

## BIRKHOFF–JAMES ORTHOGONALITY IN CERTAIN TENSOR PRODUCTS OF BANACH SPACES

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*Abstract.* In this article, the relationship between Birkhoff-James orthogonality of elementary tensors in certain tensor product spaces with the Birkhoff-James orthogonality of individual elements in their respective spaces is studied.

### 1. Introduction

Orthogonality plays a vital role in the study of Hilbert spaces. In the last few decades, there have been several generalizations of the notion of orthogonality in the setting of normed spaces, one may refer to the survey article [1] for more details. Among them, Birkhoff-James orthogonality ([5, 9]) has been studied extensively and has found various significant relationships with other geometric properties of normed spaces. For a normed space  $X$  over the scalar field  $\mathbb{F}$  ( $\mathbb{R}$  or  $\mathbb{C}$ ) and  $x, y \in X$ , we say that  $x$  is *Birkhoff-James orthogonal* to  $y$  (denoted by  $x \perp_{BJ} y$ ) if

$$\|x + \alpha y\| \geq \|x\|, \text{ for all } \alpha \in \mathbb{F}.$$

It is closely related to the notion of best approximation to a point in a subspace as is evident from the fact that for a normed space  $V$  and a subspace  $W$  of  $V$ , a point  $w \in W$  is best approximation to  $v$  in  $V$  if and only if  $v - w \perp_{BJ} W$ . It is also useful to obtain some distance formulae in normed spaces. For more details on the geometric significance of BJ-orthogonality, one may refer to [3, 5, 8, 9, 16] and the references therein.

On the other hand, as has been observed with various categories, the theories of tensor products of Banach spaces and of  $C^*$ -algebras are indispensable to the understanding of these objects. Thus, it is quite natural to analyze the notion of Birkhoff-James orthogonality in tensor product spaces. Our interest in fact grew from this very basic curiosity. Quite surprisingly, this aspect has been hardly touched upon and there are no substantial results in the literature. Closely related to our interest, Light and

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Cheney (in [12]) discussed the theory of approximation in terms of proximality in various tensor products of spaces which can be linked with the Birkhoff-James orthogonality to some extent. For instance, they proved that for subspaces  $G$  and  $H$  having linear proximity maps in Banach spaces  $X$  and  $Y$  respectively, the subspace  $\overline{X \otimes H^\alpha} + \overline{G \otimes Y^\alpha}$  also has a linear proximity map, and is proximal in  $X \otimes_\alpha Y$ , where  $\otimes_\alpha$  is a uniform cross norm. In particular, for finite-dimensional subspaces  $G$  and  $H$  of  $L_1(S)$  and  $L_1(T)$ , respectively,  $S$  and  $T$  being finite measure spaces, the subspace  $L_1(S) \otimes H + G \otimes L_1(T)$  is proximal in  $L_1(S \times T)$ .

It is natural to ask how Birkhoff-James orthogonality of elementary tensors in various tensor products of Banach spaces is related to the Birkhoff-James orthogonality of the individual elements in their respective spaces. As a first step in this direction, for Banach spaces  $X$  and  $Y$ , any cross norm  $\|\cdot\|_\alpha$  on their algebraic tensor product  $X \otimes Y$  and elements  $x_1, x_2 \in X$ ,  $y_1, y_2 \in Y$ , one can ask the following questions:

1. If  $x_1 \perp_{BJ} x_2$  and(or)  $y_1 \perp_{BJ} y_2$  then is  $x_1 \otimes y_1 \perp_{BJ} x_2 \otimes y_2$ ?
2. If  $x_1 \otimes y_1 \perp_{BJ} x_2 \otimes y_2$  in  $X \otimes_\alpha Y$  then is  $x_1 \perp_{BJ} x_2$  and(or)  $y_1 \perp_{BJ} y_2$ ?
3. For subspaces  $X_1$  and  $Y_1$  of  $X$  and  $Y$  respectively, and  $x \otimes y \perp_{BJ} X_1 \otimes Y_1$ , can we say that  $x \perp_{BJ} X_1$  or  $y \perp_{BJ} Y_1$ ?

