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FURTHER REFINEMENTS OF SOME NUMERICAL RADIUS INEQUALITIES FOR OPERATORS

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Abstract. In this work, we give refinements of some well-known numerical radius inequalities. Also, we present an improvement of the triangle inequality for the operator norm.

1. Introduction

Let $\mathbb{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$. For $A\in\mathbb{B}(\mathcal{H})$, let $w(A)$ and $\|A\|$ denote the numerical radius and the operator norm of A, respectively. Recall that $w(A)=\sup\{|\langle Ax,x\rangle|,\ x\in\mathcal{H},\ \|x\|=1\}$ or $w(A)=\sup\limits_{\theta\in\mathbb{R}}\|Re(e^{i\theta}A)\|$ and $\|A\|=\sup\{|\langle Ax,y\rangle|,\ x,y\in\mathcal{H},\ \|x\|=\|y\|=1\}$.

The spectral radius of A, denoted by $\rho(A)$, is defined by $\rho(A)=\sup\{|\lambda|,\lambda\in\sigma(A)\}$, where $\sigma(A)$ is the spectrum of A.

The Aluthge transform of A, denoted by $\tilde{A}$, is defined as $\tilde{A}=|A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$, where $|A|=(A^*A)^{\frac{1}{2}}$.

There are many existing papers dealing with bounding the numerical radius for operators, we refer the readers to [1], [3], [4], [9] and the references therein.

It is well-known that $w(\cdot)$ defines a norm on $\mathbb{B}(\mathcal{H})$, which is equivalent to the operator norm. In fact, we have

$$\frac{1}{2}\|A\|\leqslant w(A)\leqslant\|A\|. \tag{1.1}$$

In [11], Kittaneh refined the inequalities in (1.1) and obtained the following result

$$\frac{1}{4}\|AA^*+A^*A\|\leqslant w^2(A)\leqslant\frac{1}{2}\|AA^*+A^*A\|. \tag{1.2}$$

In [6], Dragomir gave the following results

$$\frac{\sqrt{2}}{2}\max\left\{\left\|\frac{(1-\mathrm{i})A+(1+\mathrm{i})A^*}{2}\right\|,\left\|\frac{(1+\mathrm{i})A+(1-\mathrm{i})A^*}{2}\right\|\right\}\leqslant w(A) \tag{1.3}$$

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and

$$\frac{1}{4}\left\|A^2+(A^*)^2\right\| \leqslant w^2(A). \tag{1.4}$$

Kittaneh, Moslehian and Yamazaki [13] have obtained the following inequalities

$$\frac{1}{2}\|A-A^*\| \leqslant w(A) \tag{1.5}$$

and

$$\frac{1}{2}\|A+A^*\| \leqslant w(A). \tag{1.6}$$

In [16], Yamazaki gave the following inequality

$$w(A) \leqslant \frac{1}{2}\|A\| + \frac{1}{2}w(\tilde{A}). \tag{1.7}$$

Abu-Omar and Kittaneh [1] refined the second inequality in (1.1) and have obtained the following result

$$w(A) \leqslant \frac{1}{2}\sqrt{\|A^*A+AA^*\|+2w(A^2)}. \tag{1.8}$$

El-Haddad and Kittaneh, see [7], proved that

$$2^{(\frac{-r}{2}-1)}\||H+K|^r+|H-K|^r\| \leqslant w^r(A) \quad \text{for } r \geqslant 2, \tag{1.9}$$

where $A = H + iK$ is the Cartesian decomposition of A.

Bhunia, Bag and Paul [5] established the following inequality

$$w(A) \leqslant \sqrt{\|Re(A)\|^2+\|Im(A)\|^2}, \tag{1.10}$$

where $Re(A) = \dfrac{A+A^*}{2}$ and $Im(A) = \dfrac{A-A^*}{2i}$.

