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REMARKS ON SCALABLE FRAMES

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Abstract. This paper investigates scalable frame in $\mathbb{R}^n$. We define the reduced diagram matrix of a frame and use it to classify scalability of the frame under some conditions. We give a new approach to the scaling problem by breaking the problem into two smaller ones, each of which is easier to solve, giving a simpler way to check scaling. Finally, we study the scalability of dual frames.

1. Introduction

A Parseval frame $\mathscr{X} = \{x_i\}_{i=1}^m$ for $\mathbb{R}^n$ is a set of vectors in $\mathbb{R}^n$ which has a property that every vector $x \in \mathbb{R}^n$ can be recovered via the painless reconstruction formula: $x = \sum_{i=1}^m \langle x, x_i \rangle x_i$ (see Section 2 for the definitions). This property shared with orthonormal bases, among others, makes them very desirable in applications. When a frame is not Parseval, the reconstruction formula depends on the inverse of the frame operator, which may be difficult or impossible to calculate. Thus, a key question in frame theory is the following: given a frame $\mathscr{X} = \{x_i\}_{i=1}^m$ for $\mathbb{R}^n$, can the frame vectors be modified so that the resulting system forms a Parseval frame? Since a frame is typically designed to accommodate certain requirements of an application, this modification process should be done in such a way as to not change the basic properties of the system. One way to do so is by scaling each frame vector in such a way to obtain a Parseval frame. Frame scaling is a noninvasive procedure, and properties such as erasure resilience or sparse expansions are left untouched by this modification. Unfortunately frame scaling is a very challenging problem in frame theory. Much work has been done on this problem [1, 2, 5, 6, 7, 8, 9, 10, 11].

In this paper we find new necessary and sufficient conditions which ensure the scalability of frames in $\mathbb{R}^n$. Specifically, in Section 3 we define the reduced diagram matrix of a given frame and use it to classify the scalability of the frame under some conditions. In Section 4, we break up the scaling problem into two smaller ones: normalized scaling and mixed scaling. This gives a simpler way for checking scalability. Finally, in Section 5, we present some results about the scalability of dual frames.

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2. Preliminaries

In this section, we recall some basic facts about finite frame theory. For more information on the subject, see the books [4, 12] and references therein.

DEFINITION 2.1. A sequence of vectors $\mathscr{X} = \{x_i\}_{i=1}^m$ in $\mathbb{R}^n$ is a *frame* for $\mathbb{R}^n$ if there are constants $0 < A \leqslant B < \infty$ such that

$$A\|x\|^2 \leqslant \sum_{i=1}^{m} |\langle x, x_i\rangle|^2 \leqslant B\|x\|^2, \text{ for all } x \in \mathbb{R}^n.$$

The constants A and B are called the *lower and upper frame bounds*, respectively. $\mathscr{X}$ is said to be a *tight frame* or an A*-tight frame* if $A = B$, and if $A = B = 1$, it is called a *Parseval frame*. If all the frame elements have the same norm, this is an *equal-norm frame* and if the frame elements have norm one, we call it a *unit-norm frame*. The numbers $\{\langle x, x_i\rangle\}_{i=1}^m$ are the frame coefficients of the vector $x \in \mathbb{R}^n$. It is well known that $\mathscr{X}$ is a frame for $\mathbb{R}^n$ if and only if it spans the space.

Given a frame $\mathscr{X} = \{x_i\}_{i=1}^m$ for $\mathbb{R}^n$, the corresponding *synthesis operator*, also denoted by $\mathscr{X}$, is the $n \times m$ matrix whose the i-th column is x_i. The adjoint matrix $\mathscr{X}^*$ is called the *analysis operator*, and the *frame operator* of $\mathscr{X}$ is then $S := \mathscr{X}\mathscr{X}^*$. It is known that S is a positive, self-adjoint, invertible operator and satisfies: $Sx = \sum_{i=1}^m \langle x, x_i\rangle x_i$ for all $x \in \mathbb{R}^n$. We recover vectors by the formula:

$$x = S^{-1}Sx = \sum_{i=1}^{m} \langle x, x_i\rangle S^{-1}x_i = \sum_{i=1}^{m} \langle x, S^{-1/2}x_i\rangle S^{-1/2}x_i.$$

It follows that $\{S^{-1/2}x_i\}_{i=1}^m$ is a Parseval frame for $\mathbb{R}^n$.

If the frame is A-tight, then its frame operator is a multiple of the identity operator. In this case, we have the following useful reconstruction formula:

$$x = \frac{1}{A}\sum_{i=1}^{m} \langle x, x_i\rangle x_i, \text{ for all } x \in \mathbb{R}^n.$$

The following characterization of tight frames is well known [4].

THEOREM 2.2. *A frame $\mathscr{X}$ for $\mathbb{R}^n$ is tight if and only if the row vectors of its synthesis matrix are orthogonal and have the same norm.*

Let us recall the definition of scalable frames from [10].

DEFINITION 2.3. A frame $\mathscr{X} = \{x_i\}_{i=1}^m$ for $\mathbb{R}^n$ is *scalable* if there exist non-negative constants $\{a_i\}_{i=1}^m$ for which $\{a_i x_i\}_{i=1}^m$ is a tight frame. If all the a_i's are positive, we say that the frame is *strictly scalable*.

Recently, a weakened version of frame scaling was introduced [3] called *piecewise scaling*.

Throughout, for any natural number m, we use the notation $[m]$ to denote the set $[m] := \{1, 2, \ldots, m\}$, and we write $x = (x(1), x(2), \ldots, x(n))$ to represent the coordinates of a vector $x \in \mathbb{R}^n$.

3. Reduced diagram matrices and scaling problem

Given a vector $x \in \mathbb{R}^n$, the *diagram vector* $\tilde{x}$ of x is defined in [8] as follows.

$$\tilde{x} := \frac{1}{\sqrt{n-1}} \begin{pmatrix} (x(1))^2 - (x(2))^2 \\ \vdots \\ (x(n-1))^2 - (x(n))^2 \\ \sqrt{2n}x(1)x(2) \\ \vdots \\ \sqrt{2n}x(n-1)x(n) \end{pmatrix} \in \mathbb{R}^{n(n-1)},$$

where the difference of squares $(x(i))^2 - (x(j))^2$ and the product $x(i)x(j)$ occur exactly once for $1 \leqslant i < j \leqslant n$.

