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TRACE INEQUALITIES RELATED TO 2×2 BLOCK SECTOR MATRICES

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Abstract. We extend several trace inequalities for 2×2 block positive semi-definite matrices to the class of matrices whose numerical range is contained in a sector. In the meanwhile, some related results are obtained.

1. Introduction

Let $\mathbb{M}_n$ be the set of all $n\times n$ complex matrices. For $A\in\mathbb{M}_n$, the singular values and eigenvalues of A are denoted by $\sigma_j(A)$ and $\lambda_j(A)$, respectively, $j=1,\dots,n$. The singular values are always arranged in nonincreasing order $\sigma_1^{\downarrow}(A)\geqslant\cdots\geqslant\sigma_n^{\downarrow}(A)$. When A is Hermitian, all eigenvalues of A are real and ordered as $\lambda_1^{\downarrow}(A)\geqslant\cdots\geqslant\lambda_n^{\downarrow}(A)$. Note that the singular values of A are the eigenvalues of $|A|$, where $|A|=(A^*A)^{\frac{1}{2}}$, i.e., $\sigma_j(A)=\lambda_j(|A|)$, $j=1,\dots,n$. Let $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)\in\mathbb{R}^n$. Let $x^{\downarrow}=(x_1^{\downarrow},\dots,x_n^{\downarrow})$ and $y^{\downarrow}=(y_1^{\downarrow},\dots,y_n^{\downarrow})$ be the vectors obtained by rearranging the coordinates of x and y in the nonincreasing order, respectively. Then we can write $x_1^{\downarrow}\geqslant\cdots\geqslant x_n^{\downarrow}$ and $y_1^{\downarrow}\geqslant\cdots\geqslant y_n^{\downarrow}$. If

$$\sum_{i=1}^{k}x_i^{\downarrow}\leqslant\sum_{i=1}^{k}y_i^{\downarrow},\quad k=1,\dots,n,$$

we say that x is weakly majorized by y, in symbols $x\prec_{\omega}y$. If, in addition,

$$\sum_{i=1}^{n}x_i^{\downarrow}=\sum_{i=1}^{n}y_i^{\downarrow}.$$

We say that x is majorized by y, written as $x\prec y$, see [2, p. 28-29]. Given Hermitian matrices $A,B\in\mathbb{M}_n$, A is positive semi-definite (definite, resp.), which is denoted by $A\geqslant 0$ ($A>0$, resp.). In particular, $A\geqslant B$ ($A>B$, resp.) means that $A-B\geqslant 0$ ($A-B>0$, resp.). For $A\in\mathbb{M}_n$, we can write

$$A=\Re A+i\Im A,$$

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where

$$\Re A = \frac{A+A^*}{2}, \ \Im A = \frac{A-A^*}{2i}.$$

The numerical range of $A \in \mathbb{M}_n$ is defined by

$$W(A) = \{x^*Ax | x \in \mathbb{C}^n, x^*x = 1\}.$$

For $\alpha \in [0, \frac{\pi}{2})$, we define a sector on the complex plane

$$S_\alpha = \{z \in \mathbb{C} | \Re z > 0, |\Im z| \leqslant (\Re z)\tan\alpha\}.$$

Sector matrices is a class of matrices whose numerical ranges are contained in S_α ($W(A) \subseteq S_\alpha$). This class of matrices has been the subject of recent research [3, 9, 11, 12, 13]. Consider $M \in \mathbb{M}_{2n}$ partitioned as

$$M = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in \mathbb{M}_{2n}$$

with each block in $\mathbb{M}_n$, its partial transpose is defined by

$$M^\tau = \begin{bmatrix} A & X^* \\ X & B \end{bmatrix}.$$

Now we extend the notion to sector matrices. Let

$$M = \begin{bmatrix} A & X \\ Y^* & B \end{bmatrix} \in \mathbb{M}_{2n}$$

with each block in $\mathbb{M}_n$ and its partial transpose

$$M^\tau = \begin{bmatrix} A & Y^* \\ X & B \end{bmatrix}.$$

M is said to be sectorial partial transpose (i.e., SPT) if $W(M) \subseteq S_\alpha$, $W(M^\tau) \subseteq S_\alpha$. Motivated by the subadditivity of q-entropies in the theory of Quantum information, Besenyei [1] gave the following trace inequality involving positive semi-definite block matrices:

$$\mathrm{tr}(AB) - \mathrm{tr}(X^*X) \leqslant \mathrm{tr}(A)\mathrm{tr}(B) - |\mathrm{tr}(X)|^2. \tag{1.1}$$

