, 0<\nu\leqslant\frac{\tau}{2}\right\} \bigcup \left\{(\nu,\tau)\in[0,1]^2 \,:\, \frac{\tau}{2}\leqslant\nu<1\right\}.$$

At this point, we distinguish the following two situations.

If $\nu\in\left(0,\frac{\tau}{2}\right]$, then $\mathrm{r}_0 = \frac{\nu}{\tau}$, and so (9) becomes

$$\begin{aligned} b_1 - a_3 &= \left(1-\nu-\frac{\nu}{\tau}\right)f(0) + \nu f(1) - \frac{\nu}{\tau}f(\tau) \\ &= \left(1-\nu-\frac{\nu}{\tau}\right)f(0) + \nu f(1) - \frac{\nu}{\tau}f(\tau) + \frac{\nu}{\tau}\left(f(0)\nabla_\tau f(1)\right) - \frac{\nu}{\tau}\left(f(0)\nabla_\tau f(1)\right) \\ &\geqslant \left(\frac{\tau-2\nu}{\tau}\right)f(0) \\ &\geqslant 0. \end{aligned}$$

If $\nu\in\left[\frac{\tau}{2},1\right)$, then $\mathrm{r}_0 = 1-\frac{\nu}{\tau}$. We may write

$$\begin{aligned} b_1 - a_3 &= \left(\frac{\nu}{\tau}-\nu\right)f(0) + \nu f(1) - \left(1-\frac{\nu}{\tau}\right)f(\tau) \\ &= \left(\frac{\nu}{\tau}-\nu\right)f(0) + \nu f(1) - \left(1-\frac{\nu}{\tau}\right)f(\tau) + \left(1-\frac{\nu}{\tau}\right)\left(f(0)\nabla_\tau f(1)\right) \\ &\quad - \left(1-\frac{\nu}{\tau}\right)\left(f(0)\nabla_\tau f(1)\right) \\ &\geqslant \left(\frac{(1-\tau)(2\nu-\tau)}{\tau}\right)f(0) + (2\nu-\tau)f(1) \\ &\geqslant 0. \end{aligned}$$

This implies that $b_1 \geqslant a_3$. Furthermore, we have $a_3 \geqslant b_3$, and so $b_1 \geqslant b_3$. Hence $b_1^* = b_1$. Moreover, by the previous notes, we have $a_i \leqslant b_1$ for every $i = 1,2,3$. Whence $a_1^* \leqslant b_1^*$.

The third inequality comes from Lemma 2 and we have

$$a_1 + a_2 + a_3 \leqslant b_1 + b_2 + b_3. \tag{10}$$

To prove the second inequality (7), the following inequalities should be shown.

$$a_1 + a_2 \leqslant b_1 + b_2, \tag{11}$$

$$a_1 + a_3 \leqslant b_1 + b_3, \tag{12}$$

$$a_2 + a_3 \leqslant b_1 + b_2. \tag{13}$$

The first inequality (11) comes easily from the first inequality in (5) for $\lambda = 1$. For the second inequality (12), since $b_2 \leqslant a_2$ together with (10), we get that

$$a_1 + a_3 \leqslant b_1 + b_3 - (a_2 - b_2) \leqslant b_1 + b_3.$$

Now, let us treat our last inequality (13). We discuses the following two cases. If $\nu \in \left(0, \frac{\tau}{2}\right]$, then $\mathrm{r}_0 = \frac{\nu}{\tau}$. We have

$$\begin{aligned} b_1 + b_2 - (a_2 + a_3) &= f(0)\nabla_\nu f(1) + \frac{\nu}{\tau} f(\tau) - \left(\frac{\nu}{\tau} f(0)\nabla_\tau f(1) + \frac{2\nu}{\tau}(f(0)\nabla f(\tau))\right) \\ &= \left(\frac{\tau - 2\nu}{\tau}\right) f(0) \\ &\geqslant 0. \end{aligned}$$

If $\nu \in \left[\frac{\tau}{2}, 1\right)$, then $\mathrm{r}_0 = 1 - \frac{\nu}{\tau}$. Hence

$$\begin{aligned} &b_1 + b_2 - (a_2 + a_3) \\ &\quad = f(0)\nabla_\nu f(1) + \frac{\nu}{\tau} f(\tau) - \left(\frac{\nu}{\tau} f(0)\nabla_\tau f(1) + 2\left(1 - \frac{\nu}{\tau}\right)(f(0)\nabla f(\tau))\right) \\ &\quad = \left(\frac{2\nu - \tau}{\tau}\right) f(\tau) \\ &\quad \geqslant 0. \end{aligned}$$

Thus complete the proof. □

In the following, we state our first main results. Our arguments are influenced by the ones given in [10]. The following results as mentioned before generalize the results given by Sababheh in [18].

THEOREM 2. *Let f be a convex function on $[0,1]$ and ϕ be a strictly increasing convex function defined on $\mathbb{R}^+$. Then for $0 < \nu \leqslant \tau < 1$, we have*

$$\begin{aligned} \phi\left(f(0)\nabla_\nu f(1)\right) \geqslant \phi \circ f(\nu) &+ \phi\left(\frac{\nu}{\tau}(f(0)\nabla_\tau f(1))\right) \\ &- \phi\left(\frac{\nu}{\tau} f(\tau)\right) + \phi\left(2\mathrm{r}_0(f(0)\nabla f(\tau))\right) \\ &- \phi\left(2\mathrm{r}_0 f\left(\frac{\tau}{2}\right)\right). \end{aligned} \tag{14}$$

where $\mathrm{r}_0 = \min\{\frac{\nu}{\tau}, 1 - \frac{\nu}{\tau}\}$.

Proof. Let $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ two vectors in $\mathbb{R}^3$ where the component are the same as Lemma 3. Let ϕ be a strictly increasing convex function defined on $\mathbb{R}^+$ and let $f : [0,1] \to [0, \infty)$. Since $a \prec_w b$, by applying Lemma 1, we get that

$$\phi(a_1) + \phi(a_2) + \phi(a_3) \leqslant \phi(b_1) + \phi(b_2) + \phi(b_3),$$

or equivalently,

$$\phi(b_1) \geqslant \phi(a_1) + (\phi(a_2) - \phi(b_2)) + (\phi(a_3) - \phi(b_3))$$

Thus complete the proof. □

The following theorem represent the reversed version of the previous one. Here we use the previous theorem and some special variable change to get the proof. We also mention that we can prove the following theorem using the same ideas as in Theorem 2.

THEOREM 3. *Let f be a convex function on $[0,1]$ and ϕ be a strictly increasing convex function defined on $\mathbb{R}^+$. Then for $0 < \nu \leqslant \tau < 1$, we have*

$$\phi\left(f(0)\nabla_\tau f(1)\right) \geqslant \phi \circ f(\tau) + \phi\left(\frac{1-\tau}{1-\nu}(f(0)\nabla_\nu f(1))\right) - \phi\left(\frac{1-\tau}{1-\nu} f(\nu)\right) \\ -\phi\Big(2\mathrm{R}_0(f(1)\nabla f(\nu))\Big) - \phi\left(2\mathrm{R}_0 f\left(\frac{1+\nu}{2}\right)\right), \tag{15}$$

where $\mathrm{R}_0 = \min\left\{\frac{1-\tau}{1-\nu}, 1 - \frac{1-\tau}{1-\nu}\right\}$.

Proof. First notice that if the function $f(x)$ is convex in $[0,1]$, then $f(1-x)$ is convex in $[0,1]$. If $0 \leqslant \nu \leqslant \tau \leqslant 1$, then we have $0 \leqslant 1-\tau \leqslant 1-\nu \leqslant 1$. So by changing $f(x)$, ν and τ into $f(1-x)$, $1-\tau$ and $1-\nu$, respectively in the Theorem 2. We obtain the desired results. □

Notice that Theorems 2 and 3, represent the general version and the general reversed version of Lemma 2, respectively. In the following, by selecting some appropriate convex functions, we gain some very nice and interesting refinement for the correspondent inequalities for convex and log-convex functions that improve the main results of [18].

Replacing f into $\log f$, we obtain the following inequalities for log-convex functions.