The normalization of the components of the diagram vector is chosen to preserve unit vectors. The following result appeared in [8].

PROPOSITION 3.1. *For any $x, y \in \mathbb{R}^n$, we have that*

$$(n-1)\langle \tilde{x}, \tilde{y} \rangle = n|\langle x, y \rangle|^2 - \|x\|^2 \|y\|^2.$$

DEFINITION 3.2. Given a set $\mathscr{X} = \{x_i\}_{i=1}^m$ in $\mathbb{R}^n$, the *diagram matrix* $\theta_{\mathscr{X}}$ is the $n(n-1) \times m$ matrix whose the i-th column is the diagram vector of x_i.

In [8], the authors use the diagram vectors to classify scalable frame. In the following, we will show that this can be done by using the reduced diagram vectors. Before giving the definition, we need a lemma.

LEMMA 3.3. *The rank of the diagram matrix $\theta_{\mathscr{X}}$ of a set of vectors $\mathscr{X} = \{x_i\}_{i=1}^m \subset \mathbb{R}^n$ is $\leqslant \frac{(n-1)(n+2)}{2}$.*

Proof. The rows $n, n+1, \ldots, \frac{n(n-1)}{2}$ of the matrix $\theta_{\mathscr{X}}$ can be obtained from the first $n-1$ rows. Specifically, if $1 < i < j \leqslant n$, the rows whose elements are

$$\left((x_1(i))^2 - (x_1(j))^2, (x_2(i))^2 - (x_2(j))^2, \ldots, (x_m(i))^2 - (x_m(j))^2\right)$$

is the difference of the $(j-1)$-th and the $(i-1)$-th rows of $\theta_{\mathscr{X}}$. Thus, the rank of $\theta_{\mathscr{X}}$ is $\leqslant \frac{n(n-1)}{2} + (n-1) = \frac{(n-1)(n+2)}{2}$. □

DEFINITION 3.4. The *reduced diagram matrix* of a set of vectors $\mathscr{X} = \{x_i\}_{i=1}^m$ in $\mathbb{R}^n$ is the matrix $\tilde{\theta}_{\mathscr{X}}$ obtained by removing the rows $n, n+1, \ldots, \frac{n(n-1)}{2}$ from the diagram matrix $\theta_{\mathscr{X}}$. The reduced diagram vector of x_i, that we still denote with $\tilde{x}_i$, is the i-th column of the reduced diagram matrix $\tilde{\theta}_{\mathscr{X}}$.

By Lemma 3.3, the rank of $\tilde{\theta}_{\mathscr{X}}$ is $\leqslant \min\left\{\frac{(n-1)(n+2)}{2}, m\right\}$.

We are now ready to present the following important classification of scalable frames. The main idea of this theorem is not new (see [8, 11]), but we will restate it here with our notation.

THEOREM 3.5. *Let $\mathscr{X} = \{x_i\}_{i=1}^m$ be a unit-norm frame for $\mathbb{R}^n$ and $\tilde{\theta}_{\mathscr{X}}$ be the reduced diagram matrix of $\mathscr{X}$. Then the following are equivalent.*

1. *$\mathscr{X}$ is scalable.*
2. *There is a non-negative, non-zero vector $c = (c_1, c_2, \dots, c_m)$ such that c is orthogonal to every row of $\tilde{\theta}_{\mathscr{X}}$.*
3. *There is a non-negative, non-zero vector $c = (c_1, c_2, \dots, c_m)$ in the null space of the Gram matrix of $\tilde{\theta}_{\mathscr{X}}$.*

Proof. $(1) \Leftrightarrow (2)$: By definition, $\mathscr{X}$ is scalable if there exist non-negative scalars $a = \{a_i\}_{i=1}^m$ so that $a\mathscr{X} := \{a_i x_i\}_{i=1}^m$ is a tight frame. By Theorem 2.2, this is equivalent to the fact that the rows of the synthesis matrix $a\mathscr{X}$ are orthogonal and have the same norm. Let R_j denote the j-th row of the synthesis matrix $a\mathscr{X}$ and let $\tilde{R}_j$ denote the j-th row of the reduced diagram matrix of $a\mathscr{X}$. The rows R_j and R_1 have the same norm if and only if

$$\sum_{i=1}^m [a_i x_i(j)]^2 = \sum_{i=1}^m [a_i x_i(1)]^2, \quad j = 2, \dots, n.$$

But this is equivalent to $\langle c, \tilde{R}_j \rangle = 0$ for all $j \in [n-1]$, where $c = (a_1^2, \dots, a_m^2)$. A similar argument shows that the rows of the synthesis matrix $a\mathscr{X}$ are orthogonal if and only if $\langle c, \tilde{R}_j \rangle = 0$ for all $j \geqslant n$.

$(2) \Leftrightarrow (3)$: Let G be the Gram matrix of $\tilde{\theta}_{\mathscr{X}}$. If there exists a non-negative, non-zero vector $c = (c_1, \dots, c_m)$ that is orthogonal to every row of $\tilde{\theta}_{\mathscr{X}}$, then $\tilde{\theta}_{\mathscr{X}} c = 0$. Thus, $Gc = \tilde{\theta}_{\mathscr{X}}^* \tilde{\theta}_{\mathscr{X}} c = 0$, and we have shown that (2) implies (3). Conversely, if $\tilde{\theta}_{\mathscr{X}}^* \tilde{\theta}_{\mathscr{X}} c = 0$, then $\|\tilde{\theta}_{\mathscr{X}} c\|^2 = \langle \tilde{\theta}_{\mathscr{X}} c, \tilde{\theta}_{\mathscr{X}} c \rangle = \langle \tilde{\theta}_{\mathscr{X}}^* \tilde{\theta}_{\mathscr{X}} c, c \rangle = 0$. So $\tilde{\theta}_{\mathscr{X}} c = 0$ and (2) is proved. □

REMARK 3.6. The statement "$(1) \Leftrightarrow (3)$" was stated in [8] for the Gram matrix of the diagram matrix of $\mathscr{X}$. Also, the statement "$(1) \Leftrightarrow (2)$" appeared in [11] when they classified k-scalable and strictly k-scalable frames. Recall that a frame $\{x_i\}_{i=1}^m$ for $\mathbb{R}^n$ is said to be k-scalable (resp., strictly k-scalable) if there is a set $I \subset [m], |I| = k$ such that $\{x_i\}_{i \in I}$ is a scalable frame (resp., a strictly scalable frame) for $\mathbb{R}^n$.