In this article we make an attempt to answer these questions by considering different cross norms on  $X \otimes Y$ . We show that the first assertion is always true for a reasonable cross norm, whereas we provide an example (see Example 3.2) to show that Question 2 does not have an affirmative answer in general. This also leads to a negative answer of Question 3 (see, 3.3). More precisely, we demonstrate that it fails in  $C_{\mathbb{C}}(K_1) \otimes^\alpha C_{\mathbb{C}}(K_2)$ , where  $C_{\mathbb{F}}(K)$  denotes the space of  $\mathbb{F}$ -valued continuous functions defined on a compact Hausdorff space  $K$  equipped with the sup norm and  $\|\cdot\|_\alpha$  is any reasonable cross norm. However, this is not a general phenomenon. By deploying different techniques, we discuss various spaces and tensor products for which Question 2 in fact has a positive answer. In particular, we prove that it has an affirmative answer for the following spaces:

- $X \otimes^\lambda Y$ , where  $X$  and  $Y$  are real normed spaces and  $\otimes^\lambda$  is the injective tensor product;
- $L^p(S, \mu) \otimes^{\Delta p} L^p(T, \nu)$ , where  $1 < p < \infty$ ,  $(S, \mu)$  and  $(T, \nu)$  are positive measure spaces and  $\otimes^{\Delta p}$  is the natural norm on the tensor product of  $p$ -integrable functions (see (1));
- $B(H) \otimes^{\min} B(K)$  (for elementary tensors of rank one operators),  $H$  and  $K$  being Hilbert spaces;
- $C_{\mathbb{C}}(K_1) \otimes^\lambda C_{\mathbb{C}}(K_2)$ , with Birkhoff-James orthogonality being replaced by strong Birkhoff-James orthogonality.

### 2. Preliminaries

Let us first recall some definitions and different notions of tensor norms. Given Banach spaces  $X$  and  $Y$ , *Banach space injective tensor norm* on the algebraic tensor product  $X \otimes Y$  is defined as

$$\|u\|_\lambda = \sup \left\{ \left| \sum_{i=1}^n f(x_i)g(y_i) \right| : f \in B_{X^*}, g \in B_{Y^*} \right\}, \quad u = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y,$$

where  $B_{X^*}$  denotes the closed unit ball of  $X^*$ . The completion of  $X \otimes Y$  with respect to  $\|\cdot\|_\lambda$  is called the *Banach space injective tensor product* of  $X$  and  $Y$  and is denoted by  $X \otimes^\lambda Y$ . A norm  $\|\cdot\|_\alpha$  on  $X \otimes Y$  is said to be a *cross norm* if  $\|x \otimes y\|_\alpha = \|x\| \|y\|$  for all  $x \in X, y \in Y$ . We say that  $\|\cdot\|_\alpha$  is *reasonable* if for every  $f \in X^*, g \in Y^*, f \otimes g$  is a bounded linear functional on  $X \otimes_\alpha Y$  with  $\|f \otimes g\| = \|f\| \|g\|$ . It is known that the injective norm  $\|\cdot\|_\lambda$  is the smallest reasonable cross norm (see, for instance, [12, Lemma 1.6]). Similarly, given two  $C^*$ -algebras  $A$  and  $B$ , the minimal tensor norm on the algebraic tensor product  $A \otimes B$  is defined as

$$\|u\|_{\min} = \sup \{ \|(\pi_1 \otimes \pi_2)(u)\| \}, \quad u \in A \otimes B,$$

where the supremum runs over all the representations  $\pi_1$  and  $\pi_2$  of  $A$  and  $B$  respectively. The completion of  $A \otimes B$  with respect to  $\|\cdot\|_{\min}$  is called the minimal  $C^*$ -tensor product of  $A$  and  $B$  and is denoted by  $A \otimes^{\min} B$ . For a detailed study of these norms one may refer to [12, 15, 18].

We now recall and collect some important results and identifications which will be useful in the later discussion. The first in the list is a basic yet an important and useful characterization of BJ-orthogonality in terms of the linear functionals.

**THEOREM 2.1.** [9, Theorem 2.1] *In a normed space  $X$ ,  $x \perp_{BJ} y$  if and only if there exists a linear functional  $f \in X^*$  such that  $|f(x)| = \|f\| \|x\|$  and  $f(y) = 0$ .*

It is a well known fact that the tensor product of continuous functions over compact spaces can be treated as a continuous function on an appropriate compact space. More precisely,

**THEOREM 2.2.** [6, §I.4.2] *Let  $K_1, K_2$  be compact Hausdorff spaces. The injective tensor product  $C_{\mathbb{F}}(K_1) \otimes^\lambda C_{\mathbb{F}}(K_2)$  is isometrically isomorphic to the space  $C_{\mathbb{F}}(K_1 \times K_2)$  via the identification  $f_1 \otimes f_2 \rightarrow f_1 f_2$  where  $f_1 f_2 : K_1 \times K_2 \rightarrow \mathbb{F}$  is defined as  $f_1 f_2(k_1, k_2) = f_1(k_1) f_2(k_2)$  for  $k_i \in K_i$ .*

There is an interesting geometric characterization of BJ-orthogonality in  $C_{\mathbb{C}}(K)$  which follows immediately from [10, Corollary 2.1].

