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UNCERTAINTY INEQUALITIES FOR WEIGHTED SPACES OF ANALYTIC FUNCTIONS ON THE UNIT DISK

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Abstract. In this paper we establish uncertainty inequality of Heisenberg type for Hardy space $\mathscr{H}$, Dirichlet space $\mathscr{D}$ and Bergman space $\mathscr{B}$, respectively. Next, we introduce a weighted Hardy space $\mathscr{H}_\beta$. This space which gives a generalization of some Hilbert spaces of analytic functions on the unit disk like, the Hardy space $\mathscr{H}$, the Dirichlet space $\mathscr{D}$ and the Bergman space $\mathscr{B}$, it plays a background to our contribution. Especially, we study the derivative operator D and its adjoint operator L_β on $\mathscr{H}_\beta$, and we deduce a general uncertainty inequality of Heisenberg type for this space.

1. Introduction

Heisenberg's uncertainty principle is a key principle in quantum mechanics [14]. Very roughly, it states that if we know everything about where a particle is located (the uncertainty of position is small), we know nothing about its momentum (the uncertainty of momentum is large), and vice versa. There exist many similar uncertainty principles, in quantum physics [3, 7, 15, 24], and in harmonic analysis [6, 16], that are based on position, momentum, energy, time, and so on. In this paper we are going to prove a version of the uncertainty principle in the context of Hardy space, Dirichlet space and Bergman space. These spaces of analytic functions on the unit disk are one of the complex analysis tools used in harmonic analysis [17, 28, 30].

Let $\mathbb{D} := \mathbb{D}(0,1) = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk. The Hardy space $\mathscr{H}$ is the set of all analytic functions f in the unit disk $\mathbb{D}$ such that

$$\|f\|^2_{\mathscr{H}} := \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2 \mathrm{d}\theta < \infty.$$

This space is the goal of many works in various field of mathematics [22, 29] and in certain parts of quantum mechanics [2, 20]. In the first section of this paper, we introduce and study the derivative operator $D = \frac{\mathrm{d}}{\mathrm{d}z}$ and its adjoint operator $L_{\mathscr{H}} = z^2\frac{\mathrm{d}}{\mathrm{d}z} + z$ on the Hardy space $\mathscr{H}$, and we deduce the following uncertainty inequality of Heisenberg-type for this space:

$$\|(D + L_{\mathscr{H}} - a)f\|_{\mathscr{H}} \|(D - L_{\mathscr{H}} + ib)f\|_{\mathscr{H}} \geqslant \|f\|^2_{\mathscr{H}}, \quad a, b \in \mathbb{C}.$$

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In the second section of this paper, we introduce and study the operator $D_0 f(z) := f'(z) - f'(0)$ and its adjoint operator $L_{\mathscr{D}} f(z) := z^2 f'(z)$ on the Dirichlet space $\mathscr{D}$, and we establish the following uncertainty inequality:

$$\|(D_0 + L_{\mathscr{D}} - a)f\|_{\mathscr{D}} \|(D_0 - L_{\mathscr{D}} + ib)f\|_{\mathscr{D}} \geqslant 2(\|f\|_{\mathscr{D}}^2 - |f(0)|^2), \quad a, b \in \mathbb{C}.$$

As in the same way in the third section of this paper, we introduce and study the derivative operator $D = \frac{\mathrm{d}}{\mathrm{d}z}$ and its adjoint operator $L_{\mathscr{B}} = z^2 \frac{\mathrm{d}}{\mathrm{d}z} + 2z$ on the Bergman space $\mathscr{B}$, and we establish an uncertainty inequality of Heisenberg-type for this space.

In the last section of this paper, we introduce the weighted Hardy space $\mathscr{H}_\beta$, which is the set of all analytic functions f in the unit disk $\mathbb{D}$, with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, such that

$$\|f\|_{\mathscr{H}_\beta}^2 := \sum_{n=0}^{\infty} \beta_n |a_n|^2 < \infty,$$

where $\beta = \{\beta_n\}$ is a positive sequence so that $\limsup_{n\to\infty} (\beta_n)^{-1/n} = 1$.

The space $\mathscr{H}_\beta$ is a reproducing kernel Hilbert space that gives a generalization of some Hilbert spaces of analytic functions on the unit disk like, the Dirichlet space $\mathscr{D}$ (see [4, 11, 12, 19]) when $\beta_n = n$, and the Bergman space $\mathscr{B}$ (see [10, 13]) when $\beta_n = \frac{1}{n+1}$. For $f \in \mathscr{H}_\beta$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, we define the derivative operator $Df(z) := \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$ and its adjoint operator $L_{\mathscr{H}_\beta} f(z) := \sum_{n=1}^{\infty} \frac{n\beta_{n-1}}{\beta_n} a_{n-1} z^n$, and we deduce the following general uncertainty inequality for the space $\mathscr{H}_\beta$:

$$\|(D + L_{\mathscr{H}_\beta} - a)f\|_{\mathscr{H}_\beta} \|(D - L_{\mathscr{H}_\beta} + ib)f\|_{\mathscr{H}_\beta} \geqslant \frac{\beta_0}{\beta_1} \|f\|_{\mathscr{H}_\beta}^2, \quad a, b \in \mathbb{C}.$$

Recently, the analog uncertainty inequality is also proved, for the Fock space [5, 31], for the Dunkl-type Fock spaces [25, 26], and for the Bessel-type Fock spaces [21, 27].

