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MAXIMAL NUMERICAL RANGE OF THE BIMULTIPLICATION $M_{2,A,B}$

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Dedicated to Professor M. Barraa

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Abstract. Let $\mathscr{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators acting on a complex Hilbert space $\mathscr{H}$. For $A,B \in \mathscr{B}(\mathscr{H})$, define the bimultiplication operator $M_{2,A,B}$ on the class of Hilbert-Schmidt operators by $M_{2,A,B}(X) = AXB$. It is known [5] that if either A or B is hyponormal, then

$$\overline{W(M_{2,A,B})} = \overline{co\big(W(A)W(B)\big)},$$

where the bar and co stand for the closure and the convex hull, respectively and $W(\cdot)$ denotes the numerical range. In this paper, we give some conditions satisfied by A and B to have the following equality

$$W_0(M_{2,A,B}) = co\big(W_0(A)W_0(B)\big),$$

where $W_0(\cdot)$ denotes the maximal numerical range.

1. Introduction

Let $\mathscr{H}$ be a Hilbert space over the complex field $\mathbb{C}$ with inner product $\langle\cdot,\cdot\rangle$ and its induced norm $\|x\| = \langle x,x\rangle^{1/2}$. Denote by $\mathscr{B}(\mathscr{H})$ the C^*-algebra of all bounded linear operators acting on $\mathscr{H}$. For $A \in \mathscr{B}(\mathscr{H})$, the numerical range of A is defined as the set

$$W(A) = \{\langle Ax,x\rangle : x \in \mathscr{H},\ \|x\| = 1\}.$$

It is a celebrated result due to Toeplitz-Hausdorff that $W(A)$ is a convex subset in the complex plane and it is known that $co(\sigma(A)) \subseteq \overline{W(A)}$, where $\sigma(A)$, co, and bar stand for the spectrum of A, the convex hull and the closure, respectively. The numerical range of an operator in $\mathscr{B}(\mathscr{H})$ is closed if $\dim(\mathscr{H}) < \infty$, but it is not always closed when $\dim(\mathscr{H}) = \infty$. Let $w(A)$ denote the numerical radius of $A \in \mathscr{B}(\mathscr{H})$, i.e., $w(A) = \sup\{|\lambda| : \lambda \in W(A)\}$. It is well-known that $w(\cdot)$ defines a norm on $\mathscr{B}(\mathscr{H})$, which is equivalent to the operator norm, denoted $\|\cdot\|$. In fact, the following inequalities are well-known

$$\frac{1}{2}\|A\| \leqslant w(A) \leqslant \|A\|.$$

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For more details about the theory of numerical ranges, the reader is referred to [2, 3, 10, 11] and references therein.

A compact operator $A \in \mathscr{B}(\mathscr{H})$ is said to be a Hilbert-Schmidt operator if $\operatorname{tr}(AA^*) < \infty$, where tr and A^* stand for the usual trace functional and the adjoint operator of A, respectively. Let $\mathscr{C}_2(\mathscr{H})$ denote the class of Hilbert-Schmidt operators on $\mathscr{H}$. Recall that $\mathscr{C}_2(\mathscr{H})$ is a complex Hilbert space with the inner product $\langle A,B\rangle_2 = \operatorname{tr}(AB^*)$ and norm $\|A\|_2^2 = \operatorname{tr}(AA^*)$. For $A \in \mathscr{B}(\mathscr{H})$, the left and right multiplications $L_{2,A}$ and $R_{2,A}$ are defined on $\mathscr{C}_2(\mathscr{H})$ by $L_{2,A}(X) = AX$ and $R_{2,A}(X) = XA$, respectively. For $A,B \in \mathscr{B}(\mathscr{H})$, the bimultiplication $M_{2,A,B}$ is defined on $\mathscr{C}_2(\mathscr{H})$ by $M_{2,A,B}(X) = (L_{2,A}R_{2,B})X = AXB$. The operators $L_{2,A}$ and $R_{2,A}$ are then particular bimultiplications since $L_{2,A} = M_{2,A,I}$ and $R_{2,A} = M_{2,I,A}$, where I is the identity operator on $\mathscr{H}$. Some results concerning the norm, spectrum and numerical range of $M_{2,A,B}$ can be found in [4, 5, 7, 13, 14]. It is proved in [7] that