**THEOREM 2.3.** *Let  $K$  be a compact Hausdorff space,  $f, g \in C_{\mathbb{C}}(K)$  and  $M_f := \{t \in K : |f(t)| = \|f\|\}$ . Then  $f \perp_{BJ} g$  if and only if the set  $A = \{\overline{f(t)}g(t) : t \in M_f\}$  is not contained in an open half plane (in  $\mathbb{C}$ ) with boundary that contains the origin. Equivalently,  $f \perp_{BJ} g$  if and only if the closed convex hull of  $A$  contains the origin.*

We have an elegant characterization of BJ-orthogonality for vector valued continuous functions also. For a real normed space  $X$  and a non-zero element  $x \in X$ , we use the notations,  $x^+ = \{y \in X : \|x + \lambda y\| \geq \|x\|, \forall \lambda \geq 0\}$  and  $x^- = \{y \in X : \|x + \lambda y\| \geq \|x\|, \forall \lambda \leq 0\}$ . It is known that for any  $x, y \in X$ , either  $y \in x^+$  or  $y \in x^-$ , see [16, Proposition 2.1].

**THEOREM 2.4.** [14, Theorem 2.1] *Let  $K$  be a compact Hausdorff space,  $X$  be a real normed space and  $f, g \in C(K, X)$  be non-zero elements. Then  $f \perp_{BJ} g$  if and only if there exists  $k_1, k_2 \in M_f$  such that  $g(k_1) \in f(k_1)^+$  and  $g(k_2) \in f(k_2)^-$ .*

### 3. Main results

We start with a proof of our first assertion.

**THEOREM 3.1.** *Let  $X$  and  $Y$  be Banach spaces,  $x_1, x_2 \in X$  with  $x_1 \perp_{BJ} x_2$ . Then for any  $y_1, y_2 \in Y$ ,  $x_1 \otimes y_1 \perp_{BJ} x_2 \otimes y_2$  in  $X \otimes^\alpha Y$ , where  $\|\cdot\|_\alpha$  is any reasonable cross norm on  $X \otimes Y$ .*

*Proof.* Since  $x_1 \perp_{BJ} x_2$ , by Theorem 2.1, there exists a  $\phi_0 \in X^*$  with  $\|\phi_0\| = 1$  such that  $|\phi_0(x_1)| = \|x_1\|$  and  $\phi_0(x_2) = 0$ . Consider  $\psi_0 \in Y^*$  such that  $\|\psi_0\| = 1$  and  $\psi_0(y_1) = \|y_1\|$ . Now for any  $\mu \in \mathbb{C}$ , we have

$$\begin{aligned} \|(x_1 \otimes y_1) + \mu(x_2 \otimes y_2)\|_\alpha &\geq \|(x_1 \otimes y_1) + \mu(x_2 \otimes y_2)\|_\lambda \\ &= \sup\{|\phi(x_1)\psi(y_1) + \mu\phi(x_2)\psi(y_2)| : \phi \in B_{X^*}, \psi \in B_{Y^*}\} \\ &\geq |\phi_0(x_1)\psi_0(y_1) + \mu\phi_0(x_2)\psi_0(y_2)| \\ &= \|x_1\| \|y_1\| = \|x_1 \otimes y_1\|_\alpha. \end{aligned}$$

This completes the proof.  $\square$

In general, converse of the above result is not true.

**EXAMPLE 3.2.** Let  $K_1 = \{x_1, x_2, x_3\}$  and  $K_2 = \{y_1, y_2, y_3\}$  be compact Hausdorff spaces equipped with discrete topology. Consider  $f_1, f_2 \in C_{\mathbb{C}}(K_1)$  defined as  $f_1(x_1) = 1, f_1(x_2) = 1 + 2i, f_1(x_3) = 1 - 2i$  and  $f_2(x_i) = 1$  for  $i = 1, 2, 3$ . Also, consider  $g_1, g_2 \in C_{\mathbb{C}}(K_2)$  defined as  $g_1(y_1) = 1, g_1(y_2) = -1 + 2i, g_1(y_3) = -1 - 2i$  and  $g_2(y_i) = 2$  for all  $i = 1, 2, 3$ . Note that  $\|f_1\| = \|g_1\| = \sqrt{5}$ ,  $M_{f_1} = \{x_2, x_3\}$  and  $M_{g_1} = \{y_2, y_3\}$ . Thus, closed convex hull of the sets  $\{f_1(t)f_2(t) : t \in M_{f_1}\}$  and  $\{g_1(s)g_2(s) : s \in M_{g_1}\}$  are the line joining  $1 + 2i, 1 - 2i$  and the line joining  $-2 + 4i, -2 - 4i$ , respectively. Since both the closed convex hulls do not contain the origin, by Theorem 2.3, neither  $f_1 \perp_{BJ} f_2$  nor  $g_1 \perp_{BJ} g_2$ . On the other hand,  $\|f_1 \otimes g_1\| = \|f_1\| \|g_1\| = 5$  and  $M_{f_1 g_1} = \{(x_2, y_2), (x_2, y_3), (x_3, y_2), (x_3, y_3)\}$ . Since the closed convex hull of the set  $\{f_1(t)g_1(s)f_2(t)g_2(s) : (t, s) \in M_{f_1 g_1}\}$  is the triangle with vertices  $-10, 6 + 8i, 6 - 8i$  and it contains the origin, thus, again by Theorem 2.3,  $f_1 g_1 \perp_{BJ} f_2 g_2$  in  $C_{\mathbb{C}}(K_1 \times K_2)$ . Equivalently,  $f_1 \otimes g_1 \perp_{BJ} f_2 \otimes g_2$  in  $C_{\mathbb{C}}(K_1) \otimes^\lambda C_{\mathbb{C}}(K_2)$ . Since injective norm  $\|\cdot\|_\lambda$  is the least reasonable cross norm on  $X \otimes Y$ , it follows that  $f_1 \otimes g_1 \perp_{BJ} f_2 \otimes g_2$  in  $C_{\mathbb{C}}(K_1) \otimes^\alpha C_{\mathbb{C}}(K_2)$ , for any reasonable cross norm  $\|\cdot\|_\alpha$ .