2. The Hardy space $\mathscr{H}$

The Hardy space $\mathscr{H}$ is the set of all analytic functions f in the unit disk $\mathbb{D}$ such that

$$\|f\|_{\mathscr{H}}^2 := \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 \mathrm{d}\theta < \infty.$$

It is a Hilbert space when equipped with the inner product

$$\langle f, g \rangle_{\mathscr{H}} := \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \mathrm{d}\theta.$$

If $f, g \in \mathscr{H}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then

$$\langle f, g \rangle_{\mathscr{H}} = \sum_{n=0}^{\infty} a_n \overline{b_n}.$$

The set $\left\{z^n\right\}_{n=0}^{\infty}$ forms a Hilbert's basis for the space $\mathscr{H}$. The Szegö kernel $K_{\mathscr{H},z}$, $z \in \mathbb{D}$, given by

$$K_{\mathscr{H},z}(w) := \sum_{n=0}^{\infty} (\bar{z}w)^n = \frac{1}{1-\bar{z}w}, \quad w \in \mathbb{D},$$

is a reproducing kernel for the Hardy space $\mathscr{H}$, meaning that $K_{\mathscr{H},z} \in \mathscr{H}$, and for all $f \in \mathscr{H}$, we have $\langle f, K_{\mathscr{H},z}\rangle_{\mathscr{H}} = f(z)$.

The function $u(z) = K_{\mathscr{H},\bar{z}}(w)$ is the unique analytic solution on $\mathbb{D}$ of the initial problem

$$zDu(z) = wL_{\mathscr{H}}u(z), \quad w \in \mathbb{D}, \quad u(0) = 1,$$

where D and $L_{\mathscr{H}}$ are the operators given by

$$Du(z) := u'(z), \quad L_{\mathscr{H}}u(z) := z^2u'(z) + zu(z).$$

These operators satisfy the commutation rule

$$[D, L_{\mathscr{H}}] := DL_{\mathscr{H}} - L_{\mathscr{H}}D = I + 2E, \tag{2.1}$$

where I is the identity operator and E is the Euler operator given by

$$Ef(z) := zf'(z).$$

We define the domain of D denoted by $\mathrm{Dom}(D)$ as

$$\mathrm{Dom}(D) := \{f \in \mathscr{H} : Df \in \mathscr{H}\}.$$

And as in the same way we define $\mathrm{Dom}(L_{\mathscr{H}})$ and $\mathrm{Dom}(E)$.

We define the Hilbert space $\mathscr{U}$ as the space of all analytic functions f in the unit disk $\mathbb{D}$ such that

$$\|f\|_{\mathscr{U}}^2 := |f(0)|^2 + \frac{1}{2\pi}\int_0^{2\pi} |f'(e^{i\theta})|^2 \mathrm{d}\theta < \infty.$$

If $f \in \mathscr{U}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then

$$\|f\|_{\mathscr{U}}^2 = |a_0|^2 + \sum_{n=1}^{\infty} n^2 |a_n|^2.$$

The spaces $\mathscr{H}$ and $\mathscr{U}$ satisfy the continuous inclusion $\mathscr{U} \subset \mathscr{H}$.

LEMMA 2.1. *The operators D, $L_{\mathscr{H}}$ and E satisfy the following properties.*
(i) $Dom(D) = Dom(L_{\mathscr{H}}) = Dom(E) = \mathscr{U}$.
(ii) For $f, g \in \mathscr{U}$, one has $\langle Df, g\rangle_{\mathscr{H}} = \langle f, L_{\mathscr{H}}g\rangle_{\mathscr{H}}$.

Proof. Let $f \in \mathscr{H}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then

$$Df(z) = \sum_{n=0}^{\infty} (n+1)a_{n+1}z^n, \quad L_{\mathscr{H}}f(z) = \sum_{n=1}^{\infty} na_{n-1}z^n,$$

and

$$Ef(z) = \sum_{n=1}^{\infty} n a_n z^n.$$

Thus

$$\|Df\|_{\mathscr{H}}^2 = \sum_{n=0}^{\infty} (n+1)^2 |a_{n+1}|^2, \quad \|L_{\mathscr{H}} f\|_{\mathscr{H}}^2 = \sum_{n=0}^{\infty} (n+1)^2 |a_n|^2,$$

and

$$\|Ef\|_{\mathscr{H}}^2 = \sum_{n=1}^{\infty} n^2 |a_n|^2.$$

Therefore

$$\|Df\|_{\mathscr{H}}^2 = \|f\|_{\mathscr{U}}^2 - |f(0)|^2, \quad \|f\|_{\mathscr{U}}^2 \leqslant \|L_{\mathscr{H}} f\|_{\mathscr{H}}^2 \leqslant 4\|f\|_{\mathscr{U}}^2,$$

and

$$\|Ef\|_{\mathscr{H}}^2 = \|f\|_{\mathscr{U}}^2 - |f(0)|^2.$$

Consequently $\mathrm{Dom}(D) = \mathrm{Dom}(L_{\mathscr{H}}) = \mathrm{Dom}(E) = \mathscr{U}$.

On the other hand, for $f, g \in \mathscr{U}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, we have

$$\langle Df, g\rangle_{\mathscr{H}} = \sum_{n=0}^{\infty} (n+1) a_{n+1} \overline{b_n} = \sum_{n=1}^{\infty} n a_n \overline{b_{n-1}} = \langle f, L_{\mathscr{H}} g\rangle_{\mathscr{H}}.$$

The lemma is proved. □

LEMMA 2.2. (See [9]) *Let A and B be self-adjoint operators on a Hilbert space H ($A^* = A$, $B^* = B$). Then*

$$\|(A-a)f\|_H \|(B-b)f\|_H \geqslant \frac{1}{2} |\langle [A,B]f, f\rangle_H|,$$

for all $f \in Dom([A,B])$, and all $a, b \in \mathbb{C}$.

THEOREM 2.3. *Let $f \in \mathscr{H}$. For all $a, b \in \mathbb{C}$, one has*

$$\|(D + L_{\mathscr{H}} - a)f\|_{\mathscr{H}} \|(D - L_{\mathscr{H}} + ib)f\|_{\mathscr{H}} \geqslant \|f\|_{\mathscr{H}}^2. \tag{2.2}$$

Proof. Let $f \in \mathscr{H}$. First the inequality (2.2) is true for $f \notin \mathscr{U}$. Now, let A and B be the operators defined for $f \in \mathscr{U}$ by

$$A := (D + L_{\mathscr{H}})f, \quad B := i(D - L_{\mathscr{H}})f.$$

By (2.1) and Lemma 2.1, the operators A and B possess the following properties.