In [15], Sattari, Moslehian and Yamazaki have obtained the following upper bounds for the numerical radius

$$w^r(B^*A) \leqslant \frac{1}{4}\|(AA^*)^r+(BB^*)^r\| + \frac{1}{2}w^r(AB^*) \quad \text{for } r \geqslant 1 \tag{1.11}$$

and

$$w^{2r}(A) \leqslant \frac{1}{2}w^r(A^2) + \frac{1}{2}\|A\|^{2r} \quad \text{for } r \geqslant 1. \tag{1.12}$$

Recently, Omidvar and Moradi [14] have obtained the following upper bound of the operator norm for the sum of two operators

$$\|A+B\| \leqslant \sqrt{\|A\|^2+\|B\|^2+\|A\|\|B\|+w(B^*A)}. \tag{1.13}$$

In [2], Abu-Omar and Kittaneh have proved that

$$w(A+B) \leqslant \sqrt{w(A)^2+w(B)^2+\|A\|\|B\|+w(B^*A)}. \tag{1.14}$$

In this paper, we refine all the above numerical radius inequalities. In Section 2, we derive some new bounds of the numerical radius for operator. These bounds refine some of the previous numerical radius inequalities for operators. In Section 3, we give some bounds of the numerical radius for two operators. These bounds improve the rest of the above numerical radius inequalities.

2. Main results

Our first result can be stated as follows.

THEOREM 2.1. *Let $A \in \mathbb{B}(\mathcal{H})$ have the Cartesian decomposition $A = H + iK$. Then for $\alpha, \beta > 0$,*

$$\sup_{\alpha^2+\beta^2=1} \left\|\alpha^2 H^2 + \beta^2 K^2\right\| \leqslant w^2(A). \tag{2.1}$$

Proof. Let $x \in \mathcal{H}$ be any unit vector. Then

$$\begin{aligned} |\langle Ax, x\rangle|^2 &= \langle Hx, x\rangle^2 + \langle Kx, x\rangle^2 \\ &= \sup_{\alpha^2+\beta^2=1} (\alpha|\langle Hx, x\rangle| + \beta|\langle Kx, x\rangle|)^2 \\ &\geqslant \sup_{\alpha^2+\beta^2=1} |\langle(\alpha H \pm \beta K)x, x\rangle|^2. \end{aligned}$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we get

$$w^2(A) \geqslant \sup_{\alpha^2+\beta^2=1} \|\alpha H \pm \beta K\|^2. \tag{2.2}$$

Thus

$$\begin{aligned} 2w^2(A) &\geqslant \sup_{\alpha^2+\beta^2=1} \left(\left\|(\alpha H + \beta K)^2\right\| + \left\|(\alpha H - \beta K)^2\right\|\right) \\ &\geqslant \sup_{\alpha^2+\beta^2=1} \left\|(\alpha H + \beta K)^2 + (\alpha H - \beta K)^2\right\|. \end{aligned}$$

Hence

$$w^2(A) \geqslant \sup_{\alpha^2+\beta^2=1} \left\|\alpha^2 H^2 + \beta^2 K^2\right\|,$$

as required. □

REMARK 2.2. Taking $\alpha = \beta = \frac{1}{\sqrt{2}}$ in the inequality (2.1), gives

$$\frac{1}{4}\|A^*A + AA^*\| \leqslant \sup_{\alpha^2+\beta^2=1} \|\alpha^2 H^2 + \beta^2 K^2\| \leqslant w^2(A).$$

This proves that the inequality (2.1) is an improvement of the first inequality in (1.2).

COROLLARY 2.3. *Let $A \in \mathbb{B}(\mathcal{H})$ have the Cartesian decomposition $A = H + iK$. Then for $\alpha, \beta > 0$,*

$$\max\left\{\sup_{\alpha^2+\beta^2=1} \|\alpha H - \beta K\|, \sup_{\alpha^2+\beta^2=1} \|\alpha H + \beta K\|\right\} \leqslant w(A). \tag{2.3}$$

Proof. The inequality (2.3) follows from the inequality (2.2). □

Now, taking $\alpha=\beta=\frac{1}{\sqrt{2}}$ in the inequality (2.3), gives the inequality (1.3). Therefore, one can conclude that the inequality (2.3) is a refinement of the inequality (1.3).

THEOREM 2.4. *Let $A\in\mathbb{B}(\mathcal{H})$. Then for $\alpha,\beta>0$,*

$$\frac{1}{2}\sup_{\alpha^2+\beta^2=1} w\left(\alpha^2A^2+\beta^2(A^*)^2\right)\leqslant w^2(A). \tag{2.4}$$

Proof. Let $x\in\mathcal{H}$ be any unit vector. Then

$$\begin{aligned}2|\langle Ax,x\rangle|^2 &= |\langle Ax,x\rangle|^2+|\langle A^*x,x\rangle|^2\\ &= \sup_{\alpha^2+\beta^2=1}\left(\alpha|\langle Ax,x\rangle|+\beta|\langle A^*x,x\rangle|\right)^2\\ &\geqslant \sup_{\alpha^2+\beta^2=1}|\langle(\alpha A\pm\beta A^*)x,x\rangle|^2.\end{aligned}$$