REMARK 3.3. By considering the subspaces  $X_2$  and  $Y_2$  spanned by  $f_2$  and  $g_2$  respectively in the above example, we observe that  $f_1 \otimes g_1 \perp_{BJ} X_2 \otimes Y_2$ , but neither  $f_1 \perp_{BJ} X_2$  nor  $g_1 \perp_{BJ} Y_2$ .

It is also interesting to note that  $x_1 \otimes x_2 \perp_{BJ} y_1 \otimes y_2$  does not necessarily imply  $x_1 \perp_{BJ} y_1$  and  $y_1 \perp_{BJ} y_2$  both. This can be illustrated by the following (and various other) examples:

EXAMPLE 3.4. For the identity function  $Id \in C_{\mathbb{C}}[0, 1]$  and the constant unity function  $1$ ,  $Id \otimes 1 \perp_{BJ} 1 \otimes Id$  in  $C_{\mathbb{C}}[0, 1] \otimes^{\lambda} C_{\mathbb{C}}[0, 1]$ . This can be seen using Theorem 2.2, for any  $\mu \in \mathbb{C}$ , we have

$$\|Id \otimes 1 + \mu 1 \otimes Id\|_{\lambda} = \|Id \cdot 1 + \mu 1 \cdot Id\| = \sup_{x,y \in [0,1]} |x + \mu y| \geq 1 = \|Id \otimes 1\|.$$

However,  $Id \not\perp_{BJ} 1$  as  $\|Id - (1/2)1\| = 1/2 \not\geq \|Id\| = 1$ . The fact that  $1 \perp_{BJ} Id$  follows easily from the definition.

EXAMPLE 3.5. Consider the matrices  $A, B, C \in M_2(\mathbb{C})$  given by  $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ ,  $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  and  $C = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ . We claim that  $A \otimes B \perp_{BJ} C \otimes I$  in  $M_2(\mathbb{C}) \otimes^{\min} M_2(\mathbb{C})$ . Since  $M_2(\mathbb{C}) \otimes^{\min} M_2(\mathbb{C})$  can be identified with  $M_4(\mathbb{C})$  equipped with the spectral norm, it is equivalent to check that  $P \perp_{BJ} Q$  for  $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ ,  $Q = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ . For the unit vector  $x = [0 \ 0 \ 1 \ 0]^t \in \mathbb{C}^4$ , we have  $\|Px\| = 2 = \|P\|$ ,  $\langle Px, Qx \rangle = 0$ . Thus, by a well known characterization given by Bhatia and Semrl [3, Theorem 1.1],  $P \perp_{BJ} Q$ . Here  $A \perp_{BJ} C$  as for  $x = (0, 1)^t$ ,  $\|Ax\| = 2 = \|A\|$  and  $\langle Ax, Cx \rangle = 0$ . However,  $B \not\perp_{BJ} I$  as for any unit vector  $x = (x_1, x_2)^t$  in  $\mathbb{C}^2$ ,  $\|Bx\| = 1 = \|B\|$  will imply that  $|x_1| = 1$  but for such an  $x$ ,  $\langle Bx, x \rangle \neq 0$ .

Here is another non-trivial example.