Kittaneh and Lin [6] presented an improvement and an analogue of (1.1):

$$|\mathrm{tr}(AB) - \mathrm{tr}(X^*X)| \leqslant \mathrm{tr}(A)\mathrm{tr}(B) - |\mathrm{tr}(X)|^2, \tag{1.2}$$

$$\mathrm{tr}(AB) + \mathrm{tr}(X^*X) \leqslant \mathrm{tr}(A)\mathrm{tr}(B) + |\mathrm{tr}(X)|^2. \tag{1.3}$$

Recently, Fu and Gumus [4, Theorem 3.3] presented the refinements of (1.2) and (1.3): Let λ be the smallest eigenvalue of M. Then,

$$|\mathrm{tr}(AB) - \mathrm{tr}(X^*X)| \leqslant \mathrm{tr}(A)\mathrm{tr}(B) - |\mathrm{tr}(X)|^2 - \frac{\lambda(n-1)}{2}\mathrm{tr}(M), \tag{1.4}$$

$$\operatorname{tr}(AB)+\operatorname{tr}(X^*X)\leqslant \operatorname{tr}(A)\operatorname{tr}(B)+|\operatorname{tr}(X)|^2-\frac{\lambda(n-1)}{2}\operatorname{tr}(M). \tag{1.5}$$

Actually, the authors [4, Theorem 3.4] also gave the corresponding results with the largest eigenvalue μ of M:

$$|\operatorname{tr}(AB)-\operatorname{tr}(X^*X)|\leqslant \frac{\mu(n+1)}{2}\operatorname{tr}(M)-\operatorname{tr}(A)\operatorname{tr}(B)+|\operatorname{tr}(X)|^2, \tag{1.6}$$

$$\operatorname{tr}(A)\operatorname{tr}(B)+|\operatorname{tr}(X)|^2\leqslant \frac{\mu(n-1)}{2}\operatorname{tr}(M)+\operatorname{tr}(AB)+\operatorname{tr}(X^*X). \tag{1.7}$$

Note that the left side of (1.1) might be negative. But if M is PPT, then

$$\operatorname{tr}(AB)-\operatorname{tr}(X^*X)\geqslant 0, \tag{1.8}$$

see [8, Theorem 2.1]. Fu and Gumus [4, Theorem 3.1] derived the sharper inequality than (1.8) and new upper bound of $\operatorname{tr}(AB)$ under the PPT condition: Let λ and μ be the smallest and the largest eigenvalues of M, respectively. If M is PPT, then

$$\frac{\mu}{2}\cdot\operatorname{tr}(M)-\operatorname{tr}(X^*X)\geqslant \operatorname{tr}(AB)\geqslant \operatorname{tr}(X^*X)+\frac{\lambda}{2}\cdot\operatorname{tr}(M). \tag{1.9}$$

When M is positive semi-definite but not PPT, the result becomes

$$\frac{\tilde{\mu}}{2}\cdot\operatorname{tr}(M)-\operatorname{tr}(X^*X)\geqslant \operatorname{tr}(AB)\geqslant \operatorname{tr}(X^*X)+\frac{\tilde{\lambda}}{2}\cdot\operatorname{tr}(M), \tag{1.10}$$

where $\tilde{\lambda}$ and $\tilde{\mu}$ are the smallest and the largest eigenvalues of M^τ, respectively.

In this paper, we extend the above trace inequalities to sector matrices. Some interesting results are included.