The following is a direct consequence of Theorem 3.5.

COROLLARY 3.7. *A unit-norm frame $\mathscr{X}$ is non-scalable if one of the following conditions holds.*

1. *$\tilde{\theta}_{\mathscr{X}}$ contains at least one row whose elements are all positive or all negative.*

2. *The columns of* $\tilde{\theta}_{\mathscr{X}}$ *are linearly independent.*

Condition (1) of the corollary is verified, for instance, when the synthesis matix $\mathscr{X}$ contains two rows with the entries which are all positive or all negative.

The following result also appeared in [11], with a differrent notation. For the completeness of the paper, we will restate it here without its proof.

Recall that the *convex hull* of a set $\mathscr{X} = \{x_i\}_{i=1}^m$ in $\mathbb{R}^n$ is the set

$$\operatorname{co}(\mathscr{X}) = \left\{ \sum_{i=1}^m \alpha_i x_i : \alpha_i \geqslant 0, \sum_{i=1}^m \alpha_i = 1 \right\}.$$

THEOREM 3.8. (see also in [11]) *Let* $\mathscr{X} = \{x_i\}_{i=1}^m$ *be a frame for* $\mathbb{R}^n$ *and* $\tilde{\theta}_{\mathscr{X}}$ *be the reduced diagram matrix of* $\mathscr{X}$. *The following are equivalent.*

1. $\mathscr{X}$ *is scalable.*
2. $0 \in \operatorname{co}(\tilde{\theta}_{\mathscr{X}})$ *(the convex hull of the column vectors of* $\tilde{\theta}_{\mathscr{X}}$*).*
3. *There is no* $y \in \mathbb{R}^{\frac{(n-1)(n+2)}{2}}$ *such that* $\langle \tilde{x}_i, y \rangle > 0$ *for all* $i \in [m]$.

Before we state our next theorem, we recall some standard linear algebra results.

If $A = \{a_{i,j}\}_{i,j \in [m]}$ is a square matrix, the determinant of the $(m-1) \times (m-1)$ sub-matrix obtained from A by deleting the i-th row and j-th column is called the *minor of* $a_{i,j}$. When there is no ambiguity, this number is often denoted by $M_{i,j}$. The *cofactor of* $a_{i,j}$ is obtained by multiplying $M_{i,j}$ with $(-1)^{i+j}$ and can be denoted by $A_{i,j}$.

Note that $\det(A) = \sum_{j=1}^m a_{i,j} A_{i,j}$, for every $i \in [m]$. The following is a well known linear algebra result.

LEMMA 3.9. *For all* $i,k \in [m]$, *with* $i \neq k$, *we have that* $\sum_{j=1}^m a_{k,j} A_{i,j} = 0$. *In other words, for every* $i \in [m]$, *the vector* $(A_{i,1}, \dots, A_{i,m})$ *is orthogonal to all rows of* A *with the exception of the* i*-th row.*

THEOREM 3.10. *Let* $\mathscr{X} = \{x_i\}_{i=1}^m$ *be a unit-norm frame for* $\mathbb{R}^n$, *and let* $\tilde{\theta}_{\mathscr{X}}$ *be the reduced diagram matrix of* $\mathscr{X}$. *Assume that* $\tilde{\theta}_{\mathscr{X}}$ *has rank* $m-1$ *and let* $R_1, R_2, \dots, R_{m-1}$ *be the* $m-1$ *linearly independent rows of* $\tilde{\theta}_{\mathscr{X}}$. *Let* $E = (\alpha_1, \alpha_2, \dots, \alpha_m)$ *be a vector of coefficients, and let* A *be the matrix whose rows are* $E, R_1, \dots, R_{m-1}$. *Then* $\mathscr{X}$ *is scalable if and only if the cofactors of the* α_j*'s in the expansion of* $\det(A)$ *are all non-negative or non-positive.*

Proof. Let $c = (c_1, \dots, c_m)$ be the vector of the cofactors of the $(\alpha_1, \dots, \alpha_m)$. Then $c \neq 0$ since the R_i's are linearly independent. By Lemma 3.9, c is orthogonal to all the rows of $\tilde{\theta}_{\mathscr{X}}$. If c_i's are all non-negative or non-positive, then $\mathscr{X}$ is scalable by Theorem 3.5. Conversely, if $\mathscr{X}$ is scalable, then again by Theorem 3.5, there is a non-negative, non-zero vector $a = (a_1, \dots, a_m)$ such that a is orthogonal to all the rows of $\tilde{\theta}_{\mathscr{X}}$. Note that these rows span a hyperplane in $\mathbb{R}^m$. Therefore, c must be a multiple of a. This completes the proof. □

REMARK 3.11. We can apply Gaussian elimination to the reduced matrix $\tilde{\theta}_{\mathscr{X}}$ and use the $(m-1)$ linearly independent rows of the result matrix.

EXAMPLE 3.12. Consider a unit-norm frame $\mathscr{X} = \{x_i\}_{i=1}^3$ of three distinct vectors in $\mathbb{R}^2$, where

$$x_1 = (1,0),\ x_2 = (\cos\theta, \sin\theta),\ x_3 = (\cos\psi, \sin\psi)$$

with $0 < \theta < \psi < \pi$. The diagram matrix of $\mathscr{X}$ is the same as the reduced diagram matrix and is

$$\begin{aligned}\tilde{\theta}_{\mathscr{X}} &= \begin{pmatrix} 1 & (x_2(1))^2 - (x_2(2))^2 & (x_3(1))^2 - (x_3(2))^2 \\ 0 & 2x_2(1)x_2(2) & 2x_3(1)x_3(2) \end{pmatrix} \\ &= \begin{pmatrix} 1 & \cos(2\theta) & \cos(2\psi) \\ 0 & \sin(2\theta) & \sin(2\psi) \end{pmatrix}.\end{aligned}$$

The condition $0 < \theta < \psi < \pi$ ensures that $\tilde{\theta}_{\mathscr{X}}$ has maximum rank. The vector of the cofactors of the elements α_j's in the matrix

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ 1 & \cos(2\theta) & \cos(2\psi) \\ 0 & \sin(2\theta) & \sin(2\psi) \end{pmatrix}$$

is $c = (\sin(2(\psi-\theta)), -\sin(2\psi), \sin(2\theta))$. We can check that the components of c cannot be all non-positive. Moreover, they are all non-negative if and only if either $0 < \theta < \pi/2 \leqslant \psi \leqslant \theta + \pi/2$ or $\theta = \pi/2 < \psi < \pi$. Thus, the given frame is scalable if the angles θ and ψ satisfy these conditions. The scaling that makes the frame tight is $(\sqrt{\sin(2(\psi-\theta))}, \sqrt{-\sin(2\psi)}, \sqrt{\sin(2\theta)})$.