(i) $A^* = A$ and $B^* = B$,

(ii) $[A,B] = -2i[D, L_{\mathscr{H}}] = -2i(I + 2E)$,

(iii) $\mathrm{Dom}([A,B]) = \mathscr{U}$.

Thus, the inequality (2.2) follows from Lemma 2.2 and the fact that

$$\langle Ef, f\rangle_{\mathscr{H}} \geqslant 0.$$

This completes the proof of the theorem. □

3. The Dirichlet space $\mathscr{D}$

The Dirichlet space $\mathscr{D}$ (see [1, 4, 8, 11, 19]) is the set of all analytic functions f in the unit disk $\mathbb{D}$ such that

$$\|f\|_{\mathscr{D}}^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \frac{\mathrm{d}x\mathrm{d}y}{\pi} < \infty, \quad z = x + iy.$$

It is a Hilbert space when equipped with the inner product

$$\langle f, g \rangle_{\mathscr{D}} := f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)} \frac{\mathrm{d}x\mathrm{d}y}{\pi}.$$

If $f, g \in \mathscr{D}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then

$$\langle f, g \rangle_{\mathscr{D}} = a_0\overline{b_0} + \sum_{n=1}^{\infty} n a_n \overline{b_n},$$

and the set $\left\{1, \frac{z^n}{\sqrt{n}}\right\}_{n=1}^{\infty}$ forms an Hilbert's basis for the space $\mathscr{D}$. The function $K_{\mathscr{D},z}$, $z \in \mathbb{D}$, given by

$$K_{\mathscr{D},z}(w) := 1 + \sum_{n=1}^{\infty} \frac{(\overline{z}w)^n}{n} = 1 + \log\left(\frac{1}{1 - \overline{z}w}\right), \quad w \in \mathbb{D},$$

is a reproducing kernel for the weighted Dirichlet space $\mathscr{D}$.

The function $u(z) = K_{\mathscr{D},\overline{z}}(w)$ is the unique analytic solution on $\mathbb{D}$ of the initial problem

$$zD_0u(z) = wL_{\mathscr{D}}u(z), \quad w \in \mathbb{D}, \quad u(0) = 1,$$

where D_0 and $L_{\mathscr{D}}$ are the operators given by

$$D_0u(z) := u'(z) - u'(0), \quad L_{\mathscr{D}}u(z) := z^2u'(z).$$

These operators satisfy the commutation relation

$$[D_0, L_{\mathscr{D}}] = 2E, \tag{3.1}$$

where E is the Euler operator given in Section 2.

We define the Hilbert space $\mathscr{V}$ as the space of all analytic functions f in the unit disk $\mathbb{D}$ such that

$$\|f\|_{\mathscr{V}}^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z) + zf''(z)|^2 \frac{\mathrm{d}x\mathrm{d}y}{\pi} < \infty.$$

If $f \in \mathscr{V}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then

$$\|f\|_{\mathscr{V}}^2 = |a_0|^2 + \sum_{n=1}^{\infty} n^3 |a_n|^2.$$

The spaces $\mathscr{D}$ and $\mathscr{V}$ satisfy the continuous inclusion $\mathscr{V} \subset \mathscr{D}$.

In this section we establish an uncertainty inequality of Heisenberg-type on the space $\mathscr{D}$. We will use the following lemma.

LEMMA 3.1. *The operators D_0, $L_{\mathscr{D}}$ and E satisfy the following properties.*
(i) $Dom(D_0) = Dom(L_{\mathscr{D}}) = Dom(E) = \mathscr{V}$.
(ii) For $f,g \in \mathscr{V}$, one has $\langle D_0 f, g\rangle_{\mathscr{D}} = \langle f, L_{\mathscr{D}} g\rangle_{\mathscr{D}}$.

Proof. Let $f \in \mathscr{D}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then

$$D_0 f(z) = \sum_{n=1}^{\infty} (n+1) a_{n+1} z^n, \quad L_{\mathscr{D}} f(z) = \sum_{n=2}^{\infty} (n-1) a_{n-1} z^n,$$

and

$$E f(z) = \sum_{n=1}^{\infty} n a_n z^n.$$

Thus

$$\|D_0 f\|_{\mathscr{D}}^2 = \sum_{n=2}^{\infty} (n-1) n^2 |a_n|^2, \quad \|L_{\mathscr{D}} f\|_{\mathscr{D}}^2 = \sum_{n=1}^{\infty} (n+1) n^2 |a_n|^2,$$

and

$$\|E f\|_{\mathscr{D}}^2 = \sum_{n=1}^{\infty} n^3 |a_n|^2.$$

Therefore

$$\|f\|_{\mathscr{V}}^2 - |f(0)|^2 - |f'(0)|^2 \leqslant 2\|D_0 f\|_{\mathscr{D}}^2 \leqslant 2\|f\|_{\mathscr{V}}^2,$$
$$\|f\|_{\mathscr{V}}^2 - |f(0)|^2 \leqslant \|L_{\mathscr{D}} f\|_{\mathscr{D}}^2 \leqslant 2\|f\|_{\mathscr{V}}^2,$$

and

$$\|f\|_{\mathscr{V}}^2 - |f(0)|^2 \leqslant \|E f\|_{\mathscr{D}}^2 \leqslant \|f\|_{\mathscr{V}}^2.$$

Consequently $\mathrm{Dom}(D_0) = \mathrm{Dom}(L_{\mathscr{D}}) = \mathrm{Dom}(E) = \mathscr{V}$.