2. The trace inequalities of block sector matrices

In this section, we will provide extensions to inequalities (1.2)–(1.10). Before presenting the main results, we list some well known results as lemmas.

LEMMA 2.1. [2, p. 73] *Let $M\in\mathbb{M}_n$. Then,*

$$\lambda_j(\Re M)\leqslant \sigma_j(M),\quad j=1,2,\dots,n.$$

LEMMA 2.2. [13, Lemma 3.1] *Let $M\in\mathbb{M}_n$ have $W(M)\subseteq S_\alpha$ for some $\alpha\in[0,\frac{\pi}{2})$. Then,*

$$\sigma(M)\prec_\omega \sec(\alpha)\lambda(\Re M).$$

LEMMA 2.3. [5, p. 445] *Let $P,H\in\mathbb{M}_n$ be positive semi-definite. Then,*

$$\operatorname{tr}(PH)\geqslant 0.$$

The next lemma is a special case of [10, Proposition 2.1].

LEMMA 2.4. [10, Proposition 2.1] *Let* $M = \begin{bmatrix} A & X \\ Y^* & B \end{bmatrix} \in \mathbb{M}_{2n}$ *be a sector matrix with* $A,B,X,Y \in \mathbb{M}_n$*. Then,*

$$T = \begin{bmatrix} \operatorname{tr}(\Re A)I - \Re A & \operatorname{tr}\left(\frac{Y^*+X^*}{2}\right)I - \frac{Y^*+X^*}{2} \\ \operatorname{tr}\left(\frac{Y+X}{2}\right)I - \frac{Y+X}{2} & \operatorname{tr}(\Re B)I - \Re B \end{bmatrix} \geqslant 0. \tag{2.1}$$

LEMMA 2.5. [7, Proposition 2.2] *Let* $M = \begin{bmatrix} A & X \\ Y^* & B \end{bmatrix} \in \mathbb{M}_{2n}$ *be a sector matrix with* $A,B,X,Y \in \mathbb{M}_n$*. Then,*

$$K = \begin{bmatrix} \operatorname{tr}(\Re A)I + \Re A & \operatorname{tr}\left(\frac{Y^*+X^*}{2}\right)I + \frac{Y^*+X^*}{2} \\ \operatorname{tr}\left(\frac{Y+X}{2}\right)I + \frac{Y+X}{2} & \operatorname{tr}(\Re B)I + \Re B \end{bmatrix} \geqslant 0. \tag{2.2}$$

We also need the following unitarily similar transformations of $\Re M$.

$$N = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} \Re A & \frac{Y+X}{2} \\ \frac{Y^*+X^*}{2} & \Re B \end{bmatrix} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} = \begin{bmatrix} \Re B & -\frac{Y^*+X^*}{2} \\ -\frac{Y+X}{2} & \Re A \end{bmatrix} \geqslant 0 \tag{2.3}$$

and

$$L = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} \Re A & \frac{Y+X}{2} \\ \frac{Y^*+X^*}{2} & \Re B \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \begin{bmatrix} \Re B & \frac{Y^*+X^*}{2} \\ \frac{Y+X}{2} & \Re A \end{bmatrix} \geqslant 0. \tag{2.4}$$

For the convenience of follow-up proofs, we compute several trace inequalities below by using the positive semi-definite matrices T,K,N,L from (2.1)–(2.4). According to Lemma 2.3,

$$\operatorname{tr}(TN) = 2\operatorname{tr}(\Re A)\operatorname{tr}(\Re B) - 2\operatorname{tr}(\Re A \Re B) + 2\operatorname{tr}(Z^*Z) - 2|\operatorname{tr}(Z)|^2 \geqslant 0, \tag{2.5}$$

$$\operatorname{tr}(KN) = 2\operatorname{tr}(\Re A)\operatorname{tr}(\Re B) + 2\operatorname{tr}(\Re A \Re B) - 2\operatorname{tr}(Z^*Z) - 2|\operatorname{tr}(Z)|^2 \geqslant 0, \tag{2.6}$$

$$\operatorname{tr}(TL) = 2\operatorname{tr}(\Re A)\operatorname{tr}(\Re B) - 2\operatorname{tr}(\Re A \Re B) - 2\operatorname{tr}(Z^*Z) + 2|\operatorname{tr}(Z)|^2 \geqslant 0. \tag{2.7}$$

Now we present the extensions on inequalities (1.4)–(1.5) in the next theorem. Actually, the inequalities achieved are (1.2)–(1.3) under the special case, respectively.