REMARK 3.13. The conditions on θ and ψ are compatible with the fact that the frame is scalable if and only if the vectors do not lie in the same open quadrant cone, see [10].

In the following, we will consider the case when the orthogonal complement of the row space of the reduced diagram matrix has dimension two.

THEOREM 3.14. *Let $\mathscr{X} = \{x_i\}_{i=1}^m$ be a frame for $\mathbb{R}^n$. Assume that the reduced diagram matrix $\tilde{\theta}_{\mathscr{X}}$ has rank $(m-2)$. Let $R_1, \ldots, R_{m-2}$ be the $(m-2)$ linearly independent rows of the matrix $\tilde{\theta}_{\mathscr{X}}$. Let $w_1, w_2 \in \mathbb{R}^m$ be such that the set $\{w_1, w_2, R_1, \ldots, R_{m-2}\}$ is linearly independent. Let $E = (\alpha_1, \ldots, \alpha_m)$ be a vector of coefficients. Then $\mathscr{X}$ is scalable if and only if there exists $t \in [0, 2\pi)$ for which the cofactors of the α_j's in the expansion of the determinant of the matrix whose rows are $(E, (\cos t)w_1 + (\sin t)w_2, R_1, \ldots, R_{m-2})$ are all non-positive or non-negative.*

Proof. By Theorem 3.5, $\mathscr{X}$ is scalable if and only if there exsits a non-negative, non-zero vector $(c_1, \ldots, c_m)$ in the orthogonal complement of the row space of $\tilde{\theta}_{\mathscr{X}}$.

Let ξ_j denote the vector of the cofactors of the $(\alpha_1,\ldots,\alpha_m)$ in the matrix $A_j = (E,w_j,R_1,\ldots,R_{m-2})$. Clearly, $\xi_j \in \left(\operatorname{span}\{R_i\}_{i=1}^{m-2}\right)^\perp$ and $\xi_j \perp w_j$, $j=1,2$.

Let us prove that ξ_1,ξ_2 are linearly independent, and so they form a basis for $\left(\operatorname{span}\{R_i\}_{i=1}^{m-2}\right)^\perp$. Assume that $\alpha\xi_1+\beta\xi_2=0$ for some $\alpha,\beta\in\mathbb{R}$. By the multilinear property of the determinant, $\alpha\xi_1+\beta\xi_2$ is the vector of the cofactors of $E=(\alpha_1,\ldots,\alpha_m)$ in the matrix whose rows are $(E,\alpha w_1+\beta w_2,R_1,\ldots,R_{m-2})$. By assumption, $\alpha w_1+\beta w_2,R_1,\ldots,R_{m-2}$ are linearly independent whenever $(\alpha,\beta)\neq(0,0)$, and so the vector of the cofactors of E can only be zero if $\alpha=\beta=0$.

We have proved that every non-zero vector $\xi\in\left(\operatorname{span}\{R_i\}_{i=1}^{m-2}\right)^\perp$ can be written as $\xi=a\xi_1+b\xi_2$ for some $a,b\in\mathbb{R},(a,b)\neq(0,0)$. We can let $a=\lambda\cos t$ and $b=\lambda\sin t$, with $\lambda>0$ and $t\in[0,2\pi)$, and write $\xi=\lambda[(\cos t)\xi_1+(\sin t)\xi_2]$. Thus, ξ has non-negative components if and only if the same is true of $(\cos t)\xi_1+(\sin t)\xi_2$. By the multilinear property of the determinant, $(\cos t)\xi_1+(\sin t)\xi_2$ is the vector of the cofactors of E in the matrix A_t whose rows are $(E,(\cos t)w_1+(\sin t)w_2,R_1,\ldots,R_{m-2})$; equivalently, with the notation previously introduced, $(\cos t)\xi_1+(\sin t)\xi_2$ is the vector of the cofactors of the α_j's in the expansion of $\det A_t=\cos t\det A_1+\sin t\det A_2$. This concludes the proof of the theorem. □

REMARK 3.15. In Theorem 3.10 and Theorem 3.14, the frame $\mathscr{X}$ is strictly scalable if and only if the cofactors of the α_j's are all positive or all negative.

EXAMPLE 3.16. Let $\mathscr{X}=\{x_i\}_{i=1}^4$ be a unit-norm frame of distinct vectors in $\mathbb{R}^2$, where

$$x_1=(1,0),\ x_2=(\cos\alpha,\sin\alpha),\ x_3=(\cos\beta,\sin\beta),\ x_4=(\cos\gamma,\sin\gamma),$$

with $0<\alpha<\frac{\pi}{2}\leqslant\beta<\gamma<\pi$. We will find conditions for which the frame is strictly scalable.

The reduced diagram matrix is

$$\tilde\theta_{\mathscr{X}}=\begin{pmatrix}1 & \cos(2\alpha) & \cos(2\beta) & \cos(2\gamma)\\ 0 & \sin(2\alpha) & \sin(2\beta) & \sin(2\gamma)\end{pmatrix}=\begin{pmatrix}R_1\\R_2\end{pmatrix}.$$

The conditions on α imply that $\tilde\theta_{\mathscr{X}}$ has rank 2. We observe that the vectors $w_1=(0,0,1,0)$ and $w_2=(0,0,0,1)$ are linearly independent from R_1, R_2 since the determinant of the matrix whose rows are (w_1,w_2,R_1,R_2) is $\sin(2\alpha)\neq0$. By Theorem 3.14, the frame is strictly scalable if and only if we can find $t\in[0,2\pi)$ for which the cofactors of the α_j's in the matrix

$$\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\\ 0 & 0 & \cos t & \sin t\\ 1 & \cos(2\alpha) & \cos(2\beta) & \cos(2\gamma)\\ 0 & \sin(2\alpha) & \sin(2\beta) & \sin(2\gamma)\end{pmatrix}$$

are all positive or all negative. The determinant of the matrix above is $\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \alpha_4 A_4$, where

$$\begin{aligned}
A_1 &= \sin t \sin(2\beta - 2\alpha) + \cos t \sin(2\alpha - 2\gamma);\\
A_2 &= \cos t \sin(2\gamma) - \sin t \sin(2\beta);\\
A_3 &= \sin t \sin(2\alpha);\\
A_4 &= -\cos t \sin(2\alpha).
\end{aligned}$$

Thus, $\mathscr{X}$ is strictly scalable if and only if there exists $t \in [0, 2\pi)$ such that the A_j's are all positive or all negative. Since we have assumed $0 < \alpha < \frac{\pi}{2}$, the coefficients A_3 and A_4 are both positive if $t \in (\pi/2, \pi)$.