EXAMPLE 3.6. Consider the operators  $P, Q, R \in B(\ell^2)$  given by  $P(x_1, x_2, \dots) = (x_1, 0, 0, \dots)$ ;  $Q(x_1, x_2, \dots) = (x_1, x_2, 0, \dots)$  and  $R(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$ , and let  $I \in B(\ell^2)$  be the identity operator. We claim that  $P \otimes I \perp_{BJ} Q \otimes R$  in  $B(\ell^2) \otimes^{\min} B(\ell^2)$ . To see this, we use the inclusion  $B(\ell^2) \otimes^{\min} B(\ell^2) \subseteq B(\ell^2 \bar{\otimes} \ell^2)$ , where  $\bar{\otimes}$  represents the Hilbertian tensor product, and consider  $\lambda \in \mathbb{C}$ ,

$$\begin{aligned} \|(P \otimes I) + \lambda(Q \otimes R)\|^2 &= \sup_{h \in \ell^2 \bar{\otimes} \ell^2, \|h\|=1} \|(P \otimes I)h + \lambda(Q \otimes R)h\|^2 \\ &\geq \|(P \otimes I)(e_1 \otimes e_2) + \lambda(Q \otimes R)(e_1 \otimes e_2)\|^2 \\ &= \|e_1 \otimes e_2 + \lambda e_1 \otimes e_3\|^2 \\ &= \langle e_1 \otimes e_2 + \lambda e_1 \otimes e_3, e_1 \otimes e_2 + \lambda e_1 \otimes e_3 \rangle \\ &= 1 + |\lambda|^2 \geq 1 = \|P \otimes I\|^2 \end{aligned}$$

Here  $I \perp_{BJ} R$  as for the constant sequence of unit vectors  $\{h_n\} \in \ell^2$  where  $h_n = e_1$  for all  $n$ , we have  $\lim_{n \rightarrow \infty} \|I(h_n)\| = \|e_1\| = 1 = \|I\|$  and  $\lim_{n \rightarrow \infty} \langle I(h_n), R(h_n) \rangle = \langle e_1, e_2 \rangle = 0$  and thus orthogonality follows from [3, Remark 3.1]. However,  $P \not\perp_{BJ} Q$ , if not, then we get a sequence of unit vectors say  $\{h_n = (h_n^{(m)})\}$  in  $\ell^2$  such that  $\lim_{n \rightarrow \infty} \|P(h_n)\| = \lim_{n \rightarrow \infty} |h_n^{(1)}| = \|P\| = 1$  and  $\lim_{n \rightarrow \infty} \langle P(h_n), Q(h_n) \rangle = \lim_{n \rightarrow \infty} |h_n^{(1)}|^2 = 0$ , which is not possible.

In many spaces the converse of Theorem 3.1 holds true. We shall now discuss few of such spaces.

**THEOREM 3.7.** *Let  $X$  and  $Y$  be real Banach spaces and  $x_1, x_2 \in X$ ,  $y_1, y_2 \in Y$ . Then  $x_1 \otimes y_1 \perp_{BJ} x_2 \otimes y_2$  in  $X \otimes^\lambda Y$  if and only if either  $x_1 \perp_{BJ} x_2$  or  $y_1 \perp_{BJ} y_2$ .*

*Proof.* In view of Theorem 3.1, we only need to prove the necessary part. Suppose that  $v_1 = x_1 \otimes y_1 \perp_{BJ} x_2 \otimes y_2 = v_2$  in  $X \otimes^\lambda Y$ . By [15, Section 3.4], we know that  $X \otimes^\lambda Y$  can be embedded (isometrically) as a subspace of  $C_{\mathbb{R}}(B_{X^*} \times B_{Y^*})$  via the embedding  $(\sum_{i=1}^n x_i \otimes y_i) \mapsto \sum_{i=1}^n f_{x_i} f_{y_i}$ , where  $f_x : X^* \rightarrow \mathbb{R}$  represents the map  $f_x(\phi) = \phi(x)$ , for all  $\phi \in X^*$ ; here  $B_{X^*}, B_{Y^*}$  are closed unit balls in  $X^*$  and  $Y^*$  respectively equipped with the weak\*-topology. Note that, by Banach Alaoglu’s Theorem,  $B_{X^*} \times B_{Y^*}$  is compact,  $B_{X^*}$  and  $B_{Y^*}$  being compact.

Thus,  $v_1 \perp_{BJ} v_2$  in  $X \otimes^\lambda Y$  is equivalent to saying that  $f_{x_1} f_{y_1} \perp_{BJ} f_{x_2} f_{y_2}$  in  $C_{\mathbb{R}}(B_{X^*} \times B_{Y^*})$ .