THEOREM 2.1. *Let* $M = \begin{bmatrix} A & X \\ Y^* & B \end{bmatrix} \in \mathbb{M}_{2n}$ *with* $A,B,X,Y \in \mathbb{M}_n$*, and* $W(M) \subseteq S_\alpha$ *for some* $\alpha \in [0, \frac{\pi}{2})$*. Let* λ *be the smallest eigenvalue of* $\Re M$*. Then,*

$$|\operatorname{tr}(\Re A \Re B) - \operatorname{tr}(Z^*Z)| \leqslant \operatorname{tr}(\Re A)\operatorname{tr}(\Re B) - |\operatorname{tr}(Z)|^2 - \frac{\lambda(n-1)}{2}\frac{1}{\sec(\alpha)}\operatorname{tr}(|M|) \tag{2.8}$$

and

$$\operatorname{tr}(\Re A \Re B) + \operatorname{tr}(Z^*Z) \leqslant \operatorname{tr}(\Re A)\operatorname{tr}(\Re B) + |\operatorname{tr}(Z)|^2 - \frac{\lambda(n-1)}{2}\frac{1}{\sec(\alpha)}\operatorname{tr}(|M|), \tag{2.9}$$

where $Z = \frac{X+Y}{2}$.

Proof. By the unitary similarity, λ is also the smallest eigenvalue of N and L. Applying Lemma 2.3, we have

$$\begin{aligned}\operatorname{tr}(T(N-\lambda I)) &= \operatorname{tr}(TN)-\lambda\cdot\operatorname{tr}(T)\\ &= \operatorname{tr}(TN)-\lambda(n-1)(\operatorname{tr}(\Re M))\geqslant 0,\end{aligned} \tag{2.10}$$

$$\begin{aligned}\operatorname{tr}(K(N-\lambda I)) &= \operatorname{tr}(KN)-\lambda\cdot\operatorname{tr}(K)\\ &= \operatorname{tr}(KN)-\lambda(n+1)(\operatorname{tr}(\Re M))\geqslant 0,\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}\operatorname{tr}(T(L-\lambda I)) &= \operatorname{tr}(TL)-\lambda\cdot\operatorname{tr}(T)\\ &= \operatorname{tr}(TL)-\lambda(n-1)(\operatorname{tr}(\Re M))\geqslant 0.\end{aligned} \tag{2.12}$$

Since $\Re M\geqslant 0$, (2.11) leads to

$$\operatorname{tr}(KN)-\lambda(n-1)(\operatorname{tr}(\Re M))\geqslant 0. \tag{2.13}$$

Therefore, (2.8) follows from (2.5), (2.6), (2.10), (2.13) and Lemma 2.2. Similarly, the inequality (2.9) follows from (2.7) and (2.12). $\square$

REMARK 2.1. When M is positive semi-definite, (2.8) and (2.9) are (1.4) and (1.5), respectively. If $\Re M$ has a zero eigenvalue, then (2.8) and (2.9) reduce to (1.2) and (1.3), respectively.

As analogues of (2.8) and (2.9), we give the following theorem with the largest eigenvalue of $\Re M$.