If we assume $\pi/2 \leqslant \beta < \gamma < \alpha + \pi/2$, it is easy to verify that $A_i > 0$ for $i = 1, 2$ and for all $t \in (\pi/2, \pi)$. So, $\mathscr{X}$ is scalable with positive scalings $c = \{\sqrt{A_i}\}_{i=1}^4$. Since t is arbitrary in $(\pi/2, \pi)$, we may have infinitely many ways of choosing a scaling that makes the given frame tight.

4. A new approach to scaling

In this section, we give a new approach to scaling problem. We are going to break the scaling problem into two smaller ones, each of which is easier to solve, giving a simpler way to check scaling. First, we introduce new notation.

NOTATION 4.1. If $Z \subset \ell_2(m)$, we denote

$$Z^+ = \{a = (a_1, a_2, \ldots, a_m) \in Z : a_i \geqslant 0 \text{ for all } i \in [m]\}.$$

If $u = (u_1, u_2, \ldots, u_m),\ v = (v_1, v_2, \ldots, v_m) \in \ell_2(m)$, we let

$$u \bullet v = (u_1 v_1, u_2 v_2, \ldots, u_m v_m)$$

be the Hadamard product of u and v. We let $u^2 = u \bullet u$.

If $\mathscr{X} = \{x_i\}_{i=1}^m \subset \mathbb{R}^n$ we denote

$$\tilde{\mathscr{X}} = \{u_j := (x_1(j), x_2(j), \ldots, x_m(j)) : j \in [n]\},$$

$$\tilde{\mathscr{X}}^2 = \{u^2 : u \in \tilde{\mathscr{X}}\},$$

and

$$W(\ell_2(m)^+, \tilde{\mathscr{X}}^2) = \{a \in \ell_2(m)^+ : \langle a, u^2 \rangle = 1 \text{ for all } u \in \tilde{\mathscr{X}}\}.$$

DEFINITION 4.2. Let $\mathscr{X} = \{x_i\}_{i=1}^m$ be a frame for $\mathbb{R}^n$. If $W(\ell_2(m)^+, \tilde{\mathscr{X}}^2) \neq \emptyset$, we say that $\mathscr{X}$ is *normalized scalable*.

REMARK 4.3. For $x, y \in \ell_2(m)$, $\langle x, y \rangle = 1$ if and only if $x = \frac{y}{\|y\|^2} + v$ for some $v \in y^\perp$.

The following result is immediate from Remark 4.3.

PROPOSITION 4.4. *Let $\mathscr{X} = \{x_i\}_{i=1}^m$ be a frame for $\mathbb{R}^n$ and let $u_j, j \in [n]$ be the vectors in $\tilde{\mathscr{X}}$. Then*

$$W(\ell_2(m)^+, \tilde{\mathscr{X}}^2) = \bigcap_{j=1}^{n} \{a \in \ell_2(m)^+ : \langle a, u_j^2 \rangle = 1\}$$
$$= \bigcap_{j=1}^{n} \left(\frac{u_j^2}{\|u_j^2\|^2} + (u_j^2)^\perp \right)^+ .$$

Note that $W(\ell_2(m)^+, \tilde{\mathscr{X}}^2)$ is the intersection of convex sets and hence it is convex.

PROPOSITION 4.5. *$\mathscr{X} = \{x_i\}_{i=1}^m \subset \mathbb{R}^n$ is normalized scalable if and only if there are constants $\{a_i\}_{i=1}^m$ so that the rows of the synthesis matrix $[a_i x_i]_{i=1}^m$ all square sum to 1.*

Proof. The square sum of the j-th row, $j \in [n]$, of the synthesis matrix is:

$$\sum_{i=1}^{m} a_i^2 x_i(j)^2 = \langle a^2, u_j^2 \rangle,$$

where $u_j = (x_1(j), x_2(j), \ldots, x_m(j)) \in \tilde{\mathscr{X}}$ and $a = (a_1, a_2, \ldots, a_m) \in \ell_2(m)$. So this equals 1 for all $j \in [n]$ if and only if $a^2 \in W(\ell_2(m)^+, \tilde{\mathscr{X}}^2)$. $\square$

EXAMPLE 4.6. It is possible that $W(\ell_2(m)^+, \tilde{\mathscr{X}}^2) = \emptyset$. To see this, take a Hadamard matrix $[\pm 1]$ in $\mathbb{R}^n$ and multiply the last row by 2. Let $\mathscr{X}$ be the frame whose frame vectors are the columns of the resulting matrix and denote by u_j the j-th row of this matrix. Then $u_j = (\pm 1, \ldots, \pm 1)$ for $1 \leqslant j \leqslant n-1$ and $u_n = (\pm 2, \pm 2, \ldots, \pm 2)$. For any $a = (a_1, a_2 \ldots, a_n) \in \ell_2(n)^+$ we have

$$\langle a, u_1^2 \rangle = \sum_{i=1}^{n} a_i \cdot 1 = \sum_{i=1}^{n} a_i,$$

while

$$\langle a, u_n^2 \rangle = \sum_{i=1}^{n} 4a_i.$$

For the second part of this approach, we start with:

NOTATION 4.7. Let $\mathscr{X} = \{x_i\}_{i=1}^m$ be a frame for $\mathbb{R}^n$ with the rows of its synthesis matrix (represented in the standard basis $\{e_i\}_{i=1}^n$ of $\mathbb{R}^n$), $u_j = (x_1(j), x_2(j), \ldots, x_m(j))$, $j \in [n]$. We denote

$$\mathscr{X}^\bullet = \{u_i \bullet u_j : 1 \leqslant i \neq j \leqslant n\},$$
$$V(\ell_2(m)^+, \mathscr{X}^\bullet) = \{a \in \ell_2(m)^+ : \langle a, u \rangle = 0 \text{ for all } u \in \mathscr{X}^\bullet\}.$$

DEFINITION 4.8. A frame $\mathscr{X} = \{x_i\}_{i=1}^m$ for $\mathbb{R}^n$ is called *mixed scalable* if $V(\ell_2(m)^+, \mathscr{X}^\bullet) \neq \{0\}$.