By taking the supremum on both sides in the above inequality over $x\in\mathcal{H}$ with $\|x\|=1$, we get

$$2w^2(A)\geqslant\sup_{\alpha^2+\beta^2=1} w^2\left(\alpha A\pm\beta A^*\right). \tag{2.5}$$

Thus

$$\begin{aligned}4w^2(A) &\geqslant \sup_{\alpha^2+\beta^2=1}\left(w\left(\alpha A+\beta A^*\right)^2+w\left(\alpha A-\beta A^*\right)^2\right)\\ &\geqslant \sup_{\alpha^2+\beta^2=1} w\left((\alpha A+\beta A^*)^2+(\alpha A-\beta A^*)^2\right).\end{aligned}$$

Hence

$$2w^2(A)\geqslant\sup_{\alpha^2+\beta^2=1} w\left(\alpha^2A^2+\beta^2(A^*)^2\right),$$

as required. □

If we take $\alpha=\beta=\frac{1}{\sqrt{2}}$ in the inequality (2.4) and using the fact that the operator $\left(A^2+(A^*)^2\right)$ is self-adjoint, then we get the inequality (1.4). Therefore, we conclude that the inequality (2.4) is sharper than the inequality (1.4).

COROLLARY 2.5. *Let $A\in\mathbb{B}(\mathcal{H})$. Then for $\alpha,\beta>0$,*

$$\frac{1}{\sqrt{2}}\max\left\{\sup_{\alpha^2+\beta^2=1} w\left(\alpha A-\beta A^*\right),\ \sup_{\alpha^2+\beta^2=1} w\left(\alpha A+\beta A^*\right)\right\}\leqslant w(A). \tag{2.6}$$

Proof. The inequality (2.6) follows from the inequality (2.5). □

If we choose in the inequality (2.6), $\alpha=\beta=\frac{1}{\sqrt{2}}$ and taking into account that $A+A^*$ and $A-A^*$ are normal, then we get the inequalities (1.5) and (1.6). Therefore, one can conclude that the inequality (2.6) is a refinement of the both previous inequalities.

The following result can be found in [5, 8].

LEMMA 2.6. *Let $A, X \in \mathbb{B}(\mathcal{H})$. Then*

$$w(AX+XA) \leqslant 2w(A)\|X\|.$$

THEOREM 2.7. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$w^2(A) \leqslant \frac{1}{4}\left(\|A\|^2 + w^2(\tilde{A}) + w(|A|\tilde{A}+\tilde{A}|A|)\right). \tag{2.7}$$

Proof. Let $A = U|A|$ be the polar decomposition of A and let $\theta \in \mathbb{R}$. For any unit vector $x \in \mathcal{H}$, we have

$$\begin{aligned}
\langle e^{i\theta}Ax, x\rangle &= \langle e^{i\theta}|A|x, U^*x\rangle \\
&= \frac{1}{4}\Big(\langle |A|(e^{i\theta}+U^*)x, (e^{i\theta}+U^*)x\rangle - \langle |A|(e^{i\theta}-U^*)x, (e^{i\theta}-U^*)x\rangle\Big) \\
&\quad + \frac{i}{4}\Big(\langle |A|(e^{i\theta}+iU^*)x, (e^{i\theta}+iU^*)x\rangle - \langle |A|(e^{i\theta}-iU^*)x, (e^{i\theta}-iU^*)x\rangle\Big).
\end{aligned}$$

Thus

$$\begin{aligned}
Re\langle e^{i\theta}Ax, x\rangle &= \frac{1}{4}\Big(\langle |A|(e^{i\theta}+U^*)x, (e^{i\theta}+U^*)x\rangle - \langle |A|(e^{i\theta}-U^*)x, (e^{i\theta}-U^*)x\rangle\Big) \\
&\leqslant \frac{1}{4}\|(e^{-i\theta}+U)|A|(e^{i\theta}+U^*)\| \\
&= \frac{1}{4}\left\|(e^{-i\theta}+U)|A|^{\frac{1}{2}}\left((e^{-i\theta}+U)|A|^{\frac{1}{2}}\right)^*\right\| \\
&= \frac{1}{4}\left\|\left((e^{-i\theta}+U)|A|^{\frac{1}{2}}\right)^*(e^{-i\theta}+U)|A|^{\frac{1}{2}}\right\| \\
&= \frac{1}{4}\||A|^{\frac{1}{2}}(e^{i\theta}+U^*)(e^{-i\theta}+U)|A|^{\frac{1}{2}}\| \\
&= \frac{1}{2}\left\||A| + Re(e^{i\theta}\tilde{A})\right\|.
\end{aligned}$$