COROLLARY 1. *Let f be a log-convex function on $[0,1]$ and ϕ be a strictly increasing convex function defined on $\mathbb{R}^+$. Then for $0 < \nu \leqslant \tau < 1$, we have*

$$\phi\left(\log\left(f^{1-\nu}(0)f^{\nu}(1)\right)\right) \geqslant \phi \circ \log f(\nu) + \phi\left(\log\left(f^{1-\tau}(0)f^{\tau}(1)\right)^{\frac{\nu}{\tau}}\right) - \phi\left(\log f^{\frac{\nu}{\tau}}(\tau)\right) \\ +\phi\left(\log\left(f^{\frac{1}{2}}(0)f^{\frac{1}{2}}(\tau)\right)^{2\mathrm{r}_0}\right) - \phi\left(\log f^{2\mathrm{r}_0}\left(\frac{\tau}{2}\right)\right), \tag{16}$$

where $\mathrm{r}_0 = \min\left\{\frac{\nu}{\tau}, 1 - \frac{\nu}{\tau}\right\}$, *and*

$$\phi\left(\log\left(f^{1-\tau}(0)f^{\tau}(1)\right)\right) \geqslant \phi \circ \log f(\tau) + \phi\left(\log\left(f^{1-\nu}(0)f^{\nu}(1)\right)^{\frac{1-\tau}{1-\nu}}\right) \\ -\phi\left(\log f^{\frac{1-\tau}{1-\nu}}(\nu)\right) + \phi\left(\log\left(f^{\frac{1}{2}}(1)f^{\frac{1}{2}}(\nu)\right)^{2\mathrm{R}_0}\right) \\ -\phi\left(\log f^{2\mathrm{R}_0}\left(\frac{1+\nu}{2}\right)\right), \tag{17}$$

where $\mathrm{R}_0 = \min\{\frac{1-\tau}{1-\nu}, 1-\frac{1-\tau}{1-\nu}\}$.

Now, by selecting $\phi(x) = x^{\lambda}$ for $\lambda \geqslant 1$, in Theorems 2 and 3, we get the following corollary.

COROLLARY 2. *Let f be a convex function on $[0,1]$, $0 < \nu \leqslant \tau < 1$ and $\lambda > 1$. Then we have*

$$(f(0)\nabla_\nu f(1))^\lambda \geqslant f^\lambda(\nu) + \left(\frac{\nu}{\tau}\right)^\lambda \left((f(0)\nabla_\tau f(1))^\lambda - f^\lambda(\tau)\right) + (2\mathrm{r}_0)^\lambda \left((f(0)\nabla f(\tau))^\lambda - f^\lambda\left(\frac{\tau}{2}\right)\right), \tag{18}$$

where $\mathrm{r}_0 = \min\{\frac{\nu}{\tau}, 1-\frac{\nu}{\tau}\}$, *and*

$$(f(0)\nabla_\tau f(1))^\lambda \geqslant f^\lambda(\tau) + \left(\frac{1-\tau}{1-\nu}\right)^\lambda \left((f(0)\nabla_\nu f(1))^\lambda - f^\lambda(\nu)\right) + (2\mathrm{R}_0)^\lambda \left((f(1)\nabla f(\nu))^\lambda - f^\lambda\left(\frac{1+\nu}{2}\right)\right), \tag{19}$$

where $\mathrm{R}_0 = \min\{\frac{1-\tau}{1-\nu}, 1-\frac{1-\tau}{1-\nu}\}$.

Now, by selecting $\phi(x) = \exp(x)$, in Corollary 1, we get the following new and important refinement and reversed for log-convex functions.

COROLLARY 3. *Let f be a log-convex function on $[0,1]$. Then for $0 < \nu \leqslant \tau < 1$, we have*

$$f^{1-\nu}(0)f^\nu(1) \geqslant f(\nu) + \left((f^{1-\tau}(0)f^\tau(1))^{\frac{\nu}{\tau}} - f^{\frac{\nu}{\tau}}(\tau)\right) + \left(f^{\frac{1}{2}}(0)f^{\frac{1}{2}}(\tau)\right)^{2\mathrm{r}_0} - f^{2\mathrm{r}_0}\left(\frac{\tau}{2}\right), \tag{20}$$

where $\mathrm{r}_0 = \min\{\frac{\nu}{\tau}, 1-\frac{\nu}{\tau}\}$, *and*

$$f^{1-\tau}(0)f^\tau(1) \geqslant f(\tau) + \left((f^{1-\nu}(0)f^\nu(1))^{\frac{1-\tau}{1-\nu}} - f^{\frac{1-\tau}{1-\nu}}(\nu)\right) + \left(f^{\frac{1}{2}}(1)f^{\frac{1}{2}}(\nu)\right)^{2\mathrm{R}_0} - f^{2\mathrm{R}_0}\left(\frac{1+\nu}{2}\right), \tag{21}$$

where $\mathrm{R}_0 = \min\{\frac{1-\tau}{1-\nu}, 1-\frac{1-\tau}{1-\nu}\}$.

REMARK 2. Before proceeding to further results, we explain a little about the relation among the Corollary 2 and Theorem 1. Notice that the first inequality in Theorem 1 can be written as follow

$$\left(\frac{\nu}{\tau}\right)^\lambda \left[(f(0)\nabla_\tau f(1))^\lambda - f^\lambda(\tau)\right] \leqslant (f(0)\nabla_\nu f(1))^\lambda - f^\lambda(\nu), \tag{22}$$

with $0 \leqslant \nu < \tau \leqslant 1$ and $\lambda \geqslant 1$. While the second inequality in the same theorem can be stated in the following way

$$\left(\frac{1-\tau}{1-\nu}\right)^{\lambda}\left[(f(0)\nabla_{\nu}f(1))^{\lambda}-f^{\lambda}(\nu)\right]\leqslant(f(0)\nabla_{\tau}f(1))^{\lambda}-f^{\lambda}(\tau), \tag{23}$$

where $0 \leqslant \nu < \tau \leqslant 1$ and $\lambda \geqslant 1$. Consequently, the first inequality in Corollary 2 present one refining term of (22), while the second inequality in Corollary 2 present one refining term of (23). Therefore, Corollary 2 gives a considerable refinement of Theorem 1. Since Theorem 1 was a generalization of the results in [3, 12, 13], it follows that our results in this section provide better new estimates than the results in these references. This is the main significance of our results. In the next sections, we present explicit examples of refined inequalities for both scalars and matrices.

3. Applications to scalar means

In this section, by selecting some appropriate convex functions, we present different means inequalities that may be derived from our convexity results, which present one term refinements of the main results of [3, 12, 13].

When $a,b>0$ and $p\in\mathbb{R}\setminus\{0\}$ the function $f(x)=a\sharp_{p,x}b:=((1-x)a^p+xb^p)^{1/p}$ is convex on $[0,1]$. Applying Corollary 2 to this function we obtain the following theorem, that proves one term refinement and generalisation of the difference between the arithmetic and the power mean inequalities presented in Theorem 2.3 of [12].

THEOREM 4. *Let $a,b>0,\ 0\leqslant\nu\leqslant\tau\leqslant1,\ p\in\mathbb{R}\setminus\{0\}$ and $\lambda\geqslant1$. Then*

$$\left(\frac{\nu}{\tau}\right)^{\lambda}\left((a\nabla_{\tau}b)^{\lambda}-(a\sharp_{(p,\tau)}b)^{\lambda}\right)+\left(2\mathrm{r}_0\right)^{\lambda}\left(\left(a\nabla(a\sharp_{(p,\tau)}b)\right)^{\lambda}-(a\sharp_{(p,\frac{\tau}{2})}b)^{\lambda}\right)$$
$$\leqslant\left(a\nabla_{\nu}b\right)^{\lambda}-\left(a\sharp_{p,\nu}b\right)^{\lambda}, \tag{24}$$

where $\mathrm{r}_0=\min\{\frac{\nu}{\tau},1-\frac{\nu}{\tau}\}$, *and*

$$\left(\frac{1-\tau}{1-\nu}\right)^{\lambda}\left((a\nabla_{\nu}b)^{\lambda}-(a\sharp_{(p,\nu)}b)^{\lambda}\right)+\left(2\mathrm{R}_0\right)^{\lambda}\left(\left(b\nabla(a\sharp_{(p,\nu)}b)\right)^{\lambda}-(a\sharp_{(p,\frac{1+\nu}{2})}b)^{\lambda}\right)$$
$$\leqslant\left(a\nabla_{\tau}b\right)^{\lambda}-\left(a\sharp_{(p,\tau)}b\right)^{\lambda}, \tag{25}$$

where $\mathrm{R}_0=\min\{\frac{1-\tau}{1-\nu},1-\frac{1-\tau}{1-\nu}\}$.