$$\|M_{2,A,B}\| = \|A\|\|B\|. \tag{1.1}$$

In particular, $\|L_{2,A}\| = \|R_{2,A}\| = \|A\|$. In [14], it is proved that

$$\sigma(M_{2,A,B}) = \sigma(A)\sigma(B). \tag{1.2}$$

In [4], it is proved that if A is a nonnegative operator and $AB = BA$, then $W(AB) \subseteq W(A)W(B)$. In this case, as a consequence of the previous result, since $L_{2,A}$ is nonnegative, $L_{2,A}R_{2,B} = R_{2,B}L_{2,A}$, $W(L_{2,A}) = W(A)$ and $W(R_{2,B}) = W(B)$, we have

$$W(M_{2,A,B}) \subseteq W(A)W(B). \tag{1.3}$$

But in the general case the inclusion (1.3) does not hold. However, we have

$$W(M_{2,A,B}) \subseteq co\big(W(A)W(B)\big) + \overline{D\big(0,d(A)\big)D\big(0,d(B)\big)}, \tag{1.4}$$

see, [13]. In particular

$$w(M_{2,A,B}) \leqslant w(A)w(B) + d(A)d(B).$$

Here $D(0,\alpha)$ is the disk centred at the origin and of radius $\alpha \geqslant 0$ and

$$d(T) = \inf_{\lambda \in \mathbb{C}} \|T - \lambda\|$$

for any $T \in \mathscr{B}(\mathscr{H})$. Recently, in [5] the authors proved that if either A or B is hyponormal, then

$$\overline{W(M_{2,A,B})} = \overline{co\big(W(A)W(B)\big)}. \tag{1.5}$$

Recall that an operator $A \in \mathscr{B}(\mathscr{H})$ is said to be hyponormal if $A^*A - AA^* \geqslant 0$ (i.e., $\|Ax\| \geqslant \|A^*x\|$ for all $x \in \mathscr{H}$). Familiar examples of hyponormal operators are normal operators, those A for which $A^*A = AA^*$.

The notion of the numerical range has been generalized in different directions. One such a direction is the maximal numerical range. It is a relatively new concept in operator theory, having been introduced only in 1970 by Stampfli [16] and defined as follows.

DEFINITION 1.1. For $A \in \mathscr{B}(\mathscr{H})$, the maximal numerical range $W_0(A)$ of A is given by

$$W_0(A) = \{\lim_n \langle Ax_n, x_n\rangle : x_n \in \mathscr{H},\ \|x_n\| = 1,\ \lim_n \|Ax_n\| = \|A\|\}.$$

It was shown in [16] that $W_0(A)$ is nonempty, closed, convex and contained in the closure of the numerical range; $W_0(A) \subseteq \overline{W(A)}$. In the case of finite-dimensional spaces, the maximal numerical range is produced by maximal vectors for A (vectors $x \in \mathscr{H}$ such that $\|x\| = 1$ and $\|Ax\| = \|A\|$). Note that the notion of the maximal numerical range was introduced in [16] for the purpose of calculating the norm of the inner derivation on $\mathscr{B}(\mathscr{H})$. Recall that the inner derivation δ_A associated with $A \in \mathscr{B}(\mathscr{H})$ is defined by

$$\delta_A : \mathscr{B}(\mathscr{H}) \longrightarrow \mathscr{B}(\mathscr{H}),\ X \longmapsto AX - XA.$$

Indeed, the author of [16] established the following. For any $A \in \mathscr{B}(\mathscr{H})$

$$\|\delta_A\| = 2d(A).$$

Recently, considerable interests have been given to the maximal numerical range, see, for instance, [1, 12, 15]. The following is proved in [12].

PROPOSITION 1.2. ([12]) *Let $A \in \mathscr{B}(\mathscr{H})$. Then*

$$W_0(A^*) = W_0(A)^*,$$

where $L^* := \{\overline{z} : z \in L\}$ for any subset $L \subset \mathbb{C}$.

In [15], the author gives a description of the maximal numerical range of a normal operator and in [1] the result is generalized to a hyponormal one as follows.