Then

$$\begin{aligned}
Re\langle e^{i\theta}Ax, x\rangle &\leqslant \frac{1}{2}\left\|\left(|A| + Re(e^{i\theta}\tilde{A})\right)^2\right\|^{\frac{1}{2}} \\
&= \frac{1}{2}\left\||A|^2 + Re^2(e^{i\theta}\tilde{A}) + Re\left(e^{i\theta}(|A|\tilde{A}+\tilde{A}|A|)\right)\right\|^{\frac{1}{2}}.
\end{aligned}$$

By taking the supremum on the both sides in the above inequality over $\theta \in \mathbb{R}$, gives

$$|\langle Ax, x\rangle|^2 \leqslant \frac{1}{4}\left(\|A\|^2 + w^2(\tilde{A}) + w(|A|\tilde{A}+\tilde{A}|A|)\right).$$

Therefore, the desired result follows by taking the supremum in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$. $\square$

REMARK 2.8. The inequality (2.7) is better than the inequality (1.7). Indeed, using Lemma 2.6, we obtain

$$\begin{aligned}\frac{1}{4}\left(\|A\|^2+w^2(\tilde{A})+w(|A|\tilde{A}+\tilde{A}|A|)\right) &\leqslant \frac{1}{4}\left(\|A\|^2+w^2(\tilde{A})+2w(\tilde{A})\|A\|\right)\\ &=\left(\frac{1}{2}\|A\|+\frac{1}{2}w(\tilde{A})\right)^2.\end{aligned}$$

LEMMA 2.9. [12] *Let* $A_1,A_2,B_1,B_2\in\mathbb{B}(\mathcal{H})$. *Then*

$$\begin{aligned}\rho(A_1B_1+A_2B_2) \leqslant\ &\frac{1}{2}\left(\|B_1A_1\|+\|B_2A_2\|\right)\\ &+\frac{1}{2}\sqrt{\left(\|B_1A_1\|-\|B_2A_2\|\right)^2+4\|B_1A_2\|\|B_2A_1\|}.\end{aligned}$$

THEOREM 2.10. *Let* $A\in\mathbb{B}(\mathcal{H})$. *Then*

$$w^2(A)\leqslant\frac{1}{8}\left(2w(A^2)+\|S\|+\sqrt{(2w(A^2)-\|S\|)^2+8\sup_{\theta\in\mathbb{R}}\|SRe(e^{2i\theta}A^2)\|}\right),\qquad(2.8)$$

where $S=AA^*+A^*A$.

Proof. Let $x\in\mathcal{H}$ be any unit vector. It is well-known that

$$|\langle Ax,x\rangle|=\sup_{\theta\in\mathbb{R}}\frac{1}{2}\left|e^{i\theta}\langle Ax,x\rangle+e^{-i\theta}\langle A^*x,x\rangle\right|.$$

By taking the supremum on both sides in the above inequality over $x\in\mathcal{H}$ with $\|x\|=1$, we obtain

$$\begin{aligned}w^2(2A)&\leqslant\sup_{\theta\in\mathbb{R}}\left\|(e^{i\theta}A+e^{-i\theta}A^*)(e^{i\theta}A+e^{-i\theta}A^*)^*\right\|\\ &=\sup_{\theta\in\mathbb{R}}\left\|2Re(e^{2i\theta}A^2)+AA^*+A^*A\right\|\\ &=\sup_{\theta\in\mathbb{R}}\rho\left(2Re(e^{2i\theta}A^2)+AA^*+A^*A\right).\end{aligned}$$

By choosing $A_1=I, A_2=2Re(e^{2i\theta}A^2), B_1=S$ and $B_2=I$ in Lemma 2.9, we get

$$w^2(A)\leqslant\sup_{\theta\in\mathbb{R}}\frac{1}{8}\left\{\|2Re(e^{2i\theta}A^2)\|+\|S\|+\sqrt{(\|2Re(e^{2i\theta}A^2)\|-\|S\|)^2+8\|SRe(e^{2i\theta}A^2)\|}\right\}.$$