We claim that either  $f_{x_1} \perp_{BJ} f_{x_2}$  in  $C_{\mathbb{R}}(B_{X^*})$  or  $f_{y_1} \perp_{BJ} f_{y_2}$  in  $C_{\mathbb{R}}(B_{Y^*})$ . Let, if possible, neither  $f_{x_1} \perp_{BJ} f_{x_2}$  nor  $f_{y_1} \perp_{BJ} f_{y_2}$ . Then, by Theorem 2.4, either  $f_{x_2}(\phi) \in f_{x_1}(\phi)^+$  for all  $\phi \in M_{f_{x_1}}$  or  $f_{x_2}(\phi) \in f_{x_1}(\phi)^-$  for all  $\phi \in M_{f_{x_1}}$ . Similarly, either  $f_{y_2}(\psi) \in f_{y_1}(\psi)^+$  for all  $\psi \in M_{f_{y_1}}$  or  $f_{y_2}(\psi) \in f_{y_1}(\psi)^-$  for all  $\psi \in M_{f_{y_1}}$ . Let us discuss the case when  $f_{x_2}(\phi) \in f_{x_1}(\phi)^+$  and  $f_{y_2}(\psi) \in f_{y_1}(\psi)^+$  for all  $\phi \in M_{f_{x_1}}, \psi \in M_{f_{y_1}}$ . Then, we have  $f_{x_1}(\phi) f_{x_2}(\phi) > 0$  and  $f_{y_1}(\psi) f_{y_2}(\psi) > 0$  for all  $\phi \in M_{f_{x_1}}, \psi \in M_{f_{y_1}}$ . This gives  $f_{x_1}(\phi) f_{x_2}(\phi) f_{y_1}(\psi) f_{y_2}(\psi) > 0$  for all  $(\phi, \psi) \in M_{f_{x_1}} \times M_{f_{y_1}} = M_{f_{x_1} f_{y_1}}$  which contradicts the hypothesis  $f_{x_1} f_{y_1} \perp_{BJ} f_{x_2} f_{y_2}$ . So this case is not possible. Similarly, we will get contradiction in rest of the three cases.

Now, assume that  $f_{x_1} \perp_{BJ} f_{x_2}$ . Therefore for any scalar  $\mu \in \mathbb{R}$ , we have  $\sup\{|f_{x_1}(\phi) + \mu f_{x_2}(\phi)| : \phi \in B_{X^*}\} \geq \sup\{|f_{x_1}(\phi)| : \phi \in B_{X^*}\}$  which, using a consequence of Hahn Banach Theorem, gives that  $\|x_1 + \mu x_2\| \geq \|x_1\|$  and thus  $x_1 \perp_{BJ} x_2$ . Similarly  $f_{y_1} \perp_{BJ} f_{y_2}$  would imply that  $y_1 \perp_{BJ} y_2$ .  $\square$

Birkhoff-James orthogonality behaves well for the injective tensor norm on the real Banach spaces. We next move to a natural norm on the space of  $p$ -integrable functions. For an arbitrary measure space  $(S, \mu)$  and a Banach space  $X$ , let  $L^p(\mu, X)$ ,  $1 \leq p < \infty$ , denote the space of Bochner  $p$ -integrable functions from  $S$  to  $X$ . Consider the norm on  $L^p(S, \mu) \otimes X$ , written as  $\Delta_p$ , induced via the injective map  $\phi : L^p(S, \mu) \otimes X \rightarrow L^p(\mu, X)$  given by  $\phi(f \otimes x) = f(\cdot)x$ , that is,

$$\|u\|_{\Delta_p} = \|\phi(u)\|_{L^p(\mu, X)}, \quad \text{for all } u \in L^p(S, \mu) \otimes X. \tag{1}$$

We take  $X = L^p(T, \nu)$  and assume that  $L^p(S, \mu) \otimes^{\Delta p} L^p(T, \nu)$  denotes the completion of  $L^p(S, \mu) \otimes L^p(T, \nu)$  with respect to the  $\Delta_p$ -norm. It is known that the space  $L^p(S, \mu) \otimes^{\Delta p} L^p(T, \nu)$  is isometrically isomorphic to  $L^p(S \times T, \mu \times \nu)$ ,  $1 < p < \infty$ , see [6, §1.7.2].

We state a result which we shall require for BJ-orthogonality of  $p$ -integrable functions.

**THEOREM 3.8.** [17, Theorem 1.11] *Let  $(S, \mu)$  be a positive measure space,  $G$  be a subspace of  $L^p(S, \mu)$ ,  $1 < p < \infty$  and  $f \in L^p(S, \mu) \setminus \overline{G}$ . An element  $g \in G$  is best approximation to  $f$  if and only if*

$$\int_S h(s)|f(s) - g(s)|^{p-1} \text{sign}(f(s) - g(s)) d\mu(s) = 0, \text{ for all } h \in G.$$

**THEOREM 3.9.** *Let  $(S, \mu)$  and  $(T, \nu)$  be positive measure spaces. Then  $f_1 \otimes f_2 \perp_{BJ} g_1 \otimes g_2$  in  $L^p(S, \mu) \otimes^{\Delta p} L^p(T, \nu)$ ,  $1 < p < \infty$  if and only if either  $f_1 \perp_{BJ} g_1$  or  $f_2 \perp_{BJ} g_2$ .*