THEOREM 1.3. ([1]) *Let $A \in \mathscr{B}(\mathscr{H})$ be a hyponormal operator. Then*

$$W_0(A) = co\big(\sigma_n(A)\big),$$

where $\sigma_n(A) := \{\lambda \in \sigma(A) : |\lambda| = \|A\|\}$.

In this paper, we are interested in the equality (1.5) when replacing the numerical range by the maximal numerical range. In Section 2, we begin by showing that for any $A, B \in \mathscr{B}(\mathscr{H})$

$$co\big(W_0(A)W_0(B)\big) \subseteq W_0(M_{2,A,B}).$$

An inclusion analogous to (1.4) is also given for the maximal numerical range. Next, we give some conditions satisfied by the operators $A, B \in \mathscr{B}(\mathscr{H})$ to have

$$W_0(M_{2,A,B}) = co\big(W_0(A)W_0(B)\big). \tag{1.6}$$

Indeed, we show that if A has a normal dilation N on some complex Hilbert space $\mathscr{K}$ with $\sigma(N) \subseteq \sigma(A)$ and B is hyponormal, then the equality (1.6) holds. Recall that if S and T are bounded linear operators on complex Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, respectively, the operator T is said to be a dilation of the operator S (or S is dilated to T) if there is an isometry V from $\mathscr{H}$ to $\mathscr{K}$ such that $S = V^*TV$. Using the fact that any hyponormal operator $A \in \mathscr{B}(\mathscr{H})$ has a normal dilation N on some complex Hilbert space $\mathscr{K}$ with $\sigma(N) \subseteq \sigma(A)$ (see, [9]), we deduce that the equality (1.6) remains true if either A or A^* is hyponormal and either B or B^* is hyponormal.

2. Main results

Before stating our results, for the sake of completeness and for the convenience of the reader, we shall show here that $M_{2,A,B}^* = M_{2,A^*,B^*}$ for any operators $A, B \in \mathscr{B}(\mathscr{H})$. Let $X, Y \in \mathscr{C}_2(\mathscr{H})$, we have

$$\langle M_{2,A,B}X, Y\rangle_2 = \langle AXB, Y\rangle_2 = \operatorname{tr}(AXBY^*).$$

Since X, $BY^* \in \mathscr{C}_2(\mathscr{H})$, by [6, Proposition 18.8], the operator XBY^* is trace class. So, by [6, Theorem 18.11], $\operatorname{tr}(AXBY^*) = \operatorname{tr}(XBY^*A)$. Therefore

$$\begin{aligned}\langle M_{2,A,B}X, Y\rangle_2 &= \operatorname{tr}(XBY^*A)\\ &= \operatorname{tr}(X(A^*YB^*)^*)\\ &= \langle X, A^*YB^*\rangle_2\\ &= \langle X, M_{2,A^*,B^*}Y\rangle_2.\end{aligned}$$

We start with the following.

THEOREM 2.1. *Let $A, B \in \mathscr{B}(\mathscr{H})$, then*

$$co\big(W_0(A)W_0(B)\big) \subseteq W_0\left(M_{2,A,B}\right).$$

Proof. Let $\lambda \in W_0(A)$, then there exists a sequence of unit vectors $x_n \in \mathscr{H}$ such that

$$\lambda = \lim_n \langle Ax_n, x_n\rangle \quad \text{and} \quad \lim_n \|Ax_n\| = \|A\|.$$

Let $\mu \in W_0(B)$, then Proposition 1.2 implies that $\overline{\mu} \in W_0(B^*)$. Therefore, there exists a sequence of unit vectors $y_n \in \mathscr{H}$ such that

$$\overline{\mu} = \lim_n \langle B^*y_n, y_n\rangle \quad \text{and} \quad \lim_n \|B^*y_n\| = \|B^*\| = \|B\|.$$

Recall that for all n, $x_n \otimes y_n \in \mathscr{C}_2(\mathscr{H})$ and $\|x_n \otimes y_n\|_2 = 1$. We have

$$\lim_n \left\langle M_{2,A,B}\left(x_n \otimes y_n\right), x_n \otimes y_n\right\rangle_2 = \lim_n \left\langle Ax_n, x_n\right\rangle \left\langle By_n, y_n\right\rangle = \lambda\mu.$$