On the other hand, for $f,g \in \mathscr{V}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, one has

$$\langle D_0 f, g\rangle_{\mathscr{D}} = \sum_{n=1}^{\infty} n(n+1) a_{n+1} \overline{b_n} = \sum_{n=2}^{\infty} n(n-1) a_n \overline{b_{n-1}} = \langle f, L_{\mathscr{D}} g\rangle_{\mathscr{D}}.$$

The lemma is proved. □

THEOREM 3.2. *Let $f \in \mathscr{D}$ with $f(0) = 0$. For all $a,b \in \mathbb{C}$, we have*

$$\|(D_0 + L_{\mathscr{D}} - a) f\|_{\mathscr{D}} \|(D_0 - L_{\mathscr{D}} + ib) f\|_{\mathscr{D}} \geqslant 2\|f\|_{\mathscr{D}}^2. \tag{3.2}$$

Proof. Let $f \in \mathscr{D}$ with $f(0) = 0$. First the inequality (3.2) is true for $f \notin \mathscr{V}$. Now, let A and B be the operators defined for $f \in \mathscr{V}$ by

$$A := (D_0 + L_{\mathscr{D}}) f, \quad B := i(D_0 - L_{\mathscr{D}}) f.$$

By (3.1) and Lemma 3.1, the operators A and B possess the following properties.
(i) $A^* = A$ and $B^* = B$,
(ii) $[A,B] = -2i[D_0, L_{\mathscr{D}}] = -4iE$,

(iii) $\mathrm{Dom}([A,B]) = \mathscr{V}$.

Thus, the inequality (3.2) follows from Lemma 2.2 and the fact that

$$\langle Ef, f\rangle_{\mathscr{D}} \geqslant \|f\|^2_{\mathscr{D}}.$$

This completes the proof of the theorem. □

Let $\alpha \geqslant 0$, the weighted Dirichlet space $\mathscr{D}_\alpha$ (see [12, 22]) is the set of all analytic functions f in the unit disk $\mathbb{D}$ such that

$$\|f\|^2_{\mathscr{D}_\alpha} := (\alpha+1)\int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)^\alpha \frac{\mathrm{d}x\mathrm{d}y}{\pi} < \infty, \quad z = x+iy.$$

It is a Hilbert space when equipped with the inner product

$$\langle f, g\rangle_{\mathscr{D}_\alpha} := f(0)\overline{g(0)} + (\alpha+1)\int_{\mathbb{D}} f'(z)\overline{g'(z)}(1-|z|^2)^\alpha \frac{\mathrm{d}x\mathrm{d}y}{\pi} < \infty, \quad z = x+iy.$$

If $f, g \in \mathscr{D}_\alpha$ with $f(z) = \sum_{n=0}^\infty a_n z^n$ and $g(z) = \sum_{n=0}^\infty b_n z^n$, then

$$\langle f, g\rangle_{\mathscr{D}_\alpha} = a_0\overline{b_0} + (\alpha+1)\sum_{n=1}^\infty \frac{nn!}{(\alpha+1)_n} a_n \overline{b_n},$$

where $(\alpha+1)_n = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}$, and the set $\left\{1, \sqrt{\frac{(\alpha+1)_n}{(\alpha+1)nn!}} z^n\right\}_{n=1}^\infty$ forms an Hilbert's basis for the space $\mathscr{D}_\alpha$. The function $K_{\mathscr{D}_\alpha,z}$, $z \in \mathbb{D}$, given by

$$K_{\mathscr{D}_\alpha,z}(w) := 1 + \frac{1}{\alpha+1}\sum_{n=1}^\infty \frac{(\alpha+1)_n}{nn!}(\overline{z}w)^n, \quad w \in \mathbb{D},$$

is a reproducing kernel for the weighted Dirichlet space $\mathscr{D}_\alpha$. In the case $\alpha = 0$, $\mathscr{D}_0$ is the Dirichlet space $\mathscr{D}$.

The function $u(z) = K_{\mathscr{D}_\alpha,\overline{z}}(w)$ is the unique analytic solution on $\mathbb{D}$ of the initial problem

$$zD_0u(z) = wL_{\mathscr{D}_\alpha}u(z), \quad w \in \mathbb{D}, \quad u(0) = 1,$$

where $L_{\mathscr{D}_\alpha}$ is the operator given by

$$L_{\mathscr{D}_\alpha}u(z) := z^2u'(z) + \alpha z u(z) - \alpha\int_{[0,z]} u(s)\mathrm{d}s,$$

and $[0,z] := \{tz, t \in [0,1]\}$ is the line segment joining 0 and z.

The operators D_0 and $L_{\mathscr{D}_\alpha}$ satisfy the commutation relation

$$[D_0, L_{\mathscr{D}_\alpha}] = 2E_\alpha,$$

where

$$E_\alpha f(z) := zf'(z) + \frac{\alpha}{2}(f(z) - f(0)).$$

As in the same way of Theorem 3.2 we obtain the following uncertainty inequality for the space $\mathscr{D}_\alpha$.

THEOREM 3.3. *Let $f \in \mathscr{D}_\alpha$ with $f(0) = 0$. For all $a, b \in \mathbb{C}$, we have*

$$\|(D_0 + L_{\mathscr{D}_\alpha} - a)f\|_{\mathscr{D}_\alpha}\|(D_0 - L_{\mathscr{D}_\alpha} + ib)f\|_{\mathscr{D}_\alpha} \geqslant 2\|f\|^2_{\mathscr{D}_\alpha}.$$

4. The Bergman space $\mathscr{B}$

The weighted Bergman space $\mathscr{B}$ (see [10, 13]) is the set of all analytic functions f in the unit disk $\mathbb{D}$ for which

$$\|f\|_{\mathscr{B}}^2 := \int_{\mathbb{D}} |f(z)|^2 \frac{\mathrm{d}x\mathrm{d}y}{\pi} < \infty, \quad z = x+iy.$$