By taking the limit for $p\longrightarrow0$ in Theorem 4, we obtain the following theorem, which present one refinement term of the main result of [3].

THEOREM 5. *Let $a,b>0,\ 0\leqslant\nu\leqslant\tau\leqslant1$ and $\lambda\geqslant1$. Then we have*

$$\left(\frac{\nu}{\tau}\right)^{\lambda}\left((a\nabla_{\tau}b)^{\lambda}-(a\sharp_{\tau}b)^{\lambda}\right)+\left(2\mathrm{r}_0\right)^{\lambda}\left((a\nabla(a\sharp_{\tau}b))^{\lambda}-(a\sharp_{\frac{\tau}{2}}b)^{\lambda}\right)$$
$$\leqslant\left(a\nabla_{\nu}b\right)^{\lambda}-\left(a\sharp_{\nu}b\right)^{\lambda}, \tag{26}$$

where $\mathrm{r}_0 = \min\{\frac{\nu}{\tau}, 1-\frac{\nu}{\tau}\}$, *and*

$$\left(\frac{1-\tau}{1-\nu}\right)^{\lambda}\Big((a\nabla_{\nu}b)^{\lambda}-(a\sharp_{\nu}b)^{\lambda}\Big)+\Big(2\mathrm{R}_0\Big)^{\lambda}\Big((b\nabla(a\sharp_{\nu}b))^{\lambda}-(a\sharp_{\frac{1+\nu}{2}}b)^{\lambda}\Big) \leqslant \Big(a\nabla_{\tau}b\Big)^{\lambda}-\Big(a\sharp_{\tau}b\Big)^{\lambda}, \tag{27}$$

where $\mathrm{R}_0 = \min\{\frac{1-\tau}{1-\nu}, 1-\frac{1-\tau}{1-\nu}\}$.

Take $p=-1$ in Theorem 4, we obtain the following theorem, which is the corresponding refinement of theorem 2.1 in [13].

THEOREM 6. *Let* $a,b>0,\ 0\leqslant\nu\leqslant\tau\leqslant 1$ *and* $\lambda\geqslant 1$. *Then we have*

$$\left(\frac{\nu}{\tau}\right)^{\lambda}\Big((a\nabla_{\tau}b)^{\lambda}-(a!_{\tau}b)^{\lambda}\Big)+\Big(2\mathrm{r}_0\Big)^{\lambda}\Big((a\nabla(a!_{\tau}b))^{\lambda}-(a!_{\frac{\tau}{2}}b)^{\lambda}\Big) \leqslant \Big(a\nabla_{\nu}b\Big)^{\lambda}-\Big(a!_{\nu}b\Big)^{\lambda}, \tag{28}$$

where $\mathrm{r}_0 = \min\{\frac{\nu}{\tau}, 1-\frac{\nu}{\tau}\}$, *and*

$$\left(\frac{1-\tau}{1-\nu}\right)^{\lambda}\Big((a\nabla_{\nu}b)^{\lambda}-(a!_{\nu}b)^{\lambda}\Big)+\Big(2\mathrm{R}_0\Big)^{\lambda}\Big((b\nabla(a!_{\nu}b))^{\lambda}-(a!_{\frac{1+\nu}{2}}b)^{\lambda}\Big) \leqslant \Big(a\nabla_{\tau}b\Big)^{\lambda}-\Big(a!_{\tau}b\Big)^{\lambda}, \tag{29}$$

where $\mathrm{R}_0 = \min\{\frac{1-\tau}{1-\nu}, 1-\frac{1-\tau}{1-\nu}\}$.

4. Some refinements of matrix inequalities

In this section, we present some new matrix inequalities, that extend some known results in the literature. We point out that the results in this section are valid for the algebra $\mathscr{B}(\mathscr{H})$ instead of $\mathbf{M}_n$. However, our discussion will be limited to $\mathbf{M}_n$ only. We start this section by the following Lemma quoted from [16, p. 3].

LEMMA 4. *Let* $A\in\mathbf{M}_n$ *be Hermitian. If* f *and* g *are both continuous real valued functions on an interval that contains the spectrum of* A, *with* $f(t)\geqslant g(t)$ *for* $t\in Sp(A)$ (*where* $Sp(A)$ *stands for the spectrum of* A), *then* $f(A)\geqslant g(A)$.

The following theorem represents the matrix version of Theorem 4.

THEOREM 7. *Let* $A,B\in\mathbf{M}_n^{++},\ 0<\nu\leqslant\tau<1,\ p\in\mathbb{R}\setminus\{0\}$ *and* $\lambda\geqslant 1$. *Then we have*

$$\left(\frac{\nu}{\tau}\right)^{\lambda}\Big(A\sharp_{\lambda}(A\nabla_{\tau}B)-A\sharp_{\lambda}(A\sharp_{p,\tau}B)\Big)+\Big(2\mathrm{r}_0\Big)^{\lambda}\Big(A\sharp_{\lambda}(A\nabla(A\sharp_{p,\tau}B))-A\sharp_{p,\frac{\tau}{2}}B)\Big) \leqslant A\sharp_{\lambda}(A\nabla_{\nu}B)-A\sharp_{\lambda}(A\sharp_{p,\nu}B), \tag{30}$$

where $\mathrm{r}_0 = \min\{\frac{\nu}{\tau}, 1-\frac{\nu}{\tau}\}$, *and*

$$\left(\frac{1-\tau}{1-\nu}\right)^{\lambda}\left(A\sharp_{\lambda}(A\nabla_{\nu}B) - A\sharp_{\lambda}(A\sharp_{p,\nu}B)\right) + \left(2\mathrm{R}_0\right)^{\lambda}\left(A\sharp_{\lambda}\left(B\nabla(A\sharp_{p,\tau}B)\right) - A\sharp_{p,\frac{1+\nu}{2}}B\right)\right) \leqslant A\sharp_{\lambda}(A\nabla_{\tau}B) - A\sharp_{\lambda}(A\sharp_{p,\tau}B), \tag{31}$$

where $\mathrm{R}_0 = \min\{\frac{1-\tau}{1-\nu}, 1-\frac{1-\tau}{1-\nu}\}$.

Proof. Let $a=1$ in inequality (24), then

$$\begin{aligned}&\left(\frac{\nu}{\tau}\right)^{\lambda}\left(((1-\tau)+\tau b)^{\lambda} \;-((1-\tau)+\tau b^{p})^{\frac{\lambda}{p}}\right)\\&+(2\mathrm{r}_0)^{\lambda}\left[\left(\frac{1+((1-\tau)+\tau b^{p})^{\frac{1}{p}}}{2}\right)^{\lambda} - \left(\left(1-\frac{\tau}{2}\right)+\frac{\tau}{2}b^{p}\right)^{\frac{\lambda}{p}}\right]\\&\leqslant ((1-\nu)+\nu b)^{\lambda} - ((1-\nu)+\nu b^{p})^{\frac{\lambda}{p}}.\end{aligned} \tag{32}$$

The matrix $C := A^{\frac{-1}{2}}BA^{\frac{-1}{2}}$ has a positive spectrum, then by Lemma 4 and inequality (32) we get

$$\begin{aligned}&\left(\frac{\nu}{\tau}\right)^{\lambda}\left(((1-\tau)I+\tau C)^{\lambda} \;-((1-\tau)I+\tau C^{p})^{\frac{\lambda}{p}}\right)\\&+(2\mathrm{r}_0)^{\lambda}\left[\left(\frac{I+((1-\tau)I+\tau C^{p})^{\frac{1}{p}}}{2}\right)^{\lambda} - \left(\left(1-\frac{\tau}{2}\right)I+\frac{\tau}{2}C^{p}\right)^{\frac{\lambda}{p}}\right]\\&\leqslant ((1-\nu)I+\nu C)^{\lambda} - ((1-\nu)I+\nu C^{p})^{\frac{\lambda}{p}}.\end{aligned} \tag{33}$$

Finally, multiply the inequality (33) by $A^{\frac{1}{2}}$ on the left and right hand sides we get

$$\left(\frac{\nu}{\tau}\right)^{\lambda}\left(A\sharp_{\lambda}(A\nabla_{\tau}B) - A\sharp_{\lambda}(A\sharp_{p,\tau}B)\right) + \left(2\mathrm{r}_0\right)^{\lambda}\left(A\sharp_{\lambda}\left(A\nabla(A\sharp_{p,\tau}B)\right) - A\sharp_{p,\frac{\tau}{2}}B\right)\right) \leqslant A\sharp_{\lambda}(A\nabla_{\nu}B) - A\sharp_{\lambda}(A\sharp_{p,\nu}B).$$

Using the same technique, we can obtain the other inequality. This completes the proof. □

The following lemma sets up the essential features for proving the next corollary.