On the other hand,

$$\begin{aligned}\lim_n \|M_{2,A,B}x_n \otimes y_n\|_2^2 &= \lim_n \langle M_{2,A,B}(x_n \otimes y_n), M_{2,A,B}(x_n \otimes y_n)\rangle_2 \\ &= \lim_n \left\langle M_{2,A,B}^* M_{2,A,B}(x_n \otimes y_n), x_n \otimes y_n\right\rangle_2 \\ &= \lim_n \langle M_{2,A^*,B^*} M_{2,A,B}(x_n \otimes y_n), x_n \otimes y_n\rangle_2 \\ &= \lim_n \langle M_{2,A^*A,BB^*}(x_n \otimes y_n), x_n \otimes y_n\rangle_2 \\ &= \lim_n \langle A^*Ax_n, x_n\rangle \langle BB^*y_n, y_n\rangle \\ &= \lim_n \|Ax_n\|^2 \|B^*y_n\|^2 \\ &= \|A\|^2\|B\|^2 \\ &= \|M_{2,A,B}\|^2 \text{ (by the equality (1.1)).}\end{aligned}$$

Thus $\lambda\mu \in W_0(M_{2,A,B})$ and so $co(W_0(A)W_0(B)) \subseteq W_0(M_{2,A,B})$. $\square$

Let $A \in \mathscr{B}(\mathscr{H})$. A linear functional f on $\mathscr{B}(\mathscr{H})$ is said to be maximal for A if $f(I) = \|f\| = 1$ and $f(A^*A) = \|A\|^2$. Let $\mathscr{S}_{max}(A)$ denote the set of all maximal linear functionals for A. The following result, which is from [8], asserts that if $A \in \mathscr{B}(\mathscr{H})$, then

$$W_0(A) = \{f(A) : f \in \mathscr{S}_{max}(A)\}.$$

Using Theorem 2.1 and the preceding result, we have the following.

COROLLARY 2.2. *For any $A \in \mathscr{B}(\mathscr{H})$, $W_0(L_{2,A}) = W_0(R_{2,A}) = W_0(A)$.*

Proof. Theorem 2.1 implies that $W_0(A) \subseteq W_0(M_{2,A,I}) = W_0(L_{2,A})$. Now, we show that $W_0(L_{2,A}) \subseteq W_0(A)$. Therefore, let $\lambda \in W_0(L_{2,A})$, then there is $f \in \mathscr{S}_{max}(L_{2,A})$ such that $\lambda = f(L_{2,A})$. Define the map h on $\mathscr{B}(\mathscr{H})$ by $h(T) = f(L_{2,T})$. We claim that $h \in \mathscr{S}_{max}(A)$. Everything but $h(A^*A) = \|A\|^2$ is obvious. So, $h(A^*A) = f(L_{2,A^*A}) = f(L_{2,A^*}L_{2,A}) = f(L_{2,A}^*L_{2,A}) = \|L_{2,A}\|^2 = \|A\|^2$. Since $\lambda = f(L_{2,A}) = h(A)$, it follows that $\lambda \in W_0(A)$ and hence $W_0(L_{2,A}) \subseteq W_0(A)$.

Similarly, we only have to show that $W_0(R_{2,A}) \subseteq W_0(A)$. For this, let $\mu \in W_0(R_{2,A})$. Then, by Proposition 1.2, $\overline{\mu} \in W_0(R_{2,A})^* = W_0(R_{2,A^*})$. So, there is $g \in \mathscr{S}_{max}(R_{2,A^*})$ such that $\overline{\mu} = g(R_{2,A^*})$. Define the map k on $\mathscr{B}(\mathscr{H})$ by $k(T) = g(R_{2,T})$. By a similar argument as in the first part, we show that $k \in \mathscr{S}_{max}(A)$ and $k(A) = \mu$. Therefore, $\mu \in W_0(A)$ and hence $W_0(R_{2,A}) \subseteq W_0(A)$ as desired. $\square$