It is a Hilbert space when equipped with the inner product

$$\langle f, g\rangle_{\mathscr{B}} := \int_{\mathbb{D}} f(z)\overline{g(z)} \frac{\mathrm{d}x\mathrm{d}y}{\pi}, \quad z = x+iy.$$

If $f, g \in \mathscr{B}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then

$$\langle f, g\rangle_{\mathscr{B}} = \sum_{n=0}^{\infty} \frac{a_n \overline{b_n}}{n+1}.$$

The set $\left\{\sqrt{n+1} z^n\right\}_{n\in\mathbb{N}}$ forms an Hilbert's basis for the space $\mathscr{B}$. The function $K_{\mathscr{B},z}$, $z \in \mathbb{D}$, given by

$$K_{\mathscr{B},z}(w) := \frac{1}{(1-\overline{z}w)^2}, \quad w \in \mathbb{D},$$

is a reproducing kernel for the weighted Bergman space $\mathscr{B}$.

The function $u(z) = K_{\mathscr{B},\overline{z}}(w)$ is the unique analytic solution on $\mathbb{D}$ of the initial problem

$$zDu(z) = wL_{\mathscr{B}}u(z), \quad w \in \mathbb{D}, \quad u(0) = 1,$$

where D is the derivative operator given in Section 2 and $L_{\mathscr{B}}$ is the operator defined by

$$L_{\mathscr{B}} := z^2 \frac{\mathrm{d}}{\mathrm{d}z} + 2z.$$

The operators D and $L_{\mathscr{B}}$ satisfy the commutation relation

$$[D, L_{\mathscr{B}}] = 2(I+E). \tag{4.1}$$

We define the Hilbert space $\mathscr{W}$ as the space of analytic functions f in the unit disk $\mathbb{D}$ such that

$$\|f\|_{\mathscr{W}}^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 |z|^2 \frac{\mathrm{d}x\mathrm{d}y}{\pi} < \infty, \quad z = x+iy.$$

If $f \in \mathscr{W}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then

$$\|f\|_{\mathscr{W}}^2 = |a_0|^2 + \sum_{n=1}^{\infty} \frac{n^2}{n+1} |a_n|^2.$$

The spaces $\mathscr{B}$ and $\mathscr{W}$ satisfy the continuous inclusion $\mathscr{W} \subset \mathscr{B}$.

In this section we establish an uncertainty inequality of Heisenberg-type on the space $\mathscr{B}$. We will use the following lemma.

LEMMA 4.1. *The operators D, $L_{\mathscr{B}}$ and E satisfy the following properties.*
(i) $Dom(D) = Dom(L_{\mathscr{B}}) = Dom(E) = \mathscr{W}$.
(ii) For $f,g \in \mathscr{W}$ we have $\langle Df, g\rangle_{\mathscr{B}} = \langle f, L_{\mathscr{B}} g\rangle_{\mathscr{B}}$.

Proof. Let $f \in \mathscr{B}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then

$$Df(z) = \sum_{n=0}^{\infty}(n+1)a_{n+1}z^n, \quad L_{\mathscr{B}}f(z) = \sum_{n=1}^{\infty}(n+1)a_{n-1}z^n,$$

and

$$Ef(z) = \sum_{n=1}^{\infty} na_n z^n.$$

Thus

$$\|Df\|_{\mathscr{B}}^2 = \sum_{n=1}^{\infty} n|a_n|^2, \quad \|L_{\mathscr{B}}f\|_{\mathscr{B}}^2 = \sum_{n=0}^{\infty}(n+2)|a_n|^2,$$

and

$$\|Ef\|_{\mathscr{B}}^2 = \sum_{n=1}^{\infty} \frac{n^2}{n+1}|a_n|^2.$$

Therefore

$$\|f\|_{\mathscr{W}}^2 - |f(0)|^2 \leqslant \|Df\|_{\mathscr{B}}^2 \leqslant 2\|f\|_{\mathscr{W}}^2, \quad \|f\|_{\mathscr{W}}^2 \leqslant \|L_{\mathscr{B}}f\|_{\mathscr{B}}^2 \leqslant 6|f\|_{\mathscr{W}}^2,$$

and

$$\|f\|_{\mathscr{W}}^2 - |f(0)|^2 \leqslant \|Ef\|_{\mathscr{B}}^2 \leqslant \|f\|_{\mathscr{W}}^2.$$

Consequently $\mathrm{Dom}(D) = \mathrm{Dom}(L_{\mathscr{B}}) = \mathrm{Dom}(E) = \mathscr{W}$.

On the other hand, for $f,g \in \mathscr{W}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, one has

$$\langle Df, g\rangle_{\mathscr{B}} = \sum_{n=0}^{\infty} a_{n+1}\overline{b_n} = \langle f, L_{\mathscr{B}}g\rangle_{\mathscr{B}}.$$

The lemma is proved. □

THEOREM 4.2. *Let $f \in \mathscr{B}$. For all $a,b \in \mathbb{C}$, we have*

$$\|(D+L_{\mathscr{B}}-a)f\|_{\mathscr{B}}\|(D-L_{\mathscr{B}}+ib)f\|_{\mathscr{B}} \geqslant 2\|f\|_{\mathscr{B}}^2. \tag{4.2}$$

Proof. Let $f \in \mathscr{B}$. First the inequality (4.2) is true for $f \notin \mathscr{W}$. Now, let A and B be the operators defined for $f \in \mathscr{W}$ by

$$A := (D+L_{\mathscr{B}})f, \quad B := i(D-L_{\mathscr{B}})f.$$

By (4.1) and Lemma 4.1, the operators A and B possess the following properties.
(i) $A^* = A$ and $B^* = B$,
(ii) $[A,B] = -2i[D,L_{\mathscr{B}}] = -4i(I+E)$,
(iii) $\mathrm{Dom}([A,B]) = \mathscr{W}$.