THEOREM 2.2. *Let* $M=\begin{bmatrix} A & X\\ Y^* & B\end{bmatrix}\in\mathbb{M}_{2n}$ *be a sector matrix with* $A,B,X,Y\in\mathbb{M}_n$. *Let* μ *be the largest eigenvalue of* $\Re M$. *Then,*

$$|\operatorname{tr}(\Re A\Re B)-\operatorname{tr}(Z^*Z)|\leqslant\frac{\mu(n+1)}{2}\operatorname{tr}(|M|)-\operatorname{tr}(\Re A)\operatorname{tr}(\Re B)+|\operatorname{tr}(Z)|^2 \tag{2.14}$$

and

$$\operatorname{tr}(\Re A)\operatorname{tr}(\Re B)+|\operatorname{tr}(Z)|^2\leqslant\frac{\mu(n-1)}{2}\operatorname{tr}(|M|)+\operatorname{tr}(\Re A\Re B)+\operatorname{tr}(Z^*Z), \tag{2.15}$$

where $Z=\frac{X+Y}{2}$.

Proof. By unitary similarity, μ is also the largest eigenvalue of N and L. Applying Lemma 2.3, we have

$$\begin{aligned}\operatorname{tr}(T(\mu I-N)) &= \mu\cdot\operatorname{tr}(T)-\operatorname{tr}(TN)\\ &= \mu(n-1)\operatorname{tr}(\Re M)-\operatorname{tr}(TN)\geqslant 0,\end{aligned} \tag{2.16}$$

$$\begin{aligned} \operatorname{tr}(K(\mu I - N)) &= \mu \cdot \operatorname{tr}(K) - \operatorname{tr}(KN) \\ &= \mu(n+1)\operatorname{tr}(\Re M) - \operatorname{tr}(KN) \geqslant 0, \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} \operatorname{tr}(T(\mu I - L)) &= \mu \cdot \operatorname{tr}(T) - \operatorname{tr}(TL) \\ &= \mu(n-1)\operatorname{tr}(\Re M) - \operatorname{tr}(TL) \geqslant 0. \end{aligned} \tag{2.18}$$

Since $\Re M \geqslant 0$, (2.16) implies that

$$\mu(n+1)\operatorname{tr}(\Re M) - \operatorname{tr}(TN) \geqslant 0. \tag{2.19}$$

Thus, (2.14) follows from (2.5), (2.6), (2.17), (2.19) and Lemma 2.1. The inequality (2.15) follows from (2.7) and (2.18). □

REMARK 2.2. When M is positive semi-definite, (2.14) and (2.15) are (1.6) and (1.7), respectively.

Next, we extend the inequalities (1.9)–(1.10) to the class of sector matrices.

THEOREM 2.3. *Let* $M = \begin{bmatrix} A & X \\ Y^* & B \end{bmatrix} \in \mathbb{M}_{2n}$ *be SPT with* $A, B, X, Y \in \mathbb{M}_n$. *Let* λ *and* μ *be the smallest and the largest eigenvalues of* $\Re M$*, respectively. Then,*

$$\frac{\mu}{2} \cdot \operatorname{tr}(|M|) - \operatorname{tr}(Z^*Z) \geqslant \operatorname{tr}(\Re A \Re B) \geqslant \operatorname{tr}(Z^*Z) + \frac{\lambda}{2} \cdot \frac{1}{\sec(\alpha)} \operatorname{tr}(|M|),$$

where $Z = \frac{X+Y}{2}$.

Proof. Observe that λ is also the smallest eigenvalue of N. Thus, $N - \lambda I \geqslant 0$. By Lemma 2.3,

$$\operatorname{tr}\big(\Re(M^\tau)(N - \lambda I)\big) = 2\operatorname{tr}(\Re A \Re B) - 2\operatorname{tr}(Z^*Z) - \lambda \cdot \operatorname{tr}(\Re M) \geqslant 0.$$

Applying Lemma 2.2, we have

$$\operatorname{tr}(\Re A \Re B) \geqslant \operatorname{tr}(Z^*Z) + \frac{\lambda}{2} \cdot \frac{1}{\sec(\alpha)} \operatorname{tr}(|M|). \tag{2.20}$$

Note that μ is also the largest eigenvalue of L. Thus, $\mu I - L \geqslant 0$. Thus,

$$\operatorname{tr}\big(\Re(M^\tau)(\mu I - L)\big) = -2\operatorname{tr}(\Re A \Re B) - 2\operatorname{tr}(Z^*Z) + \mu \cdot \operatorname{tr}(\Re M) \geqslant 0.$$

Then by Lemma 2.1,

$$\frac{\mu}{2} \cdot \operatorname{tr}(|M|) - \operatorname{tr}(Z^*Z) \geqslant \operatorname{tr}(\Re A \Re B). \tag{2.21}$$

The result follows from (2.20) and (2.21). □

REMARK 2.3. Obviously, if M is PPT in Theorem 2.3, our result is inequality (1.9).