PROPOSITION 2.3. *Let $A,B \in \mathscr{B}(\mathscr{H})$ with $\|AB\| = \|A\|\|B\|$ and $AB = BA$. Then*

$$W_0(AB) \subseteq W_0(A)W_0(B) + \overline{D(0,d(A))D(0,d(B))}.$$

Proof. Let $\lambda \in W_0(AB)$, then $\lambda = \lim_n \langle ABx_n, x_n\rangle$ and $\lim_n \|ABx_n\| = \|AB\|$ for some sequence of unit vectors $x_n \in \mathscr{H}$. Let $y_n \in \mathscr{H}$ be unit vectors with $x_n \perp y_n$ for all n and such that

$$Bx_n = \langle Bx_n, x_n\rangle x_n + \langle Bx_n, y_n\rangle y_n.$$

Then

$$\lambda = \lim_n \langle Ax_n, x_n\rangle \langle Bx_n, x_n\rangle + \lim_n \langle Ay_n, x_n\rangle \langle Bx_n, y_n\rangle .$$

We have $\|A\|\|B\| = \lim_n \|ABx_n\| \leqslant \|A\| \lim_n \|Bx_n\| \leqslant \|A\|\|B\|$. This implies that $\lim_n \|Bx_n\| = \|B\|$ and so $\lim_n \langle Bx_n, x_n\rangle \in W_0(B)$. Similarly, $\lim_n \langle Ax_n, x_n\rangle \in W_0(A)$. According to [13, Lemma 7], $\lim_n \langle Ay_n, x_n\rangle \in \overline{D(0,d(A))}$ and $\lim_n \langle Bx_n, y_n\rangle \in \overline{D(0,d(B))}$. The desired result follows. $\square$

REMARK 2.4. *In the previous proposition, the condition $\|AB\| = \|A\|\|B\|$ is necessary as is shown in the following example. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. We have $W_0(AB) = \{0\}$, $W_0(A) = W_0(B) = \{1\}$ and $d(A) = d(B) = \frac{1}{2}$. Then $0 \notin W_0(A)W_0(B) + \overline{D\big(0,d(A)\big)D\big(0,d(B)\big)} = D\left(1, \frac{1}{4}\right)$.*

COROLLARY 2.5. *Let $A, B \in \mathscr{B}(\mathscr{H})$. Then*

$$W_0\left(M_{2,A,B}\right) \subseteq W_0(A)W_0(B) + \overline{D\big(0,d(A)\big)D\big(0,d(B)\big)}.$$

Proof. First, note that $d\left(L_{2,A}\right) = \inf_{\lambda\in\mathbb{C}} \|L_{2,A} - \lambda\| = \inf_{\lambda\in\mathbb{C}} \left\|L_{2,A-\lambda}\right\| = \inf_{\lambda\in\mathbb{C}} \|A - \lambda\| = d(A)$ and similarly $d\left(R_{2,A}\right) = d(A)$. Since $L_{2,A}R_{2,B} = R_{2,B}L_{2,A}$ and $\|L_{2,A}R_{2,B}\| = \|L_{2,A}\|\,\|R_{2,B}\|$, the result follows from the previous proposition. $\square$

Note that from Corollary 2.2, we have for any $A, B \in \mathscr{B}(\mathscr{H})$

$$W_0(M_{2,A,I}) = W_0(A)W_0(I)$$

and

$$W_0(M_{2,I,B}) = W_0(I)W_0(B).$$

Then the following question arises. When is the equality (1.6) true? In the following, we give some conditions satisfied by the operators $A, B \in \mathscr{B}(\mathscr{H})$ to answer this question.

THEOREM 2.6. *Let $A, B \in \mathscr{B}(\mathscr{H})$ with dim $\mathscr{H} = m$ and A is normal. Then*