Thus

$$
\begin{aligned}
w^2(A) &\leqslant \frac{1}{4}\left\| \begin{bmatrix} \sup_{\theta\in\mathbb{R}} \|2Re(e^{2i\theta}A^2)\| & \sup_{\theta\in\mathbb{R}} \sqrt{\|2SRe(e^{2i\theta}A^2)\|} \\ \sup_{\theta\in\mathbb{R}} \sqrt{\|2SRe(e^{2i\theta}A^2)\|} & \sup_{\theta\in\mathbb{R}} \|S\| \end{bmatrix} \right\| \\
&= \frac{1}{4}\left\| \begin{bmatrix} 2w(A^2) & \sup_{\theta\in\mathbb{R}} \sqrt{\|2SRe(e^{2i\theta}A^2)\|} \\ \sup_{\theta\in\mathbb{R}} \sqrt{\|2SRe(e^{2i\theta}A^2)\|} & \|S\| \end{bmatrix} \right\| \\
&= \frac{1}{8}\left(2w(A^2)+\|S\|\right)+\frac{1}{8}\sqrt{(2w(A^2)-\|S\|)^2+8\sup_{\theta\in\mathbb{R}}\|SRe(e^{2i\theta}A^2)\|},
\end{aligned}
$$

as required. □

REMARK 2.11. Setting

$$
c_0=\frac{1}{8}\left(2w(A^2)+\|S\|\right)+\frac{1}{8}\sqrt{(2w(A^2)-\|S\|)^2+8\sup_{\theta\in\mathbb{R}}\|SRe(e^{2i\theta}A^2)\|}.
$$

Then

$$
\begin{aligned}
c_0 &\leqslant \frac{1}{8}\left(2\,w(A^2)+\|S\|\right)+\frac{1}{8}\sqrt{(2w(A^2)-\|S\|)^2+8w(A^2)\|S\|} \\
&= \frac{1}{2}w(A^2)+\frac{1}{4}\|AA^*+A^*A\|.
\end{aligned}
$$

This proves that the inequality (2.8) is an improvement of the inequality (1.8).

A generalization of Theorem 2.1 can be stated as follows.

THEOREM 2.12. *Let $A\in\mathbb{B}(\mathcal{H})$ have the Cartesian decomposition $A=H+iK$, and let $r\geqslant 2$. Then for $\alpha,\beta>0$,*

$$
\sup_{\alpha^2+\beta^2=1}\frac{1}{2}\left\||\alpha H+\beta K|^r+|\alpha H-\beta K|^r\right\|\leqslant w^r(A). \tag{2.9}
$$

Proof. From the inequality (2.2), we get

$$
\begin{aligned}
w^r(A) &\geqslant \sup_{\alpha^2+\beta^2=1}\left\|(\alpha H\pm\beta K)^2\right\|^{\frac{r}{2}} \\
&= \sup_{\alpha^2+\beta^2=1}\left\||\alpha H\pm\beta K|^r\right\|.
\end{aligned}
$$

Hence

$$
w^r(A)\geqslant \sup_{\alpha^2+\beta^2=1}\frac{1}{2}\left\||\alpha H+\beta K|^r+|\alpha H-\beta K|^r\right\|. \quad \square
$$

REMARK 2.13. If we take $\alpha = \beta = \frac{1}{\sqrt{2}}$ in the inequality (2.9), then we obtain

$$\sup_{\alpha^2+\beta^2=1} \frac{1}{2} \| |\alpha H + \beta K|^r + |\alpha H - \beta K|^r \| \geqslant 2^{\frac{-r}{2}-1} \| |H+K|^r + |H-K|^r \| .$$

This means that the inequality (2.9) is a refinement of the inequality (1.9).