Moreover, without the SPT condition in Theorem 2.3, the following result is obtained.

THEOREM 2.4. *Let* $M=\begin{bmatrix} A & X \\ Y^* & B\end{bmatrix}\in\mathbb{M}_{2n}$ *with* $A,B,X,Y\in\mathbb{M}_n$*, and* $W(M)\subseteq S_\alpha$ *for some* $\alpha\in[0,\frac{\pi}{2})$*. Let* $\tilde{\lambda}$ *and* $\tilde{\mu}$ *be the smallest and the largest eigenvalues of* $\Re(M^\tau)$*, respectively. Then,*

$$\frac{\tilde{\mu}}{2}\cdot\operatorname{tr}(|M|)-\operatorname{tr}(Z^*Z)\geqslant\operatorname{tr}(\Re A\Re B)\geqslant\operatorname{tr}(Z^*Z)+\frac{\tilde{\lambda}}{2}\cdot\frac{1}{\sec(\alpha)}\operatorname{tr}(|M|),$$

where $Z=\frac{X+Y}{2}$.

Proof. Note that $\Re M=\begin{bmatrix}\Re A & \frac{X+Y}{2}\\ \frac{X^*+Y^*}{2} & \Re B\end{bmatrix}$ is positive semi-definite and $\Re(M^\tau)-\tilde{\lambda}I\geqslant 0$, $\tilde{\mu}I-\Re(M^\tau)\geqslant 0$. By Lemma 2.3,

$$\operatorname{tr}\big((\Re(M^\tau)-\tilde{\lambda}I)N\big)=2\operatorname{tr}(\Re A\Re B)-2\operatorname{tr}(Z^*Z)-\tilde{\lambda}\cdot\operatorname{tr}(\Re A+\Re B)\geqslant 0$$

and

$$\operatorname{tr}\big((\tilde{\mu}I-\Re(M^\tau))L\big)=-2\operatorname{tr}(\Re A\Re B)-2\operatorname{tr}(Z^*Z)+\tilde{\mu}\cdot\operatorname{tr}(\Re A+\Re B)\geqslant 0.$$

Thus,

$$\operatorname{tr}(\Re A\Re B)-\operatorname{tr}(Z^*Z)\geqslant\frac{\tilde{\lambda}}{2}\cdot\operatorname{tr}(\Re M)$$

and

$$\operatorname{tr}(\Re A\Re B)+\operatorname{tr}(Z^*Z)\leqslant\frac{\tilde{\mu}}{2}\cdot\operatorname{tr}(\Re M).$$

By Lemmas 2.1 and 2.2,

$$\operatorname{tr}(\Re M)\leqslant\operatorname{tr}(|M|)$$

and

$$\operatorname{tr}(\Re M)\geqslant\frac{1}{\sec(\alpha)}\operatorname{tr}(|M|).$$

Hence, we have

$$\operatorname{tr}(\Re A\Re B)\geqslant\operatorname{tr}(Z^*Z)+\frac{\tilde{\lambda}}{2}\cdot\frac{1}{\sec(\alpha)}\operatorname{tr}(|M|)$$

and

$$\operatorname{tr}(\Re A\Re B)\leqslant\frac{\tilde{\mu}}{2}\cdot\operatorname{tr}(|M|)-\operatorname{tr}(Z^*Z),$$

which complete the proof. □

REMARK 2.4. When M is positive semi-definite (i.e., $\alpha = 0$), our result is inequality (1.10).

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