$$W_0\left(M_{2,A,B}\right) = co\big(W_0(A)W_0(B)\big).$$

Proof. Let $\lambda \in W_0\left(M_{2,A,B}\right)$. Then, as in the proof of Theorem 2.1, there is $X \in \mathscr{C}_2(\mathscr{H})$ $(= \mathscr{B}(\mathscr{H}))$ with $\|X\|_2 = 1$, $\lambda = \langle M_{2,A,B}X, X\rangle_2$ and $\|M^*_{2,A,B}X\|_2 = \|A\|\|B\|$. We know that $\left\{e_i \otimes e_j : i,j = 1,\dots,m\right\}$ is a basis of $\mathscr{C}_2(\mathscr{H})$ where $\{e_i : i = 1,\dots,m\}$ is an orthonormal basis of $\mathscr{H}$ such that $\left\langle Ae_i, e_j\right\rangle = a_i\delta_{i,j}$, where a_i are the eigenvalues of A and the symbol $\delta_{i,j}$ stands for the Kronecker delta. Write $X = \sum_{i,j=1}^{m} b_{i,j} e_i \otimes e_j$ with $\sum_{i,j=1}^{m} |b_{i,j}|^2 = 1$ and set $\alpha_i := \left[\sum_{k=1}^{m} |b_{i,k}|^2\right]^{1/2}$. Define

$$y_i := \begin{cases} \sum_{j=1}^{m} \frac{b_{ij}}{\alpha_i} e_j, \text{ if } \alpha_i \neq 0, \\ e_i, \text{ otherwise.} \end{cases}$$

Then $X = \sum_{i=1}^{m} \alpha_i e_i \otimes y_i$ with $\|y_i\| = 1$ and $\sum_{i=1}^{m} \alpha_i^2 = 1$. Therefore, we get

$$\lambda = \langle M_{2,A,B} X, X \rangle = \sum_{i=1}^{m} \alpha_i^2 \langle Ae_i, e_i \rangle \langle By_i, y_i \rangle$$

and

$$\begin{aligned} \|A\|^2 \|B\|^2 = \left\langle M_{2,A,B}^* X, M_{2,A,B}^* X \right\rangle_2 &= \langle M_{2,AA^*,B^*B} X, X \rangle_2 \\ &= \langle M_{2,A^*A,B^*B} X, X \rangle_2 \\ &= \sum_{i=1}^{m} \alpha_i^2 \langle A^*Ae_i, e_i \rangle \langle B^*By_i, y_i \rangle \\ &= \sum_{i=1}^{m} \alpha_i^2 \|Ae_i\|^2 \|By_i\|^2. \end{aligned}$$

From this, since $\sum_{i=1}^{m} \alpha_i^2 = 1$, we get $\|Ae_i\| = \|A\|$ and $\|By_i\| = \|B\|$ for all i such that $\alpha_i \neq 0$. Hence $\langle Ae_i, e_i \rangle \in W_0(A)$ and $\langle By_i, y_i \rangle \in W_0(B)$. It results that $\lambda \in co\big(W_0(A)W_0(B)\big)$ and so, $W_0(M_{2,A,B}) \subseteq co\big(W_0(A)W_0(B)\big)$. The other inclusion is given by Theorem 2.1. □

THEOREM 2.7. *Let $A, B \in \mathscr{B}(\mathscr{H})$ such that B is hyponormal. If A has a normal dilation N on some complex Hilbert space $\mathscr{K}$ with $\sigma(N) \subseteq \sigma(A)$, then*

$$W_0(M_{2,A,B}) = co\big(W_0(A)W_0(B)\big).$$

To prove this theorem, we need the following auxiliary lemmas.

LEMMA 2.8. *Let $A \in \mathscr{B}(\mathscr{H})$. Then*

$$co\big(\sigma_n(A)\big) \subseteq W_0(A). \tag{2.1}$$

Proof. Let $\lambda \in \sigma_n(A)$, then $\lambda \in \overline{W(A)}$. So, there is a sequence of unit vectors $x_n \in \mathscr{H}$ such that $\lambda = \lim_n \langle Ax_n, x_n \rangle$. But, $\|A\| = |\lambda| = \left|\lim_n \langle Ax_n, x_n \rangle\right| \leqslant \lim_n \|Ax_n\| \leqslant \|A\|$, then $\lim_n \|Ax_n\| = \|A\|$. Consequently, $\lambda \in W_0(A)$ and so $\sigma_n(A) \subseteq W_0(A)$. The desired result follows by the convexity of $W_0(A)$. □

LEMMA 2.9. *Let $A \in \mathscr{B}(\mathscr{H})$. If there exists a hyponormal operator H on some complex Hilbert space $\mathscr{K}$ and an isometry V from $\mathscr{H}$ to $\mathscr{K}$ such that $A = V^*HV$ and $\sigma(H) \subseteq \sigma(A)$, then*

$$W_0(A) \subseteq W_0(H).$$

Proof. The proof is similar to the one of [1, Lemma 3.1]. □

Now, we are ready to prove the theorem.