Thus, the inequality (4.2) follows from Lemma 2.2 and the fact that

$$\langle Ef, f\rangle_{\mathscr{B}} \geqslant 0.$$

This completes the proof of the theorem. □

Let $\alpha > -1$, the weighted Bergman space $\mathscr{B}_\alpha$ (see [18, 23]) is the set of all analytic functions f in the unit disk $\mathbb{D}$ for which

$$\|f\|^2_{\mathscr{B}_\alpha} := (\alpha+1)\int_{\mathbb{D}} |f(z)|^2(1-|z|^2)^\alpha \frac{\mathrm{d}x\mathrm{d}y}{\pi} < \infty, \quad z = x+iy.$$

It is a Hilbert space when equipped with the inner product

$$\langle f, g\rangle_{\mathscr{B}_\alpha} := (\alpha+1)\int_{\mathbb{D}} f(z)\overline{g(z)}(1-|z|^2)^\alpha \frac{\mathrm{d}x\mathrm{d}y}{\pi}, \quad z = x+iy.$$

If $f, g \in \mathscr{B}_\alpha$ with $f(z) = \sum_{n=0}^\infty a_n z^n$ and $g(z) = \sum_{n=0}^\infty b_n z^n$, then

$$\langle f, g\rangle_{\mathscr{B}_\alpha} = \sum_{n=0}^\infty a_n \overline{b_n} \frac{n!}{(\alpha+2)_n},$$

where $(\alpha+2)_n = \frac{\Gamma(n+\alpha+2)}{\Gamma(\alpha+2)}$. The set $\left\{\sqrt{\frac{(\alpha+2)_n}{n!}} z^n\right\}_{n\in\mathbb{N}}$ forms an Hilbert's basis for the space $\mathscr{B}_\alpha$. The function $K_{\mathscr{B}_\alpha, z}$, $z \in \mathbb{D}$, given by

$$K_{\mathscr{B}_\alpha, z}(w) := \frac{1}{(1-\overline{z}w)^{\alpha+2}}, \quad w \in \mathbb{D},$$

is a reproducing kernel for the weighted Bergman space $\mathscr{B}_\alpha$. In the case $\alpha = 0$, $\mathscr{B}_0$ is the Bergman space $\mathscr{B}$.

The function $u(z) = K_{\mathscr{B}_\alpha, \overline{z}}(w)$ is the unique analytic solution on $\mathbb{D}$ of the initial problem

$$zDu(z) = wL_{\mathscr{B}_\alpha} u(z), \quad w \in \mathbb{D}, \quad u(0) = 1,$$

where $L_{\mathscr{B}_\alpha}$ is the operator given by

$$L_{\mathscr{B}_\alpha} := z^2 \frac{\mathrm{d}}{\mathrm{d}z} + (\alpha+2)z.$$

The operators D and $L_{\mathscr{B}_\alpha}$ satisfy the commutation relation

$$[D, L_{\mathscr{B}_\alpha}] = (\alpha+2)I + 2E.$$

As in the same way of Theorem 4.2 we obtain the following uncertainty inequality for the space $\mathscr{B}_\alpha$.

THEOREM 4.3. *Let $f \in \mathscr{B}_\alpha$. For all $a, b \in \mathbb{C}$, we have*

$$\|(D + L_{\mathscr{B}_\alpha} - a)f\|_{\mathscr{B}_\alpha} \|(D - L_{\mathscr{B}_\alpha} + ib)f\|_{\mathscr{B}_\alpha} \geqslant (\alpha+2)\|f\|^2_{\mathscr{B}_\alpha}.$$

5. The weighted Hardy space $\mathscr{H}_\beta$

We consider a sequence $\beta = \{\beta_n\}$, with $\beta_n > 0$, such that

$$\limsup_{n\to\infty}(\beta_n)^{-1/n} = 1.$$

The weighted Hardy space $\mathscr{H}_\beta$ is the set of all analytic functions f in the disk $\mathbb{D}$, with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, such that

$$\|f\|^2_{\mathscr{H}_\beta} := \sum_{n=0}^{\infty} \beta_n |a_n|^2 < \infty.$$

It is a Hilbert space when equipped with the inner product

$$\langle f, g\rangle_{\mathscr{H}_\beta} = \sum_{n=0}^{\infty} \beta_n a_n \overline{b_n},$$

where $f,g \in \mathscr{H}_\beta$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$. The set $\left\{\frac{z^n}{\sqrt{\beta_n}}\right\}_{n=0}^{\infty}$ forms a Hilbert's basis for the space $\mathscr{H}_\beta$. The function $K_{\mathscr{H}_\beta,z}$, $z \in \mathbb{D}$, given by

$$K_{\mathscr{H}_\beta,z}(w) := \sum_{n=0}^{\infty} \frac{(\overline{z}w)^n}{\beta_n}, \quad w \in \mathbb{D},$$

is a reproducing kernel for the weighted Hardy space $\mathscr{H}_\beta$.

If $\beta_n = 1$ the corresponding weighted Hardy space is the Hardy space $\mathscr{H}$.

If $\beta_0 = 1$ and $\beta_n = n$, $n \geqslant 1$, the corresponding weighted Hardy space is the Dirichlet space $\mathscr{D}$.

And, if $\beta_n = \frac{1}{n+1}$ the corresponding weighted Hardy space is the Bergman space $\mathscr{B}$.