*Proof.* Using the identification mentioned above, we know that  $f_1 \otimes f_2 \perp_{BJ} g_1 \otimes g_2$  in  $L^p(S, \mu) \otimes^{\Delta p} L^p(T, \nu)$  if and only if  $f_1 f_2 \perp_{BJ} g_1 g_2$  in  $L^p(S \times T, \mu \times \nu)$ . Let if possible, neither  $f_1 \perp_{BJ} g_1$  nor  $f_2 \perp_{BJ} g_2$ . Now,  $f_1 \not\perp_{BJ} g_1$  gives that 0 is not a best approximation to  $f_1$  from the subspace  $\text{span}\{g_1\}$ . By Theorem 3.8, we have

$$\int_S g_1(s)|f_1(s)|^{p-1} \text{sign}(f_1(s)) d\mu(s) \neq 0,$$

and similarly the other part gives

$$\int_T g_2(t)|f_2(t)|^{p-1} \text{sign}(f_2(t)) d\nu(t) \neq 0.$$

Combining both we have

$$\begin{aligned} & \iint_{S \times T} g_1(s)g_2(t)|f_1(s)f_2(t)|^{p-1} \text{sign}(f_1(s)f_2(t)) (d\mu(s) \times d\nu(t)) = \\ & \left( \int_S g_1(s)|f_1(s)|^{p-1} \text{sign}(f_1(s)) d\mu(s) \right) \left( \int_T g_2(t)|f_2(t)|^{p-1} \text{sign}(f_2(t)) d\nu(t) \right) \neq 0, \end{aligned}$$

which implies that  $f_1 f_2 \not\perp_{BJ} g_1 g_2$  in  $L^p(S \times T, \mu \times \nu)$ . This completes the proof.  $\square$

We now turn our attention back to the space of continuous complex valued functions. It was illustrated in Example 3.2 that the BJ-orthogonality does not work well for the tensor product  $C_{\mathbb{C}}(K_1) \otimes^{\alpha} C_{\mathbb{C}}(K_2)$  with respect to any reasonable cross norm  $\alpha$ . Although in a special situation it does work well.

PROPOSITION 3.10. *Let  $f_1 \in C_{\mathbb{C}}(K_1)$  and  $g_1 \in C_{\mathbb{C}}(K_2)$  both attain their norms at a unique point. Then, for any  $f_2 \in C_{\mathbb{C}}(K_1)$  and  $g_2 \in C_{\mathbb{C}}(K_2)$ ,  $f_1 \otimes g_1 \perp_{BJ} f_2 \otimes g_2$  in  $C_{\mathbb{C}}(K_1) \otimes^{\lambda} C_{\mathbb{C}}(K_2)$  implies  $f_1 \perp_{BJ} f_2$  or  $g_1 \perp_{BJ} g_2$ .*

*Proof.* It follows easily from Theorem 2.3.  $\square$

However, if we treat  $C_{\mathbb{C}}(K)$ ,  $K$  being a compact Hausdorff space, as a  $C^*$ -algebra then we have the desired result with BJ-orthogonality being replaced by the strong BJ-orthogonality.

Recall that, if  $H$  is a right Hilbert  $C^*$ -module over a  $C^*$ -algebra  $A$ , then for  $h_1, h_2 \in H$ , we say that  $h_1$  is strongly Birkhoff-James orthogonal to  $h_2$  (written as  $h_1 \perp_{BJ}^s h_2$ ), if  $\|h_1 + h_2a\| \geq \|h_1\|$  for all  $a \in A$ , see [2]. It is obvious that any  $C^*$ -algebra is a Hilbert  $C^*$ -module over itself. Also, it is worth mentioning that for commutative  $C^*$ -algebras, the injective norm coincides with the minimal  $C^*$ -tensor norm [18, Theorem 4.14]. More precisely,  $C_{\mathbb{C}}(K_1) \otimes^{\lambda} C_{\mathbb{C}}(K_2)$  is isometrically isomorphic to  $C_{\mathbb{C}}(K_1) \otimes^{\min} C_{\mathbb{C}}(K_2)$  and the latter is again a  $C^*$ -algebra.