THEOREM 2.14. *Let $A \in \mathbb{B}(\mathcal{H})$ have the Cartesian decomposition $A = H + iK$. Then*

$$w^2(A) \leqslant \frac{1}{2}\left\{ \|H\|^2 + \|K\|^2 + \sqrt{(\|H\|^2 - \|K\|^2)^2 + \|HK+KH\|^2} \right\}. \tag{2.10}$$

Proof. We have

$$\begin{aligned} w^2(A) &= \sup_{\alpha^2+\beta^2=1} \|\alpha H + \beta K\|^2 \\ &= \sup_{\alpha^2+\beta^2=1} \|\alpha^2 H^2 + \beta^2 K^2 + \alpha\beta(HK+KH)\| \\ &\leqslant \sup_{\alpha^2+\beta^2=1} \left(\alpha^2 \|H\|^2 + \beta^2 \|K\|^2 + |\alpha\beta| \|HK+KH\|\right) \\ &= \frac{1}{2}\left\{ \|H\|^2 + \|K\|^2 + \sqrt{(\|H\|^2 - \|K\|^2)^2 + \|HK+KH\|^2} \right\}. \quad \square \end{aligned}$$

It is easy to check that the inequality (2.10) is a refinement of the inequality (1.10).

3. Numerical radius inequalities of the product and the sum for two operators

In the following theorems, we present some numerical radius inequalities of the product and the sum for two operators. Some well-known numerical radius inequalities are reobtained.

THEOREM 3.1. *Let $A, B \in \mathbb{B}(\mathcal{H})$. Then*

$$w^2(B^*A) \leqslant \frac{1}{4} w^2(AB^*) + \frac{1}{8} w(PAB^* + AB^*P) + \frac{1}{16}\|P\|^2, \tag{3.1}$$

where $P = AA^* + BB^*$.

Proof. Let $x \in \mathcal{H}$ be any unit vector. For any $\theta \in \mathbb{R}$, we have

$$\begin{aligned}
Re\langle e^{i\theta} B^* Ax, x\rangle &= Re\langle e^{i\theta} Ax, Bx\rangle \\
&= \frac{1}{4}\left\|(e^{i\theta}A+B)x\right\|^2 - \frac{1}{4}\left\|(e^{i\theta}A-B)x\right\|^2 \\
&\leqslant \frac{1}{4}\left\|e^{i\theta}A+B\right\|^2 \\
&= \frac{1}{4}\left\|P+2Re(e^{i\theta}AB^*)\right\| \\
&= \frac{1}{4}\left\|P^2+4Re^2(e^{i\theta}(AB^*))+2Re(e^{i\theta}(PAB^*+AB^*P))\right\|^{\frac{1}{2}} \\
&\leqslant \frac{1}{4}\left(\|P\|^2+4\|Re(e^{i\theta}(AB^*))\|^2+2\|Re(e^{i\theta}(PAB^*+AB^*P))\|\right)^{\frac{1}{2}}.
\end{aligned}$$

By taking the supremum on the both sides in the above inequality over $\theta \in \mathbb{R}$, gives

$$|\langle B^*Ax, x\rangle|^2 \leqslant \frac{1}{4}w^2(AB^*) + \frac{1}{8}w(PAB^*+AB^*P) + \frac{1}{16}\|P\|^2.$$

Therefore, the desired inequality follows by taking the supremum in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$. □

REMARK 3.2. Using Lemma 2.6, it follows that

$$\begin{aligned}
\frac{1}{4}w^2(AB^*) + \frac{1}{8}w(PAB^*+AB^*P) + \frac{1}{16}\|P\|^2 &\leqslant \frac{1}{4}w^2(AB^*) + \frac{1}{4}w(AB^*)\|P\| + \frac{1}{16}\|P\|^2 \\
&= \left(\frac{1}{4}\|P\| + \frac{1}{2}w(AB^*)\right)^2.
\end{aligned}$$

This proves that the inequality (3.1) is sharper than the inequality (1.11) for $r = 1$.

The following lemma is known as the mixed Schwarz inequality, it can be found in [10, pp. 75–76].

LEMMA 3.3. *Let* $A \in \mathbb{B}(\mathcal{H})$. *Then*

$$|\langle Ax, y\rangle| \leqslant \langle |A|x, x\rangle^{\frac{1}{2}} \langle |A^*|y, y\rangle^{\frac{1}{2}} \quad \text{for all } x, y \in \mathcal{H}.$$

THEOREM 3.4. *Let* $A, B \in \mathbb{B}(\mathcal{H})$. *Then*

$$w(BA) \leqslant \min\left\{\frac{1}{2}\|\, A^*|B|A + |B^*|\,\|, \frac{1}{2}\|\, B|A^*|B^* + |A|\,\|\right\}.$$