For $f \in \mathscr{H}_\beta$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we define the operators D and $L_{\mathscr{H}_\beta}$ on $\mathscr{H}_\beta$ by

$$Df(z) := f'(z) = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n, \quad L_{\mathscr{H}_\beta} f(z) := \sum_{n=1}^{\infty} \frac{n\beta_{n-1}}{\beta_n} a_{n-1} z^n. \tag{5.1}$$

The function $u(z) = K_{\mathscr{H}_\beta,\overline{z}}(w)$ is the unique analytic solution on $\mathbb{D}$ of the initial problem

$$zDu(z) = wL_{\mathscr{H}_\beta} u(z), \quad w \in \mathbb{D}, \quad u(0) = 1/\beta_0.$$

The operators D and $L_{\mathscr{H}_\beta}$ satisfy the commutation rule

$$[D, L_{\mathscr{H}_\beta}] = \frac{\beta_0}{\beta_1} I + 2E_\beta, \tag{5.2}$$

where E_β is the operator given by

$$E_\beta f(z) = \frac{1}{2} \sum_{n=1}^{\infty} \left[(n+1)^2 \frac{\beta_n}{\beta_{n+1}} - n^2 \frac{\beta_{n-1}}{\beta_n} - \frac{\beta_0}{\beta_1}\right] a_n z^n.$$

If $\beta_n = 1$, then $L_{\mathscr{H}_\beta} f(z) = z^2 f'(z) + zf(z)$ and $E_\beta f(z) = zf'(z)$.
If $\beta_n = n$, then $L_{\mathscr{H}_\beta} f(z) = z^2 f'(z)$ and $E_\beta f(z) = zf'(z)$.
And, if $\beta_n = \frac{1}{n+1}$, then $L_{\mathscr{H}_\beta} f(z) = z^2 f'(z) + 2zf(z)$ and $E_\beta f(z) = zf'(z)$.

We define the Hilbert space $\mathscr{U}_\beta^{(1)}$ as the space of all $f \in \mathscr{H}_\beta$ with $f(z) = \sum_{n=0}^\infty a_n z^n$ such that

$$\|f\|^2_{\mathscr{U}_\beta^{(1)}} := \frac{(\beta_0)^2}{\beta_1}|a_0|^2 + \sum_{n=1}^\infty n^2 \beta_{n-1} |a_n|^2 < \infty.$$

We define the Hilbert space $\mathscr{U}_\beta^{(2)}$ as the space of all $f \in \mathscr{H}_\beta$ with $f(z) = \sum_{n=0}^\infty a_n z^n$ such that

$$\|f\|^2_{\mathscr{U}_\beta^{(2)}} := \frac{(\beta_0)^2}{\beta_1}|a_0|^2 + \sum_{n=1}^\infty n^2 \frac{(\beta_n)^2}{\beta_{n+1}} |a_n|^2 < \infty.$$

If $\beta_n = 1$, then $\mathscr{U}_\beta^{(1)} = \mathscr{U}_\beta^{(2)} = \mathscr{U}$ ($\mathscr{U}$ the Hilbert space defined in Section 2).

In the following we suppose that the sequence $\{\beta_n\}$ satisfies the condition

$$(n+1)^2 \frac{\beta_n}{\beta_{n+1}} - n^2 \frac{\beta_{n-1}}{\beta_n} \geqslant \frac{\beta_0}{\beta_1}. \tag{5.3}$$

The condition (5.3) is verified in the following cases:
– when $\beta_n = 1$, the case where $\mathscr{H}_\beta$ is the Hardy space $\mathscr{H}$,
– when $\beta_0 = 1$ and $\beta_n = n$, $n \geqslant 1$, the case where $\mathscr{H}_\beta$ is the Dirichlet space $\mathscr{D}$,
– and, when $\beta_n = \frac{1}{n+1}$, the case where $\mathscr{H}_\beta$ is the Bergman space $\mathscr{B}$.

By condition (5.3) we obtain the inequality

$$\|f\|_{\mathscr{U}_\beta^{(1)}} \leqslant 2\|f\|_{\mathscr{U}_\beta^{(2)}}.$$

Therefore, we have the continuous inclusion $\mathscr{U}_\beta^{(2)} \subset \mathscr{U}_\beta^{(1)}$.

In this section we establish an uncertainty inequality of Heisenberg-type on the space $\mathscr{H}_\beta$. We will use the following two lemmas.

LEMMA 5.1. *The operators D and $L_{\mathscr{H}_\beta}$ satisfy the following properties.*
(i) $Dom(D) = \mathscr{U}_\beta^{(1)}$ and $Dom(L_{\mathscr{H}_\beta}) = \mathscr{U}_\beta^{(2)}$.
(ii) For $f \in \mathscr{U}_\beta^{(1)}$ and $g \in \mathscr{U}_\beta^{(2)}$, we have $\langle Df, g\rangle_{\mathscr{H}_\beta} = \langle f, L_{\mathscr{H}_\beta} g\rangle_{\mathscr{H}_\beta}$.

Proof. Let $f \in \mathscr{H}_\beta$ with $f(z) = \sum_{n=0}^\infty a_n z^n$. From (5.1) we have

$$\|Df\|^2_{\mathscr{H}_\beta} = \sum_{n=0}^\infty (n+1)^2 \beta_n |a_{n+1}|^2, \quad \|L_{\mathscr{H}_\beta} f\|^2_{\mathscr{H}_\beta} = \sum_{n=0}^\infty (n+1)^2 \frac{(\beta_n)^2}{\beta_{n+1}} |a_n|^2.$$

Therefore

$$\|Df\|^2_{\mathscr{H}_\beta} = \|f\|^2_{\mathscr{U}_\beta^{(1)}} - \frac{(\beta_0)^2}{\beta_1}|f(0)|^2, \quad \|f\|^2_{\mathscr{U}_\beta^{(2)}} \leqslant \|L_{\mathscr{H}_\beta} f\|^2_{\mathscr{H}_\beta} \leqslant 4\|f\|^2_{\mathscr{U}_\beta^{(2)}}.$$

Consequently $\mathrm{Dom}(D) = \mathscr{U}_\beta^{(1)}$ and $\mathrm{Dom}(L_{\mathscr{H}_\beta}) = \mathscr{U}_\beta^{(2)}$.