LEMMA 5. ([24]) *Let* $A,B \in \mathbf{M}_n^{++}$ *and let* α,β *two real numbers. Then*

$$A\sharp_{\alpha}(A\sharp_{\beta}B) = A\sharp_{\alpha\beta}B.$$

Take the limit as $p \longrightarrow 0$ in Theorem 7 and using Lemma 5, we obtain the following corollary which presents the matrix version of Theorem 5.

COROLLARY 4. *Let* $A,B \in \mathbf{M}_n^{++}$, $0 < \nu \leqslant \tau < 1$ *and* $\lambda \geqslant 1$. *Then we have*

$$\left(\frac{\nu}{\tau}\right)^{\lambda}\left(A\sharp_{\lambda}(A\nabla_{\tau}B) - A\sharp_{\lambda\tau}B\right) + \left(2\mathrm{r}_0\right)^{\lambda}\left(A\sharp_{\lambda}(A\nabla(A\sharp_{\tau}B)) - A\sharp_{\frac{\tau}{2}}B)\right) \leqslant A\sharp_{\lambda}(A\nabla_{\nu}B) - A\sharp_{\lambda\nu}B, \tag{34}$$

where $\mathrm{r}_0 = \min\{\frac{\nu}{\tau}, 1-\frac{\nu}{\tau}\}$, *and*

$$\left(\frac{1-\tau}{1-\nu}\right)^{\lambda}\left(A\sharp_{\lambda}(A\nabla_{\nu}B) - A\sharp_{\lambda\nu}B\right) + \left(2\mathrm{R}_0\right)^{\lambda}\left(A\sharp_{\lambda}(B\nabla(A\sharp_{\tau}B)) - A\sharp_{\frac{1+\nu}{2}}B)\right) \leqslant A\sharp_{\lambda}(A\nabla_{\tau}B) - A\sharp_{\lambda\tau}B, \tag{35}$$

where $\mathrm{R}_0 = \min\{\frac{1-\tau}{1-\nu}, 1-\frac{1-\tau}{1-\nu}\}$.

By taking $p=-1$ in Theorem 7, we get the following result, which presents the matrix version of Theorem 6.

COROLLARY 5. *Let* $A,B \in \mathbf{M}_n^{++}$, $0 < \nu \leqslant \tau < 1$ *and* $\lambda \geqslant 1$. *Then we have*

$$\left(\frac{\nu}{\tau}\right)^{\lambda}\left(A\sharp_{\lambda}(A\nabla_{\tau}B) - A\sharp_{\lambda}(A!_{\tau}B)\right) + \left(2\mathrm{r}_0\right)^{\lambda}\left(A\sharp_{\lambda}(A\nabla(A!_{\tau}B)) - A!_{\frac{\tau}{2}}B)\right) \leqslant A\sharp_{\lambda}(A\nabla_{\nu}B) - A\sharp_{\lambda}(A!_{\nu}B), \tag{36}$$

where $\mathrm{r}_0 = \min\{\frac{\nu}{\tau}, 1-\frac{\nu}{\tau}\}$, *and*

$$\left(\frac{1-\tau}{1-\nu}\right)^{\lambda}\left(A\sharp_{\lambda}(A\nabla_{\nu}B) - A\sharp_{\lambda}(A!_{\nu}B)\right) + \left(2\mathrm{R}_0\right)^{\lambda}\left(A\sharp_{\lambda}(B\nabla(A!_{\tau}B)) - A!_{\frac{1+\nu}{2}}B)\right) \leqslant A\sharp_{\lambda}(A\nabla_{\tau}B) - A\sharp_{\lambda}(A!_{\tau}B), \tag{37}$$

where $\mathrm{R}_0 = \min\{\frac{1-\tau}{1-\nu}, 1-\frac{1-\tau}{1-\nu}\}$.

5. Matrix norm inequalities

In this part of the paper, by selecting some appropriate convex and log-convex functions, we obtain new refinements of some results in [18].

5.1. Matrix norm inequalities via convexity

The classical Young inequality $a\sharp_{\nu}b \leqslant a\nabla_{\nu}b$ has been extended to matrices as follows

$$|||A^{1-\nu}XB^{\nu}||| \leqslant (1-\nu)|||AX||| + \nu|||XB|||, 0 \leqslant \nu \leqslant 1. \tag{38}$$

It is known that for two matrices $A,B \in \mathbf{M}_n^{+}$ and $X \in \mathbf{M}_n$, the function $f(\nu) = |||A^{1-\nu}XB^{\nu}|||$ is convex on $[0,1]$, for any unitarily invariant norm $|||\cdot|||$ on $\mathbf{M}_n$ (see [19]). Then by using Corollary 2 we obtain the following theorem which present one refinement term of the corresponding Young's inequality (38) for matrices.

THEOREM 8. *Let* $A,B\in \mathbf{M}_n^{++},\ X\in \mathbf{M}_n\setminus\{0\},\ 0<\nu\leqslant\tau<1$ *and* $\lambda\geqslant 1$. *Then we have*

$$\left(\frac{\nu}{\tau}\right)^{\lambda}\left((|||AX|||\nabla_{\tau}|||XB|||)^{\lambda}-(|||A^{1-\tau}XB^{\tau}|||)^{\lambda}\right) + (2\mathrm{r}_0)^{\lambda}\left(\left(|||AX|||\nabla|||A^{1-\tau}XB^{\tau}|||\right)^{\lambda}-|||A^{1-\frac{\tau}{2}}XB^{\frac{\tau}{2}}|||^{\lambda}\right) \leqslant \left((1-\nu)|||AX|||+\nu|||XB|||\right)^{\lambda}-\left(|||A^{1-\nu}XB^{\nu}|||\right)^{\lambda}. \tag{39}$$

where $\mathrm{r}_0=\min\{\frac{\nu}{\tau},1-\frac{\nu}{\tau}\}$, *and*

$$\left(\frac{1-\tau}{1-\nu}\right)^{\lambda}\left((|||AX|||\nabla_{\nu}|||XB|||)^{\lambda}-(|||A^{1-\nu}XB^{\nu}|||)^{\lambda}\right) + (2\mathrm{R}_0)^{\lambda}\left(\left(|||XB|||\nabla|||A^{1-\nu}XB^{\nu}|||\right)^{\lambda}-|||A^{1-\frac{1+\nu}{2}}XB^{\frac{1+\nu}{2}}|||^{\lambda}\right) \leqslant \left((1-\tau)|||AX|||+\tau|||XB|||\right)^{\lambda}-\left(|||A^{1-\tau}XB^{\tau}|||\right)^{\lambda}, \tag{40}$$

where $\mathrm{R}_0=\min\{\frac{1-\tau}{1-\nu},1-\frac{1-\tau}{1-\nu}\}$.

Also it is known that for two matrices $A,B\in\mathbf{M}_n^{+}$ the function $f(\nu)=\mathrm{tr}(A^{1-\nu}B^{\nu})$ is convex on $[0,1]$, (see [19]). Then by using Corollary 2, we obtain the following theorem which present one refinement term of the corresponding Young's inequality for trace of matrices.

