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LOG–MAJORIZATION OF GAN–LIU–TAM TYPE

JIAN SHI* AND YING DAI

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Abstract. In this paper, we shall obtain several extensions of log-majorization of Gan-Liu-Tam type.

1. Introduction

Throughout this paper, a capital letter, such as T, means an $n\times n$ matrix. We denote $T\geqslant 0$ if T is a positive semidefinite matrix and $T>0$ if T is positive definite, respectively. For $A>0$, $B\geqslant 0$, $0\leqslant\alpha\leqslant 1$, F. Kubo and T. Ando, in [7], introduce the α-power mean of A and B as follows,

$$A\sharp_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}.$$

Usually, $A\sharp_{\frac{1}{2}}B$ is denoted by $A\sharp B$. There are many beautiful properties of the α-power mean. For example, if $0\leqslant A\leqslant C$, $0\leqslant B\leqslant D$, then $A\sharp_t B\leqslant C\sharp_t D$ holds for $t\in[0,1]$. If $A,B\geqslant 0$, T. Ando and F. Hiai, in [1], introduce the following relationship, which is called log-majorization, denoted by $A\underset{(\log)}{\succ} B$, if

$$\prod_{i=1}^{k}\lambda_i(A)\geqslant\prod_{i=1}^{k}\lambda_i(B)\quad (k=1,2,\cdots,n-1)$$

and

$$\prod_{i=1}^{n}\lambda_i(A)=\prod_{i=1}^{n}\lambda_i(B)\quad (i.e.\quad detA=detB)$$

hold, where $\lambda_1(A)\geqslant\lambda_2(A)\geqslant\cdots\geqslant\lambda_n(A)$ and $\lambda_1(B)\geqslant\lambda_2(B)\geqslant\cdots\geqslant\lambda_n(B)$ are the eigenvalues of A and B respectively arranged in decreasing order. There are many perfect log-majorizations, see [2, 6, 9] for details.

Very recently, in [4], L. Gan, X. Liu and T.-Y. Tam obtained the following log-majorization.

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* Corresponding author.

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THEOREM 1.1. ([4], Log-majorization of Gan-Liu-Tam type) *If* $A, B > 0$ *and* $t \in [0,1]$, *we have*

$$A\natural_t B \underset{(\log)}{\prec} (A^{-1}\sharp B)^t A (A^{-1}\sharp B)^t.$$

In this paper, we shall extend the above result in several cases. In order to prove our results, we list some lemmas first.

LEMMA 1.1. ([5, 8], Löwner-Heinz inequality) *If* $A \geqslant B \geqslant 0$, *then*

$$A^p \geqslant B^p$$

holds for all $0 \leqslant p \leqslant 1$.

LEMMA 1.2. ([9, 10, 11], Tanahashi inequality) *If* $A \geqslant B \geqslant 0$ *and* $A > 0$, *we have*

(I) $A^{-t} \geqslant (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^{\frac{-t}{p-t}}$, *for all* $0 \leqslant p < t \leqslant 1,\ p \leqslant \frac{1}{2}$;

(II) $A^{2p-1-t} \geqslant (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^{\frac{2p-1-t}{p-t}}$, *for all* $\frac{1}{2} \leqslant p < t \leqslant 1$.

LEMMA 1.3. ([3], Furuta lemma) *If* $A > 0$ *and* B *is invertible, then*

$$(BAB^*)^s = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{s-1}A^{\frac{1}{2}}B^*$$

holds for all $s \in \mathbb{R}$.

LEMMA 1.4. ([3], Grand Furuta inequality) *If* $A \geqslant B \geqslant 0$ *and* $A > 0$, *then*

$$A^{1-t+r} \geqslant \{A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$$

holds for $0 \leqslant t \leqslant 1,\ p \geqslant 1,\ s \geqslant 1$ *and* $r \geqslant t$.

LEMMA 1.5. *Let* $f(A,B)$ *and* $g(A,B)$ *be positive operator-valued functions for* $A,B > 0$ *satisfying the homogeneity* $f(aI,bI) = g(aI,bI)$ *for* $a,b > 0$. *Then* $\|f(A,B)\| \leqslant \|g(A,B)\|$ *if and only if* $g(A,B) \leqslant I$ *implies* $f(A,B) \leqslant I$.