Proof. Let $x \in \mathcal{H}$ be any unit vector. Using Lemma 3.3, we have

$$\begin{aligned}
|\langle BAx, x\rangle| &\leqslant \langle |B|Ax, Ax\rangle^{\frac{1}{2}} \langle |B^*|x, x\rangle^{\frac{1}{2}} \\
&= \langle A^*|B|Ax, x\rangle^{\frac{1}{2}} \langle |B^*|x, x\rangle^{\frac{1}{2}} \\
&\leqslant \frac{1}{2}\left(\langle A^*|B|Ax, x\rangle + \langle |B^*|x, x\rangle\right).
\end{aligned}$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we get

$$w(BA) \leqslant \frac{1}{2}\||A^*|B|A + |B^*|\|.$$

Again, we have $w(BA) = w(A^*B^*) \leqslant \frac{1}{2}\||B|A^*|B^* + |A|\|$. This completes the proof. □

THEOREM 3.5. *Let* $A, B \in \mathbb{B}(\mathcal{H})$. *Then*

$$w^2(A+B) \leqslant 2w(AB) + \|AA^* + B^*B\|. \tag{3.2}$$

Proof. Let $x \in \mathcal{H}$ be any unit vector. Then

$$\begin{aligned} |\langle (A+B)x, x\rangle| &\leqslant |\langle Ax, x\rangle| + |\langle Bx, x\rangle| \\ &= \sup_{\theta \in \mathbb{R}} \left| e^{i\theta} \langle Ax, x\rangle + e^{-i\theta} \langle B^*x, x\rangle \right| \\ &= \sup_{\theta \in \mathbb{R}} \left| \langle (e^{i\theta}A + e^{-i\theta}B^*)x, x\rangle \right|. \end{aligned}$$

By taking the supremum on both sides in the above inequality over $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain

$$\begin{aligned} w^2(A+B) &\leqslant \sup_{\theta \in \mathbb{R}} \left\| (e^{i\theta}A + e^{-i\theta}B^*)(e^{i\theta}A + e^{-i\theta}B^*)^* \right\| \\ &= \sup_{\theta \in \mathbb{R}} \left\| 2Re(e^{2i\theta}AB) + AA^* + B^*B \right\| \\ &\leqslant \sup_{\theta \in \mathbb{R}} \left\| 2Re(e^{2i\theta}AB) \right\| + \|AA^* + B^*B\| \\ &= 2w(AB) + \|AA^* + B^*B\|, \end{aligned}$$

as required. □

If we take $B = A$ in the inequality (3.2), then we reobtain the inequality (1.8).

REMARK 3.6. If A, B are normal, then the inequality (3.2) is a refinement for the triangle inequality of the numerical radius. Indeed,

$$\begin{aligned} w^2(A+B) \leqslant 2w(AB) + \|AA^* + B^*B\| &\leqslant 2w(A)w(B) + w(AA^* + B^*B) \\ &\leqslant w^2(A) + w^2(B) + 2w(A)w(B) \\ &= (w(A) + w(B))^2. \end{aligned}$$

THEOREM 3.7. *Let* $A, B \in \mathbb{B}(\mathcal{H})$ *and let* $r \geqslant 1$. *Then*

$$w^{2r}(A+B) \leqslant 2^{2r-1}\left(w^r(AB) + \frac{1}{2}\|(AA^*)^r + (B^*B)^r\| \right).$$

Proof. Using the previous theorem, it follows that

$$\begin{aligned} w^{2r}(A+B) &\leqslant (2\,w(AB)+\|AA^*+B^*B\|)^r \\ &\leqslant 2^{r-1}\left(2^r w^r(AB)+2^r\left\|\left(\frac{AA^*+B^*B}{2}\right)^r\right\|\right) \\ &\leqslant 2^{2r-1}\left(w^r(AB)+\frac{1}{2}\|(AA^*)^r+(B^*B)^r\|\right). \quad \square \end{aligned}$$

COROLLARY 3.8. *Let $A \in \mathbb{B}(\mathcal{H})$ and let $r \geqslant 1$. Then*

$$w^{2r}(A) \leqslant \frac{1}{2}\left(w^r(A^2)+\frac{1}{2}\|(AA^*)^r+(A^*A)^r\|\right). \tag{3.3}$$

Note that

$$\frac{1}{2}\left(w^r(A^2)+\frac{1}{2}\|(AA^*)^r+(A^*A)^r\|\right) \leqslant \frac{1}{2}w^r(A^2)+\frac{1}{2}\|A\|^{2r}.$$

Therefore, the inequality (3.3) is an improvement of the inequality (1.12).