THEOREM 9. *Let* $A,B\in\mathbf{M}_n^{+},\ 0<\nu\leqslant\tau<1$ *and* $\lambda\geqslant 1$. *Then we have*

$$\left(\frac{\nu}{\tau}\right)^{\lambda}\left\{\mathrm{tr}^{\lambda}(A\nabla_{\tau}B)-\mathrm{tr}^{\lambda}\left(A^{1-\tau}B^{\tau}\right)\right\} + (2\mathrm{r}_0)^{\lambda}\left(\left(\mathrm{tr}(A)\nabla\mathrm{tr}(A^{1-\tau}B^{\tau})\right)^{\lambda}-tr^{\lambda}(A^{1-\frac{\tau}{2}}B^{\frac{\tau}{2}})\right) \leqslant \mathrm{tr}^{\lambda}(A\nabla_{\nu}B)-\mathrm{tr}^{\lambda}\left(A^{1-\nu}B^{\nu}\right),$$

where $\mathrm{r}_0=\min\{\frac{\nu}{\tau},1-\frac{\nu}{\tau}\}$, *and*

$$\left(\frac{1-\tau}{1-\nu}\right)^{\lambda}\left\{\mathrm{tr}^{\lambda}(A\nabla_{\nu}B)-\mathrm{tr}^{\lambda}\left(A^{1-\nu}B^{\nu}\right)\right\} + (2\mathrm{R}_0)^{\lambda}\left(\left(\mathrm{tr}(B)\nabla tr(A^{1-\nu}B^{\nu})\right)^{\lambda}-\mathrm{tr}(A^{1-\frac{1+\nu}{2}}B^{\frac{1+\nu}{2}})^{\lambda}\right) \leqslant \mathrm{tr}^{\lambda}(A\nabla_{\tau}B)-\mathrm{tr}^{\lambda}\left(A^{1-\tau}B^{\tau}\right),$$

where $\mathrm{R}_0=\min\{\frac{1-\tau}{1-\nu},1-\frac{1-\tau}{1-\nu}\}$.

Now, for two matrices $A,B\in\mathbf{M}_n^{+}$ and $X\in\mathbf{M}_n$, we consider the function $f(\nu)=|||A^{1-\nu}XB^{\nu}+A^{\nu}XB^{1-\nu}|||$. From [18], we know that this function is convex on $[0,1]$, for any unitarily invariant norm $|||.|||$ on $\mathbf{M}_n$. Consequently, we may apply Corollary 2 for this function to get the following Heinz-type inequality [1].

THEOREM 10. *Let* $A,B \in \mathbf{M}_n^+$ *and* $X \in \mathbf{M}_n, X \neq 0, 0 < \nu \leqslant \tau < 1$ *and* $\lambda \geqslant 1$. *Then, for any unitarily invariant norm* $|||.|||$, *we have*

$$\begin{aligned}|||AX+XB|||^{\lambda} \geqslant\ & |||A^{1-\nu}XB^{\nu}+A^{\nu}XB^{1-\nu}|||^{\lambda}\\ &+\left(\frac{\nu}{\tau}\right)^{\lambda}\left(|||AX+XB|||^{\lambda}-|||A^{1-\tau}XB^{\tau}+A^{\tau}XB^{1-\tau}|||^{\lambda}\right)\\ &+(2\mathrm{r}_0)^{\lambda}\Big(\left(|||AX+XB|||\nabla|||A^{1-\tau}XB^{\tau}+A^{\tau}XB^{1-\tau}|||\right)^{\lambda}\\ &-|||A^{1-\frac{\tau}{2}}XB^{\frac{\tau}{2}}+A^{\frac{\tau}{2}}XB^{1-\frac{\tau}{2}}|||^{\lambda}\Big),\end{aligned}$$

where $\mathrm{r}_0 = \min\{\frac{\nu}{\tau}, 1-\frac{\nu}{\tau}\}$, *and*

$$\begin{aligned}|||AX+XB|||^{\lambda} \geqslant\ & |||A^{1-\tau}XB^{\tau}+A^{\tau}XB^{1-\tau}|||^{\lambda}\\ &+\left(\frac{1-\tau}{1-\nu}\right)^{\lambda}\left(|||AX+XB|||^{\lambda}-|||A^{1-\nu}XB^{\nu}+A^{\nu}XB^{1-\nu}|||^{\lambda}\right)\\ &+(2\mathrm{R}_0)^{\lambda}\Big(\left(|||AX+XB|||\nabla|||A^{1-\nu}XB^{\nu}+A^{\nu}XB^{1-\nu}|||\right)^{\lambda}\\ &-|||A^{1-\frac{1+\nu}{2}}XB^{\frac{1+\nu}{2}}+A^{\frac{1+\nu}{2}}XB^{1-\frac{1+\nu}{2}}|||^{\lambda}\Big),\end{aligned}$$

where $\mathrm{R}_0 = \min\{\frac{1-\tau}{1-\nu}, 1-\frac{1-\tau}{1-\nu}\}$.

Let A and B be two Hermitian matrices and let g be convex function on $\mathbb{R}$. Then

$$f_k(t) = \sum_{j=1}^{k} \lambda_j(g((1-t)A+tB)), 1 \leqslant k \leqslant n$$

is convex function on $\mathbb{R}$ for any $k \in \{1,\dots,n\}$. By applying Corollary 2 for the function $f_k(t) = \sum_{j=1}^{k} \lambda_j((1-t)A+tB), 0 \leqslant t \leqslant 1$, we obtain the following eigenvalues version.

COROLLARY 6. *Let* A *and* B *be Hermitian in* $\mathbf{M}_n$ *and let* $0 < \nu \leqslant \tau < 1$. *Then for* $1 \leqslant k \leqslant n$ *and* $\lambda \geqslant 1$, *we have*

$$\begin{aligned}&\left(\sum_{j=1}^{k}((1-\nu)\lambda_j(A)+\nu\lambda_j(B))\right)^{\lambda}-\left(\sum_{j=1}^{k}\lambda_j((1-\nu)A+\nu B)\right)^{\lambda}\\ &-(2\mathrm{r}_0)^{\lambda}\left[\left(\left(\sum_{j=1}^{k}\lambda_j(A)\right)\nabla\left(\sum_{j=1}^{k}\lambda_j((1-\tau)A+\tau B)\right)\right)^{\lambda}\right.\\ &\left.-\left(\sum_{j=1}^{k}\lambda_j\left(\left(1-\frac{\tau}{2}\right)A+\frac{\tau}{2}B\right)\right)^{\lambda}\right]\\ &\geqslant\left(\frac{\nu}{\tau}\right)^{\lambda}\left(\left(\sum_{j=1}^{k}((1-\tau)\lambda_j(A)+\tau\lambda_j(B))\right)^{\lambda}-\left(\sum_{j=1}^{k}\lambda_j((1-\tau)A+\tau B)\right)^{\lambda}\right),\end{aligned}$$

where $\mathrm{r}_0 = \min\{\frac{\nu}{\tau}, 1-\frac{\nu}{\tau}\}$, *and*

$$\left(\sum_{j=1}^{k}((1-\tau)\lambda_j(A)+\tau\lambda_j(B))\right)^{\lambda} - \left(\sum_{j=1}^{k}\lambda_j((1-\tau)A+\tau B)\right)^{\lambda}$$
$$-(2\mathrm{R}_0)^{\lambda}\left[\left(\left(\sum_{j=1}^{k}\lambda_j(B)\right)\nabla\left(\sum_{j=1}^{k}\lambda_j((1-\nu)A+\nu B)\right)\right)^{\lambda}\right.$$
$$\left.-\left(\sum_{j=1}^{k}\lambda_j\left(\left(1-\frac{1+\nu}{2}\right)A+\frac{1+\nu}{2}B\right)\right)^{\lambda}\right]$$
$$\geqslant\left(\frac{1-\tau}{1-\nu}\right)^{\lambda}\left(\left(\sum_{j=1}^{k}((1-\nu)\lambda_j(A)+\nu\lambda_j(B))\right)^{\lambda}-\left(\sum_{j=1}^{k}\lambda_j((1-\nu)A+\nu B)\right)^{\lambda}\right),$$

where $\mathrm{R}_0 = \min\{\frac{1-\tau}{1-\nu}, 1-\frac{1-\tau}{1-\nu}\}$.