REMARK 4.9. 1) If $\{x_i\}_{i=1}^m$ is a frame for $\mathbb{R}^n$ written with respect to the eigenbasis of its frame operator, then $V(\ell_2(m)^+, \mathscr{X}^\bullet) = \ell_2(m)^+$. This is because the rows of the synthesis matrix are orthogonal.

2) $u_i \bullet u_j : 1 \leqslant i \neq j \leqslant n$ are the last $n(n-1)/2$ rows of the reduced diagram matrix defined in Section 3.

THEOREM 4.10. *If $\mathscr{X} = \{x_i\}_{i=1}^m$ is a frame for $\mathbb{R}^n$, then $V(\ell_2(m)^+, \mathscr{X}^\bullet)$ is a positive cone in $\mathbb{R}^n$.*

Proof. Let $a, b \in V(\ell_2(m)^+, \mathscr{X}^\bullet)$ and $\alpha, \beta \geqslant 0$. Then $\alpha a + \beta b \in \ell_2(m)^+$, and for all $1 \leqslant i \neq j \leqslant n$ we have that

$$\langle \alpha a + \beta b, u_i \bullet u_j \rangle = \alpha \langle a, u_i \bullet u_j \rangle + \beta \langle b, u_i \bullet u_j \rangle = 0.$$

The claim is proven. □

Now the following theorem is obvious.

THEOREM 4.11. *A frame $\mathscr{X} = \{x_i\}_{i=1}^m$ for $\mathbb{R}^n$ is scalable if and only if $W(\ell_2(m)^+, \tilde{\mathscr{X}}^2) \cap V(\ell_2(m)^+, \mathscr{X}^\bullet) \neq \emptyset$. In other words, the convex set intersects the positive cone.*

Theorem 4.11 gives an easier method to check if a frame is scalable. There are three reasons the theorem may fail:

1. Either $W(\ell_2(m)^+, \tilde{\mathscr{X}}^2) = \emptyset$ or $V(\ell_2(m)^+, \mathscr{X}^\bullet) = \{0\}$ (or both).

2. These two sets contain non-zero vectors but do not intersect.

We will look at examples of all cases. We will work in $\mathbb{R}^4$ since there are simple complete classifications for scaling in $\mathbb{R}^2$ and $\mathbb{R}^3$.

EXAMPLE 4.12. Let $\mathscr{X} = \{x_i\}_{i=1}^4$ be the frame in $\mathbb{R}^4$ given by the columns of the matrix:

$$\mathscr{X} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

The row vectors here are already orthogonal and it is easily checked that one cannot scale these vectors to make the row vectors norm 1.

EXAMPLE 4.13. Let $\mathscr{Y} = \{y_i\}_{i=1}^4$ be the frame in $\mathbb{R}^4$ given by the columns of the matrix:

$$\mathscr{Y} = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Clearly,

$$W(\ell_2(4)^+, \tilde{\mathscr{Y}}^2) = \{(a_1,a_2,a_3,a_4) \in \ell_2(4)^+ : \sum_{i=1}^4 a_i = 1\}.$$

It is easily checked that $V(\ell_2(4)^+, \mathscr{Y}^\bullet) = \{0\}$. Let $a = (a_1,a_2,a_3,a_4) \in V(\ell_2(4)^+, \mathscr{Y}^\bullet)$ and let $\{u_j\}_{j=1}^4$ be the row vectors of $\mathscr{Y}$. Then

$$\langle a, u_1 \bullet u_4 \rangle = a_1 - a_2 + a_3 + a_4 = 0 \text{ and } \langle a, u_2 \bullet u_4 \rangle = a_1 + a_2 - a_3 + a_4 = 0.$$

Adding these equations yields $a_1 + a_4 = 0$ and so $a_1 = a_4 = 0$. Similarly, $a_2 = a_3 = 0$.

EXAMPLE 4.14. Consider the frame $\mathscr{Z} = \mathscr{X} \cup \mathscr{Y}$ for $\mathbb{R}^4$, i.e.,

$$\mathscr{Z} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & -1 & 1 & 1 \\ 2 & -2 & 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & -1 \\ 0 & 0 & 2 & -2 & 1 & 1 & 1 & 1 \end{bmatrix}$$

We can check that $W(\ell_2(8)^+, \tilde{\mathscr{Z}}^2)$ and $V(\ell_2(8)^+, \mathscr{Z}^\bullet)$ contain non-zero vectors but they don't intersect. That is,

$$W(\ell_2(8)^+, \tilde{\mathscr{Z}}^2) = \{(0,0,0,0,a_5,a_6,a_7,a_8) \in \ell_2(8)^+ : \sum_{i=5}^8 a_i = 1\}$$

and $V(\ell_2(8)^+, \mathscr{Z}^\bullet)$ contains $(1,1,1,1,0,0,0,0)$. So the frame is non-scalable by Theorem 4.11.

Now we will look at an example of 6 vectors in $\mathbb{R}^4$. To check if this is scalable, one has to solve 10 equations in 6 unknowns, which is quite a feat. But we will only have to solve 4 equations in 6 unknowns and then go to Theorem 4.11.