THEOREM 3.9. *Let $A,B \in \mathbb{B}(\mathcal{H})$. Then*

$$w^4(A+B) \leqslant 4w^2(BA)+2w(BAP+PBA)+\|P\|^2,$$

where $P = A^*A+BB^*$.

Proof. We have

$$w(A+B) \leqslant \sup_{\theta\in\mathbb{R}} \|e^{i\theta}A+e^{-i\theta}B^*\|.$$

Let $\psi(\theta) = \|e^{i\theta}A+e^{-i\theta}B^*\|$. Then

$$\begin{aligned} \psi(\theta) &= \|(e^{i\theta}A+e^{-i\theta}B^*)^*(e^{i\theta}A+e^{-i\theta}B^*)\|^{\frac{1}{2}} \\ &= \|P+2Re(e^{2i\theta}(BA))\|^{\frac{1}{2}} \\ &= \|P^2+4Re^2(e^{2i\theta}(BA))+2Re(e^{2i\theta}(BAP+PBA))\|^{\frac{1}{4}} \\ &\leqslant \left(\|P\|^2+4\|Re(e^{2i\theta}(BA))\|^2+2\|Re(e^{2i\theta}(BAP+PBA))\|\right)^{\frac{1}{4}}. \end{aligned}$$

By taking the supremum on both sides in $\psi(\theta)$ over $\theta \in \mathbb{R}$, we obtain the desired inequality. $\square$

REMARK 3.10. If we put $A = B$ in the previous theorem, then we get

$$w^4(A) \leqslant \frac{1}{4}w^2(A^2)+\frac{1}{8}w(A^2R+RA^2)+\frac{1}{16}\|R\|^2,$$

where $R = A^*A+AA^*$.

This inequality has been given in [5].

COROLLARY 3.11. *Let* $A,B \in \mathbb{B}(\mathcal{H})$. *Then*

$$w^4(A+B) \leqslant 4w^2(AB) + 2w(ABT+TAB) + \|T\|^2,$$

where $T = B^*B + AA^*$.

THEOREM 3.12. *Let* $A,B \in \mathbb{B}(\mathcal{H})$. *Then*

$$\|A+B\|^2 \leqslant \min\{\|AA^*+BB^*\| + 2w(AB^*), \|A^*A+B^*B\| + 2w(B^*A)\}. \tag{3.4}$$

Proof. Let $x,y \in \mathcal{H}$ be two vectors with $\|x\| = \|y\| = 1$. Then

$$\begin{aligned} |\langle (A+B)x,y\rangle| &\leqslant |\langle Ax,y\rangle| + |\langle Bx,y\rangle| \\ &= \sup_{\theta\in\mathbb{R}} \left| e^{i\theta}\langle Ax,y\rangle + e^{-i\theta}\langle Bx,y\rangle \right| \\ &\leqslant \sup_{\theta\in\mathbb{R}} \left\| e^{i\theta}A + e^{-i\theta}B \right\| \|x\|\|y\|. \end{aligned}$$

By taking the supremum on both sides in the above inequality over $x,y \in \mathcal{H}$ with $\|x\| = 1$ and $\|y\| = 1$, we obtain

$$\|A+B\| \leqslant \sup_{\theta\in\mathbb{R}} \left\| e^{i\theta}A + e^{-i\theta}B \right\|.$$

Using the fact that $\|XX^*\| = \|X^*X\| = \|X\|^2$ and $w(X) = \sup_{\theta\in\mathbb{R}} \left\| Re\left(e^{i\theta}X\right) \right\|$ for any operator X, the desired result is obtained. □

The inequality (3.4) is a refinement of the triangle inequality. Indeed,

$$\|A+B\|^2 \leqslant \|A^*A+B^*B\| + 2w(B^*A) \leqslant \|A\|^2 + \|B\|^2 + 2\|A\|\|B\| = (\|A\|+\|B\|)^2.$$

Also, it easy to check that the inequality (3.4) is an improvement of the inequality (1.13).

COROLLARY 3.13. *Let* $A,B \in \mathbb{B}(\mathcal{H})$. *Then*

$$w^2(A+B) \leqslant \min\{\|AA^*+BB^*\| + 2w(AB^*), \|A^*A+B^*B\| + 2w(B^*A)\}. \tag{3.5}$$

It is easy to see that, if A and B are normal, then the inequality (3.5) is better than the inequality (1.14).

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