Notice that $\sum_{j=1}^{k}\lambda_j(A)$ becomes the Ky Fan's norm $\|A\|_{(k)}$ when A is positive [1]. In this case, the above inequality may be stated as follows.

COROLLARY 7. *Let* $A, B \in \mathbf{M}_n^+$ *and* $0 < \nu \leqslant \tau < 1$. *Then for* $1 \leqslant k \leqslant n$ *and* $\lambda \geqslant 1$, *we have*

$$\left(((1-\nu)\|A\|_{(k)}+\nu\|B\|_{(k)})^{\lambda}-(\|(1-\nu)A+\nu B\|_{(k)})^{\lambda}\right)$$
$$-(2\mathrm{r}_0)^{\lambda}\left[(\|A\|_{(k)}\nabla\|(1-\tau)A+\tau B\|_{(k)})^{\lambda}-\left(\left\|\left(1-\frac{\tau}{2}\right)A+\frac{\tau}{2}B\right\|_{(k)}\right)^{\lambda}\right]$$
$$\geqslant\left(\frac{\nu}{\tau}\right)^{\lambda}\left(((1-\tau)\|A\|_{(k)}+\tau\|B\|_{(k)})^{\lambda}-(\|(1-\tau)A+\tau B\|_{(k)})^{\lambda}\right),$$

where $\mathrm{r}_0 = \min\{\frac{\nu}{\tau}, 1-\frac{\nu}{\tau}\}$, *and*

$$((1-\tau)\|A\|_{(k)}+\tau\|B\|_{(k)})^{\lambda}-(\|(1-\tau)A+\tau B\|_{(k)})^{\lambda}$$
$$-(2\mathrm{R}_0)^{\lambda}\left[(\|B\|_{(k)}\nabla\|(1-\nu)A+\nu B\|_{(k)})^{\lambda}-\left(\left\|\left(1-\frac{1+\nu}{2}\right)A+\frac{1+\nu}{2}B\right\|_{(k)}\right)^{\lambda}\right]$$
$$\geqslant\left(\frac{1-\tau}{1-\nu}\right)^{\lambda}\left(((1-\nu)\|A\|_{(k)}+\nu\|B\|_{(k)})^{\lambda}-(\|(1-\nu)A+\nu B\|_{(k)})^{\lambda}\right),$$

where $\mathrm{R}_0 = \min\{\frac{1-\tau}{1-\nu}, 1-\frac{1-\tau}{1-\nu}\}$.

5.2. Matrix inequalities via log-convexity

For all $A, B \in \mathbf{M}_n$, any real number $r > 0$ and every unitarily invariant norm, Horn and Mathias [8, 9] obtained the following matrix Cauchy-Schwarz inequality

$$|||\,|A^*B|^r\,|||^2 \leqslant |||(A^*A)^r|||.|||(B^*B)^r|||. \tag{41}$$

Bhatia and Davis proved in [4] a more general form of Cauchy-Schwarz inequality; for $A,B \in \mathbf{M}_n^+, X \in \mathbf{M}_n$ and $r > 0$,

$$||| \, |A^*XB|^r \, ||| \leqslant ||| \, |AA^*X|^r \, |||.||| \, |XBB^*|^r \, |||, \tag{42}$$

which is equivalent to,

$$||| \, |A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r \, |||^2 \leqslant ||| \, |AX|^r \, |||.||| \, |XB|^r \, |||. \tag{43}$$

For $A,B \in \mathbf{M}_n^+$ and $\nu \in [0,1]$, we have the Hölder type inequality [11]

$$||| \, |A^{1-\nu}XB^{\nu}|^r \, ||| \leqslant ||| \, |AX|^r \, |||^{1-\nu}.||| \, |XB|^r \, |||^{\nu}. \tag{44}$$

Let $A,B \in \mathbf{M}_n^{++}$ and $X \in \mathbf{M}_n$, the function $f(\nu) = ||| \, |A^{1-\nu}XB^{\nu}|^r \, |||$ is log-convex on $[0,1]$, for any unitarily invariant norm $||| \cdot |||$ on $\mathbf{M}_n$ (see [11]). By applying Corollary 3, we obtain the following new refinement and reverse of the Hölder type inequality (44).

THEOREM 11. *Let $A,B \in \mathbf{M}_n^{++}$, $X \in \mathbf{M}_n$ and $0 < \nu \leqslant \tau < 1$. Then*

$$\begin{aligned} ||| \, |AX|^r \, |||^{1-\nu}.||| \, |XB|^r \, |||^{\nu} \geqslant\ & ||| \, |A^{1-\nu}XB^{\nu}|^r \, ||| \\ & + \left(||| \, |AX|^r \, |||^{1-\tau}.||| \, |XB|^r \, |||^{\tau}\right)^{\frac{\nu}{\tau}} - ||| \, |A^{1-\tau}XB^{\tau}|^r \, |||^{\frac{\nu}{\tau}} \\ & + \left(\sqrt{||| \, |AX|^r \, |||.||| \, |A^{1-\tau}XB^{\tau}|^r \, |||}\right)^{2\mathrm{r}_0} - ||| \, |A^{1-\frac{\tau}{2}}XB^{\frac{\tau}{2}}|^r \, |||^{2\mathrm{r}_0}, \end{aligned}$$

where $\mathrm{r}_0 = \min\{\frac{\nu}{\tau}, 1 - \frac{\nu}{\tau}\}$, *and*

$$\begin{aligned} ||| \, |AX|^r \, |||^{1-\tau}.||| \, |XB|^r \, |||^{\tau} \geqslant\ & ||| \, |A^{1-\tau}XB^{\tau}|^r \, ||| \\ & + \left(||| \, |AX|^r \, |||^{1-\nu}.||| \, |XB|^r \, |||^{\nu}\right)^{\frac{1-\tau}{1-\nu}} - ||| \, |A^{1-\tau}XB^{\tau}|^r \, |||^{\frac{1-\tau}{1-\nu}} \\ & + \left(\sqrt{||| \, |XB|^r \, |||.||| \, |A^{1-\nu}XB^{\nu}|^r \, |||}\right)^{2\mathrm{R}_0} - ||| \, |A^{1-\frac{1+\nu}{2}}XB^{\frac{1+\nu}{2}}|^r \, |||^{2\mathrm{R}_0}, \end{aligned}$$

where $\mathrm{R}_0 = \min\{\frac{1-\tau}{1-\nu}, 1 - \frac{1-\tau}{1-\nu}\}$.

For every $A,B \in \mathbf{M}_n^{++}$ and $X \in \mathbf{M}_n$, the function $f(\nu) = ||| \, |A^{\nu}XB^{\nu}|^r \, |||$ is log-convex on $[0,1]$, for any unitarily invariant norm $||| \cdot |||$ on $\mathbf{M}_n$ (see [11]). Applying Corollary 3, we obtain the following new refinement and reverse of the Hölder type inequality (44).