EXAMPLE 4.15. Let the frame $\mathscr{X} = \{x_i\}_{i=1}^6$ in $\mathbb{R}^4$ be given by:

$$\mathscr{X} = \begin{bmatrix} \frac{1}{\sqrt{10}} & 0 & -\frac{2}{5} & 0 & 0 & 0 \\ \frac{1}{3} & -\frac{\sqrt{2}}{3} & \frac{1}{3\sqrt{10}} & \frac{1}{\sqrt{6}} & \frac{1}{6} & \frac{1}{3\sqrt{2}} \\ \frac{2}{3\sqrt{5}} & \frac{1}{3}\sqrt{\frac{5}{2}} & \frac{\sqrt{2}}{15} & \frac{1}{2\sqrt{3}} & \frac{1}{6\sqrt{2}} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{2\sqrt{3}} & 0 \end{bmatrix}$$

We want to see if this is scalable. So we consider $\{a_i x_i\}_{i=1}^6$ and we are faced with solving 10 very complicated equations in 6 unknowns. We rely on Theorem 4.11 and let u_j be the row vectors of the matrix. We only have to set up 4 equations in 6 unknowns $\langle a, u_j^2\rangle = 1$, for $j = 1,2,3,4$. Solving these equations yields that

$$(2,1,5,1,6,3)$$

are in $W(\ell_2(6)^+, \tilde{\mathscr{X}}^2)$. We can check that this vector is also in $V(\ell_2(6)^+, \mathscr{X}^\bullet)$ and so $(\sqrt{2}, 1, \sqrt{5}, 1, \sqrt{6}, \sqrt{3})$ scales the frame.

5. Scalability of dual frames

Let $\mathscr{X} = \{x_i\}_{i=1}^m$ be a frame for $\mathbb{R}^n$ with the frame operator S. A sequence $\mathscr{Y} = \{y_i\}_{i=1}^m$ in $\mathbb{R}^n$ is called a *dual frame* for $\mathscr{X}$ if $\mathscr{Y}$ satisfies the reconstruction formula:

$$x = \sum_{i=1}^m \langle x, x_i\rangle y_i = \sum_{i=1}^m \langle x, y_i\rangle x_i, \text{ for all } x \in \mathbb{R}^n.$$

If $y_i = S^{-1}x_i$, $i \in [m]$, then $\mathscr{Y}$ is called the *canonical dual frame*, otherwise it is called an *alternate dual frame*.

In this section, we will study scalabilty of the dual frames.

THEOREM 5.1. *Every scalable frame has a scalable alternate dual frame.*

Proof. Let $\{x_i\}_{i=1}^m$ be a scalable frame for $\mathbb{R}^n$. Then after perhaps dropping those vectors which are scaled with zero constants, we may assume there are $a_i > 0$ so that $\{a_i x_i\}_{i=1}^m$ is a Parseval frame. So for every $x \in \mathbb{R}^n$ we have

$$x = \sum_{i=1}^m \langle x, a_i x_i\rangle a_i x_i = \sum_{i=1}^m \langle x, x_i\rangle a_i^2 x_i.$$

It follows that $\{a_i^2 x_i\}_{i=1}^m$ is a dual frame. But now, if we scale the dual frame by $\frac{1}{a_i}$ we get $\{a_i x_i\}_{i=1}^m$ which is Parseval. □

COROLLARY 5.2. *Let $\mathscr{X} = \{x_i\}_{i=1}^m$ be a frame for $\mathbb{R}^n$. The following are equivalent.*

1. *$\{a_i x_i\}_{i=1}^m$ is Parseval.*
2. *$\{a_i^2 x_i\}_{i=1}^m$ is a dual frame for $\{x_i\}_{i=1}^m$.*

REMARK 5.3. There are non-scalable frames (for example, non-orthogonal bases of $\mathbb{R}^n$) whose all duals are not scalable, and also examples of non-scalable frames whose canonical dual frame is scalable.

EXAMPLE 5.4. Let $\mathscr{X}$ be a frame for $\mathbb{R}^2$ whose the vectors are the columns of the following matrix:

$$\mathscr{X} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

This frame is not scalable since all the vectors lie in the first open quadrant. Its frame operator S is

$$S = \mathscr{X}\mathscr{X}^* = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix},$$

We have

$$S^{-1} = \frac{1}{11}\begin{bmatrix} 6 & -5 \\ -5 & 6 \end{bmatrix},$$

and

$$S^{-1}\mathscr{X} = \frac{1}{11}\begin{bmatrix} 7 & -4 & 1 \\ -4 & 7 & 1 \end{bmatrix}.$$

The columns of the matrix $S^{-1}\mathscr{X}$ form a scalable frame since the vectors do not lie in the same open quadrant cone.

We will provide a classification of frames whose canonical dual frame is scalable. First, we consider the scalabilty of frames under invertible operators.

THEOREM 5.5. *Let $\mathscr{X} = \{x_i\}_{i=1}^m$ be a frame for $\mathbb{R}^n$ and T be an invertible operator on $\mathbb{R}^n$. The following are equivalent:*

1. *There are constants $\{a_i\}_{i=1}^m$ so that $\{a_i T x_i\}_{i=1}^m$ is a Parseval frame for $\mathbb{R}^n$.*
2. *There are constants $\{a_i\}_{i=1}^m$ so that the frame operator for $\{a_i x_i\}_{i=1}^m$ is $(T^*T)^{-1}$.*

Proof. Let S_1 be the frame operator for $\{a_i x_i\}_{i=1}^m$. For $x \in \mathbb{R}^n$ we have

$$\sum_{i=1}^m \langle x, a_i T x_i\rangle a_i T x_i = T\left(\sum_{i=1}^m \langle T^* x, a_i x_i\rangle a_i x_i\right) = T S_1 T^* x.$$

So $\{a_i T x_i\}_{i=1}^m$ is a Parseval for $\mathbb{R}^n$ if and only if $TS_1T^* = I$. That is, $S_1 = T^{-1}(T^*)^{-1} = (T^*T)^{-1}$. □

COROLLARY 5.6. *Let $\mathscr{X} = \{x_i\}_{i=1}^m$ be a frame for $\mathbb{R}^n$ and T be an invertible operator on $\mathbb{R}^n$. If there are constants $\{a_i\}_{i=1}^m$ so that $\{a_i x_i\}_{i=1}^m$ and $\{a_i T x_i\}_{i=1}^m$ are Parseval frames for $\mathbb{R}^n$, then T is unitary.*

Proof. This is immediate from Theorem 5.5. □

The following theorem classifies when the canonical dual frame is scalable. Note that we do not require the original frame is scalable.