THEOREM 3.11. *Let  $K_1$  and  $K_2$  be compact Hausdorff spaces,  $f_1, f_2 \in C_{\mathbb{C}}(K_1)$  and  $g_1, g_2 \in C_{\mathbb{C}}(K_2)$ . Then  $f_1 \otimes g_1 \perp_{BJ}^s f_2 \otimes g_2$  in  $C_{\mathbb{C}}(K_1) \otimes^{\min} C_{\mathbb{C}}(K_2)$  if and only if either  $f_1 \perp_{BJ}^s f_2$  or  $g_1 \perp_{BJ}^s g_2$ .*

*Proof.* First suppose that  $f_1 \otimes g_1 \perp_{BJ}^s f_2 \otimes g_2$ . Using the identification as in Theorem 2.2, we have  $f_1g_1 \perp_{BJ}^s f_2g_2$  in  $C_{\mathbb{C}}(K_1 \times K_2)$ . By [2, Proposition 4.2], there exists  $(x_0, y_0) \in K_1 \times K_2$  such that  $|f_1(x_0)g_1(y_0)| = \|f_1\| \|g_1\|$  and  $f_2(x_0)g_2(y_0) = 0$ . The former condition implies that  $\|f_1\| = |f_1(x_0)|$  and  $\|g_1\| = |g_1(x_0)|$  and the latter one implies either  $f_2(x_0) = 0$  or  $g_2(y_0) = 0$ .

Thus, again by using [2, Proposition 4.2], we have  $f_1 \perp_{BJ}^s f_2$  or  $g_1 \perp_{BJ}^s g_2$ . Conversely, let us assume that  $f_1 \perp_{BJ}^s f_2$ . So there exists  $x_0 \in M_{f_1}$  such that  $f_2(x_0) = 0$ . Since  $M_{g_1} \neq \emptyset$ , by fixing any  $y_0 \in M_{g_1}$ , we have  $f_2(x_0)g_2(y_0) = 0$  and  $|f_1(x_0)g_1(y_0)| = \|f_1\| \|g_1\|$ . Thus  $f_1 \otimes g_1 \perp_{BJ}^s f_2 \otimes g_2$ .  $\square$

REMARK 3.12. Above result gives that  $f_1 \otimes g_1 \perp_{BJ}^s f_2 \otimes g_2$  implies either  $f_1 \perp_{BJ}^s f_2$  or  $g_1 \perp_{BJ}^s g_2$ . So  $f_1 \perp_{BJ} f_2h$  or  $g_1 \perp_{BJ} g_2k$  for every  $h \in C_{\mathbb{C}}(K_1)$ ,  $k \in C_{\mathbb{C}}(K_2)$ . Since strong BJ-orthogonality implies BJ-orthogonality, this gives  $f_1 \perp_{BJ} f_2$  or  $g_1 \perp_{BJ} g_2$ .

Lastly, we deviate our attention a little bit from the commutative setup to a non-commutative setting. It is quite obvious to ask what happens in the tensor product of non-commutative space  $B(H)$ , the space of bounded linear operators on a Hilbert space. In this direction, we obtain a positive result for the elementary tensors of rank one operators in  $B(H) \otimes^{\min} B(H)$ .

PROPOSITION 3.13. *Let  $H$  be a Hilbert space and let  $T_1, T_2, S_1, S_2 \in B(H)$  be rank one operators satisfying  $T_1 \otimes T_2 \perp_{BJ} S_1 \otimes S_2$  in  $B(H) \otimes^{\min} B(H)$ . Then either  $T_1 \perp_{BJ} S_1$  or  $T_2 \perp_{BJ} S_2$ .*



*Proof.* We can embed  $B(H) \otimes^{\min} B(H)$  into  $B(H \bar{\otimes} H)$  isometrically. Note that a rank one operator  $T$  in  $B(H)$  is of the form  $x \bar{\otimes} y$  for some  $x, y \in H$ , where  $(x \bar{\otimes} y)(h) = \langle h, y \rangle x$ , for all  $h \in H$ . It is easy to check that tensor product of two rank one operators is again a rank one operator on  $H \bar{\otimes} H$ , more precisely, if  $T_i = x_i \bar{\otimes} y_i$ ,  $i = 1, 2$ , then  $T_1 \otimes T_2 = (x_1 \otimes x_2) \bar{\otimes} (y_1 \otimes y_2)$ . Taking  $S_i = z_i \otimes w_i$ ,  $i = 1, 2$ , since  $T_1 \otimes T_2 \perp_{BJ} S_1 \otimes S_2$ , by [7, Example 4.5], either  $\langle x_1 \otimes x_2, z_1 \otimes z_2 \rangle = 0$  or  $\langle y_1 \otimes y_2, w_1 \otimes w_2 \rangle = 0$ . That is, either  $\langle x_1, z_1 \rangle \langle x_2, z_2 \rangle = 0$  or  $\langle y_1, w_1 \rangle \langle y_2, w_2 \rangle = 0$ . The former case implies that either  $\langle x_1, z_1 \rangle = 0$  or  $\langle x_2, z_2 \rangle = 0$ , which gives either  $T_1 \perp_{BJ} S_1$  or  $T_2 \perp_{BJ} S_2$ , using [7, Example 4.5] again. Similarly one can check for the other case.  $\square$

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