THEOREM 12. *Let $A,B \in \mathbf{M}_n^{++}$, $X \in \mathbf{M}_n$ and $0 < \nu \leqslant \tau < 1$. Then*

$$\begin{aligned} ||| \, |AXB|^r \, |||^{\nu}.||| \, |X|^r \, |||^{1-\nu} \geqslant\ & ||| \, |A^{\nu}XB^{\nu}|^r \, ||| \\ & + \left(||| \, |AXB|^r \, |||^{\tau}.||| \, |X|^r \, |||^{1-\tau}\right)^{\frac{\nu}{\tau}} - ||| \, |A^{\tau}XB^{\tau}|^r \, |||^{\frac{\nu}{\tau}} \\ & + \left(\sqrt{||| \, |AXB|^r \, |||.||| \, |A^{\tau}XB^{\tau}|^r \, |||}\right)^{2\mathrm{r}_0} - ||| \, |A^{\frac{\tau}{2}}XB^{\frac{\tau}{2}}|^r \, |||^{2\mathrm{r}_0}, \end{aligned}$$

where $\mathrm{r}_0 = \min\{\frac{\nu}{\tau}, 1-\frac{\nu}{\tau}\}$, *and*

$$\begin{aligned}||| \, |AXB|^r \, |||^{\tau}. ||| \, |X|^r \, |||^{1-\tau} \geqslant & ||| \, |A^{\tau}XB^{\tau}|^r \, ||| \\ & + \left(||| \, |AXB|^r \, |||^{\nu}. ||| \, |X|^r \, |||^{1-\nu}\right)^{\frac{1-\tau}{1-\nu}} - ||| \, |A^{\tau}XB^{\tau}|^r \, |||^{\frac{1-\tau}{1-\nu}} \\ & + \left(\sqrt{||| \, |AXB|^r \, |||. ||| \, |A^{\nu}XB^{\nu}|^r \, |||}\right)^{2\mathrm{R}_0} - ||| \, |A^{\frac{1+\nu}{2}}XB^{\frac{1+\nu}{2}}|^r \, |||^{2\mathrm{R}_0},\end{aligned}$$

where $\mathrm{R}_0 = \min\{\frac{1-\tau}{1-\nu}, 1-\frac{1-\tau}{1-\nu}\}$.

Notice that Theorem 12 present two refining terms and it reversed for the correspondent Hölder type inequalities for unitary invariant norm. In particular for $X = I$, we get the following theorem which present two refinement terms and reverses of the classical inequality $||| \, |AB|^r \, |||^{\nu} \geqslant ||| \, |A^{\nu}B^{\nu}|^r \, |||$, for $0 < \nu \leqslant 1$.

THEOREM 13. *Let* $A, B \in \mathbf{M}_n^{++}$ *and* $0 < \nu \leqslant \tau < 1$. *Then*

$$\begin{aligned}||| \, |AB|^r \, |||^{\nu} \geqslant & ||| \, |A^{\nu}B^{\nu}|^r \, ||| \\ & + (||| \, |AB|^r \, |||^{\tau}.)^{\frac{\nu}{\tau}} - ||| \, |A^{\tau}B^{\tau}|^r \, |||^{\frac{\nu}{\tau}} \\ & + \left(\sqrt{||| \, |AB|^r \, |||. ||| \, |A^{\tau}B^{\tau}|^r \, |||}\right)^{2\mathrm{r}_0} - ||| \, |A^{\frac{\tau}{2}}B^{\frac{\tau}{2}}|^r \, |||^{2\mathrm{r}_0},\end{aligned}$$

where $\mathrm{r}_0 = \min\{\frac{\nu}{\tau}, 1-\frac{\nu}{\tau}\}$, *and*

$$\begin{aligned}||| \, |AB|^r \, |||^{\tau} \geqslant & ||| \, |A^{\tau}B^{\tau}|^r \, ||| \\ & + (||| \, |AB|^r \, |||^{\nu})^{\frac{1-\tau}{1-\nu}} - ||| \, |A^{\tau}XB^{\tau}|^r \, |||^{\frac{1-\tau}{1-\nu}} \\ & + \left(\sqrt{||| \, |AB|^r \, |||. ||| \, |A^{\nu}B^{\nu}|^r \, |||}\right)^{2\mathrm{R}_0} - ||| \, |A^{\frac{1+\nu}{2}}B^{\frac{1+\nu}{2}}|^r \, |||^{2\mathrm{R}_0},\end{aligned}$$

where $\mathrm{R}_0 = \min\{\frac{1-\tau}{1-\nu}, 1-\frac{1-\tau}{1-\nu}\}$.

It was shown in [11], that the function $f(\nu) = \mathrm{tr}(A^{1-\nu}B^{\nu})$ is log-convex on $[0,1]$. Applying Corollary 3, we obtain the following new refinement and reverse of the Hölder type inequality for traces.

THEOREM 14. *Let* $A, B \in \mathbf{M}_n^{++}$, $X \in \mathbf{M}_n$ *and* $0 < \nu \leqslant \tau < 1$. *Then*

$$\begin{aligned}\mathrm{tr}^{1-\nu}(A)\mathrm{tr}^{\nu}(B) \geqslant & \mathrm{tr}(A^{1-\nu}B^{\nu}) \\ & + \left(\mathrm{tr}^{1-\tau}(A)\mathrm{tr}^{\tau}(B)\right)^{\frac{\nu}{\tau}} - \mathrm{tr}^{\frac{\nu}{\tau}}(A^{1-\tau}B^{\tau}) \\ & + \left(\sqrt{\mathrm{tr}(A)\mathrm{tr}(A^{1-\tau}B^{\tau})}\right)^{2\mathrm{r}_0} - \mathrm{tr}^{2\mathrm{r}_0}\left(A^{1-\frac{\tau}{2}}B^{\frac{\tau}{2}}\right),\end{aligned}$$

where $\mathrm{r}_0 = \min\{\frac{\nu}{\tau}, 1-\frac{\nu}{\tau}\}$, *and*

$$\begin{aligned}\mathrm{tr}^{1-\tau}(A)\mathrm{tr}^{\tau}(B) \geqslant & \mathrm{tr}(A^{1-\tau}B^{\tau}) \\ & + \left(\mathrm{tr}^{1-\nu}(A)\mathrm{tr}^{\nu}(B)\right)^{\frac{1-\tau}{1-\nu}} - \mathrm{tr}^{\frac{1-\tau}{1-\nu}}(A^{1-\nu}B^{\nu}) \\ & + \left(\sqrt{\mathrm{tr}(B)\mathrm{tr}(A^{1-\nu}B^{\nu})}\right)^{2\mathrm{r}_0} - \mathrm{tr}^{2\mathrm{r}_0}\left(A^{1-\frac{1+\nu}{2}}B^{\frac{1+\nu}{2}}\right),\end{aligned}$$

where $\mathrm{R}_0 = \min\{\frac{1-\tau}{1-\nu}, 1-\frac{1-\tau}{1-\nu}\}$.

The previous Theorem present two refining terms and it reversed for Hölder type inequalities for the trace.

6. Inequalities for generalized numerical radius

The generalized numerical radius for A, denoted by $w_N(A)$, is obtained via the supremum of the norm over the real parts of all rotations of A i.e.

$$w_N(A) = \sup_{\theta\in\mathbb{R}} N\left(\mathrm{Re}\left(e^{i\theta}A\right)\right).$$

Where $X = \mathrm{Re}(X) + i\,\mathrm{Im}(X)$ is the Cartesian decomposition of $X \in \mathbf{M}_n, \mathrm{Re}(X) = \frac{X+X^*}{2}$ and $\mathrm{Im}(X) = \frac{X-X^*}{2i}$, and X^* denotes the adjoint of X. Simple computation shows that when N is the usual operator norm inherited from the inner product on H then $w_N(\cdot)$ coincides with the usual numerical radius norm $w(\cdot)$ which is defined as

$$w(A) = \sup_{\|x\|=1} |\langle Ax, x\rangle|.$$

On the other hand, it is known that if $A, B \in \mathbf{M}_n^+$ and $X \in \mathbf{M}_n$, then the function $f(\nu) = w_N(A^{1-\nu}XB^{\nu} + A^{\nu}XB^{1-\nu})$ is convex on $[0,1]$, for any unitarily invariant norm $N(.)$ on $\mathbf{M}_n$, [1]. Consequently, we may apply Corollary 2 for this function to get the following Heinz-type inequality for the numerical radius.