THEOREM 5.7. *Let* $\mathscr{X} = \{x_i\}_{i=1}^m$ *be a frame for* $\mathbb{R}^n$ *with frame operator* S. *The following are equivalent.*

1. *There are constants* $\{a_i\}_{i=1}^m$ *so that* $\{a_i S^{-1} x_i\}_{i=1}^m$ *is a Parseval frame for* $\mathbb{R}^n$.
2. *There are constants* $\{a_i\}_{i=1}^m$ *so that the frame operator for* $\{a_i x_i\}_{i=1}^m$ *is* S^2.
3. *There are constants* $\{a_i\}_{i=1}^m$ *so that the frame* $\{a_i S^{-1/2} x_i\}_{i=1}^m$ *has* S *as its frame operator.*

Proof. $(1) \Leftrightarrow (2)$: This follows from Theorem 5.5 with $T = S^{-1}$.
$(2) \Rightarrow (3)$: Given (2), for every $x \in \mathbb{R}^n$ we have

$$\begin{aligned}\sum_{i=1}^m \langle x, a_i S^{-1/2} x_i \rangle a_i S^{-1/2} x_i &= S^{-1/2} \sum_{i=1}^m \langle S^{-1/2} x, a_i x_i \rangle a_i x_i \\ &= S^{-1/2} S^2 S^{-1/2} x \\ &= Sx.\end{aligned}$$

$(3) \Rightarrow (2)$: Let S_1 be the frame operator for $\{a_i x_i\}_{i=1}^m$. Given (3), for every $x \in \mathbb{R}^n$ we have

$$\begin{aligned}Sx &= \sum_{i=1}^m \langle x, a_i S^{-1/2} x_i \rangle a_i S^{-1/2} x_i \\ &= S^{-1/2} \sum_{i=1}^m \langle S^{-1/2} x, a_i x_i \rangle a_i x_i \\ &= S^{-1/2} S_1 S^{-1/2} x.\end{aligned}$$

So $S = S^{-1/2} S_1 S^{-1/2}$ and hence $S_1 = S^2$. □

REMARK 5.8. (1) Note that S^2 is the frame operator for the frame $\{S^{1/2} x_i\}_{i=1}^m$.

(2) If $\{S^{-1} x_i\}_{i=1}^m$ can be scaled in more than one way, for example if $\{a_i S^{-1} x_i\}_{i=1}^m$ and $\{b_i S^{-1} x_i\}_{i=1}^m$ are Parseval frames, then $\{a_i x_i\}_{i=1}^m$ and $\{b_i x_i\}_{i=1}^m$ have the same frame operator.

COROLLARY 5.9. *Let* $\mathscr{X} = \{x_i\}_{i=1}^m$ *be a frame for* $\mathbb{R}^n$ *with frame operator* S *and let* D *be the diagonal operator with* $\{a_1, a_2, \dots, a_m\}$ *on its diagonal entries. Then* $\{a_i S^{-1} x_i\}_{i=1}^m$ *is Parseval if and only if* $\mathscr{X}(D^2 - G)\mathscr{X}^* = 0$, *where* G *be the Grammian operator of* $\mathscr{X}$.

Proof. We have $S = \mathscr{X}\mathscr{X}^*$. By the Theorem 5.7, $\{a_i S^{-1} x_i\}$ is Parseval if and only if $S_1 = S^2$, where S_1 is the frame operator of $\{a_i x_i\}_{i=1}^m$.
But

$$S_1 = (\mathscr{X}D)(\mathscr{X}D)^* = \mathscr{X}D^2\mathscr{X}^*$$

Thus, $S_1 = S^2$ is equivalent to $\mathscr{X}D^2\mathscr{X}^* = (\mathscr{X}\mathscr{X}^*)^2 = \mathscr{X}G\mathscr{X}^*$, or $\mathscr{X}(D^2 - G)\mathscr{X}^* = 0$. □

PROPOSITION 5.10. *There are scalable frames whose canonical dual frames are not scalable.*

Proof. Let $[x_{i,j}]_{i,j=1}^n$ be a unitary Hadamard matrix (i.e., entries $\pm 1/\sqrt{n}$) with row vectors $\{x_i\}_{i=1}^n$. Let

$$y_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,n-2}, 2x_{i,n-1}, 3x_{i,n}) \text{ for all } i \in [n].$$

Then $\{x_i\}_{i=1}^n \cup \{y_i\}_{i=1}^n$ is a frame for $\mathbb{R}^n$. Let S be its frame operator. We see that $Se_n = 2e_n$ for $n = 1, \ldots, n-2$, $Se_{n-1} = 5e_{n-1}$, and $Se_n = 10e_n$. So the unit vectors $e_n's$ are eigenvectors with eignvalues $2, 2, \ldots, 2, 5, 10$. This frame is scalable since it contains the Parseval frame $\{x_i\}_{i=1}^n$. We proceed by way of contradiction by assuming the canonical dual frame is scalable. Then by Theorem 5.7, there are constants $\{a_i\}_{i=1}^n \cup \{b_i\}_{i=1}^n$ so that $\{a_i x_i\}_{i=1}^n \cup \{b_i y_i\}_{i=1}^n$ is a frame with frame operator S^2. It follows that this frame has the unit vectors as eigenvectors with eigenvalues:

$$\frac{1}{n}\left(\sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2\right) = 4, \text{ for the first } (n-2) \text{ eigenvectors}$$

and

$$\frac{1}{n}\left(\sum_{i=1}^n a_i^2 + 4\sum_{i=1}^n b_i^2\right) = 25, \text{ for the } (n-1)\text{-th eigenvector,}$$

and

$$\frac{1}{n}\left(\sum_{i=1}^n a_i^2 + 9\sum_{i=1}^n b_i^2\right) = 100 \text{ for the last eigenvector}.$$

So

$$\sum_{i=1}^n a_i^2 = 4n - \sum_{i=1}^n b_i^2,$$

and substituting this into the next two equations yields:

$$25n = 4n - \sum_{i=1}^n b_i^2 + 4\sum_{i=1}^n b_i^2 \text{ and so } \sum_{i=1}^n b_i^2 = \frac{21n}{3}.$$

$$100n = 4n - \sum_{i=1}^n b_i^2 + 9\sum_{i=1}^n b_i^2 \text{ and so } \sum_{i=1}^n b_i^2 = \frac{96n}{8}.$$

This contradiction completes the proof. □

REMARK 5.11. (1) Proposition 5.10 also shows that there are non-scalable frames whose canonical dual frames are scalable, namely, take the canonial dual frame of the above frame.

(2) Part (3) in Theorem 5.7 is interesting. Scaling means multiplying the frame vectors by a constant to get a Parseval frame. Theorem 5.7(3) says that we are to take the canonical Parseval frame and scale it so that the new frame has the original frame operator as its frame operator.

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