On the other hand for $f \in \mathscr{U}_\beta^{(1)}$ and $g \in \mathscr{U}_\beta^{(2)}$ with $f(z) = \sum_{n=0}^\infty a_n z^n$ and $g(z) = \sum_{n=0}^\infty b_n z^n$, we have

$$\langle Df, g\rangle_{\mathscr{H}_\beta} = \sum_{n=0}^\infty (n+1)\beta_n a_{n+1}\overline{b_n} = \sum_{n=1}^\infty n\beta_{n-1} a_n \overline{b_{n-1}} = \langle f, L_{\mathscr{H}_\beta} g\rangle_{\mathscr{H}_\beta}.$$

The lemma is proved. □

We define the Hilbert space $\mathscr{S}_\beta^{(1)}$ as the space of all $f \in \mathscr{H}_\beta$ with $f(z) = \sum_{n=0}^\infty a_n z^n$ such that

$$\|f\|^2_{\mathscr{S}_\beta^{(1)}} := \frac{(\beta_0)^3}{(\beta_1)^2}|a_0|^2 + \sum_{n=1}^\infty n^4 \frac{(\beta_{n-1})^2}{\beta_n}|a_n|^2 < \infty.$$

We define the Hilbert space $\mathscr{S}_\beta^{(2)}$ as the space of all $f \in \mathscr{H}_\beta$ with $f(z) = \sum_{n=0}^\infty a_n z^n$ such that

$$\|f\|^2_{\mathscr{S}_\beta^{(2)}} := \frac{(\beta_0)^3}{(\beta_1)^2}|a_0|^2 + \sum_{n=1}^\infty n^4 \frac{(\beta_n)^3}{(\beta_{n+1})^2}|a_n|^2 < \infty.$$

By condition (5.3) we obtain the inequalities

$$\|f\|_{\mathscr{S}_\beta^{(1)}} \leqslant 4\|f\|_{\mathscr{S}_\beta^{(2)}}, \quad \|f\|_{\mathscr{U}_\beta^{(2)}} \leqslant \sqrt{\frac{\beta_0}{\beta_1}}\|f\|_{\mathscr{S}_\beta^{(2)}}.$$

Therefore, we have the continuous inclusions $\mathscr{S}_\beta^{(2)} \subset \mathscr{S}_\beta^{(1)}$ and $\mathscr{S}_\beta^{(2)} \subset \mathscr{U}_\beta^{(2)}$.

LEMMA 5.2. *We have* $\mathrm{Dom}(L_{\mathscr{H}_\beta} D) = \mathscr{S}_\beta^{(1)}$ *and* $\mathrm{Dom}(DL_{\mathscr{H}_\beta}) = \mathscr{S}_\beta^{(2)}$.

Proof. From (5.1) we have

$$L_{\mathscr{H}_\beta} Df(z) = \sum_{n=1}^\infty n^2 \frac{\beta_{n-1}}{\beta_n} a_n z^n, \quad DL_{\mathscr{H}_\beta} f(z) = \sum_{n=0}^\infty (n+1)^2 \frac{\beta_n}{\beta_{n+1}} a_n z^n.$$

Thus

$$\|L_{\mathscr{H}_\beta} Df\|^2_{\mathscr{H}_\beta} = \sum_{n=1}^\infty n^4 \frac{(\beta_{n-1})^2}{\beta_n}|a_n|^2, \quad \|DL_{\mathscr{H}_\beta} f\|^2_{\mathscr{H}_\beta} = \sum_{n=0}^\infty (n+1)^4 \frac{(\beta_n)^3}{(\beta_{n+1})^2}|a_n|^2.$$

Therefore

$$\|L_{\mathscr{H}_\beta} Df\|^2_{\mathscr{H}_\beta} = \|f\|^2_{\mathscr{S}_\beta^{(1)}} - \frac{(\beta_0)^3}{(\beta_1)^2}|f(0)|^2, \quad \|f\|^2_{\mathscr{S}_\beta^{(2)}} \leqslant \|DL_{\mathscr{H}_\beta} f\|^2_{\mathscr{H}_\beta} \leqslant 16\|f\|^2_{\mathscr{S}_\beta^{(2)}}.$$

Consequently $\mathrm{Dom}(L_{\mathscr{H}_\beta} D) = \mathscr{S}_\beta^{(1)}$, $\mathrm{Dom}(DL_{\mathscr{H}_\beta}) = \mathscr{S}_\beta^{(2)}$. □

THEOREM 5.3. *Let $f \in \mathscr{H}_\beta$. For all $a,b \in \mathbb{C}$, we have*

$$\|(D+L_{\mathscr{H}_\beta}-a)f\|_{\mathscr{H}_\beta}\|(D-L_{\mathscr{H}_\beta}+ib)f\|_{\mathscr{H}_\beta} \geqslant \frac{\beta_0}{\beta_1}\|f\|^2_{\mathscr{H}_\beta}. \tag{5.4}$$

Proof. Let $f \in \mathscr{H}_\beta$. First the inequality (5.4) is true for $f \notin \mathscr{S}^{(2)}_\beta$. Now, let A and B be the operators defined for $f \in \mathscr{S}^{(2)}_\beta$ by

$$A := (D+L_{\mathscr{H}_\beta})f, \quad B := i(D-L_{\mathscr{H}_\beta})f.$$

By (5.2), Lemma 5.1 and Lemma 5.2, the operators A and B possess the following properties.

(i) $A^* = A$ and $B^* = B$,

(ii) $[A,B] = -2i[D,L_{\mathscr{H}_\beta}] = -2i(\frac{\beta_0}{\beta_1}I + 2E_\beta)$,

(iii) $\mathrm{Dom}([A,B]) = \mathscr{S}^{(2)}_\beta$.

Thus, the inequality (5.4) follows from Lemma 2.2 and the fact that

$$\langle E_\beta f, f\rangle_{\mathscr{H}_\beta} \geqslant 0.$$

This completes the proof of the theorem. □

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