Proof of Theorem 2.7. By hypothesis, there is an isometry V from $\mathscr{H}$ to $\mathscr{K}$ such that $A = V^*NV$. It is easy to see that $M^*_{2,A,B} = L^*_{2,V} M^*_{2,N,B} L_{2,V}$. We have

$$M_{2,N,B} M^*_{2,N,B} - M^*_{2,N,B} M_{2,N,B} = M_{2,N^*N,B^*B-BB^*}.$$

Note that N^*N and $B^*B - BB^*$ are positive, then by the equality (1.5),

$$\overline{W(M_{2,N^*N,B^*B-BB^*})} \subseteq \overline{co\big(W(N^*N)W(B^*B-BB^*)\big)}.$$

From this, we derive that M_{2,N^*N,B^*B-BB^*} is positive, that is, $M^*_{2,N,B}$ is hyponormal. Moreover, $L_{2,V}$ is an isometry and $\sigma(M_{2,N,B}) = \sigma(N)\sigma(B) \subseteq \sigma(A)\sigma(B) = \sigma(M_{2,A,B})$, that is, $\sigma(M^*_{2,N,B}) \subseteq \sigma(M^*_{2,A,B})$. Then, according to Lemma 2.9, we get $W_0(M^*_{2,A,B}) \subseteq W_0(M^*_{2,N,B})$. Therefore,

$$\begin{aligned}
W_0(M_{2,A,B})^* &= W_0(M^*_{2,A,B}) \\
&\subseteq W_0(M^*_{2,N,B}) \\
&= co\big(\sigma_n(M^*_{2,N,B})\big) \text{ (by Theorem 1.3)} \\
&= co\big(\sigma_n(M_{2,N^*,B^*})\big) \\
&= co\big(\sigma_n(N^*)\sigma_n(B^*)\big) \text{ (by the equality (1.2))} \\
&\subseteq co\big(\sigma_n(A^*)\sigma_n(B^*)\big) \text{ (since } \|A\| = \|N\| \text{ and } \sigma(N) \subseteq \sigma(A)) \\
&\subseteq co\big(W_0(A^*)W_0(B^*)\big) \text{ (by Lemma 2.8)} \\
&= \Big(co\big(W_0(A)W_0(B)\big)\Big)^* \text{ (by Proposition 1.2)}.
\end{aligned}$$

Note that in the last equality we use the fact that $co(L^*) = \big(co(L)\big)^*$ for any subset $L \subset \mathbb{C}$. We derive that $W_0(M_{2,A,B}) \subseteq co\big(W_0(A)W_0(B)\big)$ and we conclude by Theorem 2.1. □

Let $A \in \mathscr{B}(\mathscr{H})$ be a hyponormal operator. Then, according to [9], A has a normal dilation N on some complex Hilbert space $\mathscr{K}$ with $\sigma(N) \subseteq \sigma(A)$. Note that A^* has N^* as a dilation on $\mathscr{K}$ with $\sigma(N^*) \subseteq \sigma(A^*)$. From this and the previous theorem, we have the following.

COROLLARY 2.10. *Let $A, B \in \mathscr{B}(\mathscr{H})$. If either A or A^* is hyponormal and either B or B^* is hyponormal, then*

$$W_0(M_{2,A,B}) = co\big(W_0(A)W_0(B)\big).$$

Proof. Let us prove the result in the the case where A and B^* are hyponormal. So, according to [9], A has a normal dilation N on some complex Hilbert space $\mathscr{K}$ with $\sigma(N) \subseteq \sigma(A)$. Then N^* is a normal dilation of A^* on $\mathscr{K}$ with $\sigma(N^*) \subseteq \sigma(A^*)$. By Theorem 2.7, $W_0(M_{2,A^*,B^*}) = co\big(W_0(A^*)W_0(B^*)\big)$ and we conclude as above. For the other cases, we use the same argument. □

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