THEOREM 15. *Let* $A, B \in \mathbf{M}_n^+$ *and* $X \in \mathbf{M}_n, X \neq 0, 0 < \nu \leqslant \tau < 1$ *and* $\lambda \geqslant 1$. *Then, for any unitarily invariant norm* $|||.|||$, *we have*

$$\begin{aligned} w_N(AX+XB)^{\lambda} \geqslant\ & w_N(A^{1-\nu}XB^{\nu} + A^{\nu}XB^{1-\nu})^{\lambda} \\ & + \left(\frac{\nu}{\tau}\right)^{\lambda}\left(w_N(AX+XB)^{\lambda} - w_N(A^{1-\tau}XB^{\tau} + A^{\tau}XB^{1-\tau})^{\lambda}\right) \\ & + (2\mathrm{r}_0)^{\lambda}\Big(\left(w_N(AX+XB)\nabla w_N(A^{1-\tau}XB^{\tau} + A^{\tau}XB^{1-\tau})\right)^{\lambda} \\ & - w_N(A^{1-\frac{\tau}{2}}XB^{\frac{\tau}{2}} + A^{\frac{\tau}{2}}XB^{1-\frac{\tau}{2}})^{\lambda}\Big), \end{aligned}$$

where $\mathrm{r}_0 = \min\{\frac{\nu}{\tau}, 1-\frac{\nu}{\tau}\}$, *and*

$$\begin{aligned} w_N(AX+XB)^{\lambda} \geqslant\ & w_N(A^{1-\tau}XB^{\tau} + A^{\tau}XB^{1-\tau})^{\lambda} \\ & + \left(\frac{1-\tau}{1-\nu}\right)^{\lambda}\left(w_N(AX+XB)^{\lambda} - w_N(A^{1-\nu}XB^{\nu} + A^{\nu}XB^{1-\nu})^{\lambda}\right) \\ & + (2\mathrm{R}_0)^{\lambda}\Big(\left(w_N(AX+XB)\nabla w_N(A^{1-\nu}XB^{\nu} + A^{\nu}XB^{1-\nu})\right)^{\lambda} \\ & - w_N(A^{1-\frac{1+\nu}{2}}XB^{\frac{1+\nu}{2}} + A^{\frac{1+\nu}{2}}XB^{1-\frac{1+\nu}{2}})^{\lambda}\Big), \end{aligned}$$

where $\mathrm{R}_0 = \min\{\frac{1-\tau}{1-\nu}, 1-\frac{1-\tau}{1-\nu}\}$.

Let $A,B \in \mathbf{M}_n^+$ and $X \in \mathbf{M}_n$, then the function $f(\nu) = w_N(A^{1-\nu}XB^{\nu})$ is convex on $[0,1]$, for any unitarily invariant norm $N(.)$ on $\mathbf{M}_n$, (see [1]). Consequently, we may apply Corollary 2 for this function to get the following Young-type inequality for the numerical radius.

THEOREM 16. *Let $A,B \in \mathbf{M}_n^+$ and $X \in \mathbf{M}_n, X \neq 0, 0 < \nu \leqslant \tau < 1$ and $\lambda \geqslant 1$. Then, for any unitarily invariant norm $N(.)$, we have*

$$\begin{aligned}&(w_N(AX)\nabla_\nu w_N(XB))^\lambda \\ \geqslant\ & w_N^\lambda(A^{1-\nu}XB^\nu) + \left(\frac{\nu}{\tau}\right)^\lambda \left((w_N(AX)\nabla_\tau w_N(XB))^\lambda - w_N^\lambda(A^{1-\tau}XB^\tau)\right) \\ &+ (2\mathrm{r}_0)^\lambda \left(\left(w_N(AX)\nabla w_N(A^{1-\tau}XB^\tau)\right)^\lambda - w_N(A^{1-\frac{\tau}{2}}XB^{\frac{\tau}{2}})^\lambda \right),\end{aligned}$$

where $\mathrm{r}_0 = \min\{\frac{\nu}{\tau}, 1-\frac{\nu}{\tau}\}$, *and*

$$\begin{aligned}&(w_N(AX)\nabla_\tau w_N(XB))^\lambda \\ \geqslant\ & w_N^\lambda(A^{1-\tau}XB^\tau) + \left(\frac{1-\tau}{1-\nu}\right)^\lambda \left((w_N(AX)\nabla_\nu w_N(XB))^\lambda - w_N^\lambda(A^{1-\nu}XB^\nu)\right) \\ &+ (2\mathrm{R}_0)^\lambda \left(\left(w_N(XB)\nabla w_N(A^{1-\nu}XB^\nu)\right)^\lambda - w_N(A^{1-\frac{1+\nu}{2}}XB^{\frac{1+\nu}{2}})^\lambda \right),\end{aligned}$$

where $\mathrm{R}_0 = \min\{\frac{1-\tau}{1-\nu}, 1-\frac{1-\tau}{1-\nu}\}$.

On the other hand, by using [1] and Proposition 2.5 of [23] it is known that if $A,B \in \mathbf{M}_n^+$ and $X \in \mathbf{M}_n$, then the function $f(\nu) = w_N(A^\nu XB^\nu)$ is log-convex on $[0,1]$, for any unitarily invariant norm $N(.)$ on $\mathbf{M}_n$, [1]. Consequently, we may apply Corollary 2 using this function to get the following Heinz-type inequality for the numerical radius.

THEOREM 17. *Let $A,B \in \mathbf{M}_n^{++}$, $X \in \mathbf{M}_n$ and $0 < \nu \leqslant \tau < 1$. Then*

$$\begin{aligned}&w_N^{1-\nu}(X)w_N^\nu(AXB) \\ \geqslant\ & w_N(A^\nu XB^\nu) + \left(w_N^{1-\tau}(X)w_N^\tau(AXB)\right)^{\frac{\nu}{\tau}} - w_N^{\frac{\nu}{\tau}}(A^\tau XB^\tau) \\ &+ \left(\sqrt{w_N(AXB)w_N(A^\tau XB^\tau)}\right)^{2\mathrm{r}_0} - w_N^{2\mathrm{r}_0}(A^{1-\frac{\tau}{2}}XB^{\frac{\tau}{2}}),\end{aligned}$$

where $\mathrm{r}_0 = \min\{\frac{\nu}{\tau}, 1-\frac{\nu}{\tau}\}$, *and*

$$\begin{aligned}&w_N^{1-\tau}(X)w_N^\tau(AXB) \\ \geqslant\ & w_N(A^\tau XB^\tau) + \left(w_N^{1-\nu}(X)w_N^\nu(AXB)\right)^{\frac{1-\tau}{1-\nu}} - w_N^{\frac{1-\tau}{1-\nu}}(A^{1-\nu}XB^{1-\nu}) \\ &+ \left(\sqrt{w_N(X)w_N(A^\nu XB^\nu)}\right)^{2\mathrm{r}_0} - w_N^{2\mathrm{r}_0}(A^{1-\frac{1+\nu}{2}}XB^{\frac{1+\nu}{2}}),\end{aligned}$$

where $\mathrm{R}_0 = \min\{\frac{1-\tau}{1-\nu}, 1-\frac{1-\tau}{1-\nu}\}$.

In particular for $X = I$ we get the following theorem.

THEOREM 18. *Let $A, B \in \mathbf{M}_n^{++}$, $X \in \mathbf{M}_n$ and $0 < \nu \leqslant \tau < 1$. Then*

$$\begin{aligned} w_N^{\nu}(AB) \geqslant\ & w_N(A^{\nu}B^{\nu}) + (w_N^{\tau}(AB))^{\frac{\nu}{\tau}} - w_N^{\frac{\nu}{\tau}}(A^{\tau}B^{\tau}) \\ & + \left(\sqrt{w_N(AB)w_N(A^{\tau}XB^{\tau})}\right)^{2\mathrm{r}_0} - w_N^{2\mathrm{r}_0}(A^{1-\frac{\tau}{2}}B^{\frac{\tau}{2}}), \end{aligned}$$

where $\mathrm{r}_0 = \min\{\frac{\nu}{\tau}, 1 - \frac{\nu}{\tau}\}$, *and*

$$\begin{aligned} w_N^{\tau}(AB) \geqslant\ & w_N(A^{\tau}B^{\tau}) + (w_N^{\nu}(AB))^{\frac{1-\tau}{1-\nu}} - w_N^{\frac{1-\tau}{1-\nu}}(A^{1-\nu}B^{1-\nu}) \\ & + \left(\sqrt{w_N(A^{\nu}B^{\nu})}\right)^{2\mathrm{r}_0} - w_N^{2\mathrm{r}_0}(A^{1-\frac{1+\nu}{2}}B^{\frac{1+\nu}{2}}), \end{aligned}$$

where $\mathrm{R}_0 = \min\{\frac{1-\tau}{1-\nu}, 1 - \frac{1-\tau}{1-\nu}\}$.

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