2. Gan-Liu-Tam type log-majorization in the case of $0 < t \leqslant \frac{1}{2}$ and $\frac{1}{2} < t \leqslant 1$

In this section, we first shall show several extensions of Gan-Liu-Tam type log-majorization in the case of $0 < t \leqslant \frac{1}{2}$.

THEOREM 2.1. (Gan-Liu-Tam type log-majorization in the case of $0 < t \leqslant \frac{1}{2}$) *If* $A,B > 0,\ 0 \leqslant \theta \leqslant 1,\ 0 \leqslant t \leqslant 1,\ 0 \leqslant \alpha \leqslant \frac{1}{2},\ s \geqslant 1,\ p \geqslant 1,\ r \geqslant t,\ h = \frac{(1-t+r)ps\theta}{(p-t)s+r}$, *then*

$$A^{\frac{(1-t+r)\theta}{2}}\{A^{-\frac{r}{2}}[A^{\frac{t}{2}}(A^{-1}\sharp B)^{2\alpha p}A^{\frac{t}{2}}]^sA^{-\frac{r}{2}}\}^{\frac{(1-t+r)\theta}{(p-t)s+r}}A^{\frac{(1-t+r)\theta}{2}} \underset{(\log)}{\prec} \{(A^{-1}\sharp B)^{\alpha}A(A^{-1}\sharp B)^{\alpha}\}^h \tag{2.1}$$

holds and is equivalent to grand Furuta inequality.

Proof. First, we proof that (2.1) can be derived from grand Furuta inequality.

Notice that $\alpha=0$ holds obviously, we only need to prove that under the condition of $0<\alpha\leqslant\frac{1}{2}$. The following identity holds

$$\begin{aligned}&det\big(A^{\frac{(1-t+r)\theta}{2}}\{A^{-\frac{r}{2}}[A^{\frac{t}{2}}(A^{-1}\sharp B)^{2\alpha p}A^{\frac{t}{2}}]^sA^{-\frac{r}{2}}\}^{\frac{(1-t+r)\theta}{(p-t)s+r}}A^{\frac{(1-t+r)\theta}{2}}\big)\\&=det\big(\{(A^{-1}\sharp B)^{\alpha}A(A^{-1}\sharp B)^{\alpha}\}^h\big)\end{aligned}$$

because

$$\begin{aligned}&det\big(\{(A^{-1}\sharp B)^{\alpha}A(A^{-1}\sharp B)^{\alpha}\}^h\big)\\&=\big(det(A^{-1}\sharp B)^{2\alpha}(detA)\big)^h\\&=\big((detA)^{-\frac{1}{2}}(detB)^{\frac{1}{2}}\big)^{2\alpha h}(detA)^h\\&=(detA)^{(1-\alpha)h}(detB)^{\alpha h}\end{aligned}$$

and

$$\begin{aligned}&det\big(A^{\frac{(1-t+r)\theta}{2}}\{A^{-\frac{r}{2}}[A^{\frac{t}{2}}(A^{-1}\sharp B)^{2\alpha p}A^{\frac{t}{2}}]^sA^{-\frac{r}{2}}\}^{\frac{(1-t+r)\theta}{(p-t)s+r}}A^{\frac{(1-t+r)\theta}{2}}\big)\\&=(detA)^{(1-t+r)\theta}\{(detA)^{-r}[(detA)^t\big((detA)^{-\frac{1}{2}}(detB)^{\frac{1}{2}}\big)^{2\alpha p}]^s\}^{\frac{(1-t+r)\theta}{(p-t)s+r}}\\&=(detA)^{\frac{(1-\alpha)(1-t+r)ps\theta}{(p-t)s+r}}(detB)^{\frac{(1-t+r)\alpha ps\theta}{(p-t)s+r}}\\&=(detA)^{(1-\alpha)h}(detB)^{\alpha h}.\end{aligned}$$

Notice that

$$\{\big((xA)^{-1}\sharp(yB)\big)^{\alpha}\big(xA\big)\big((xA)^{-1}\sharp(yB)\big)^{\alpha}\}^h=x^{(1-\alpha)h}y^{\alpha h}\{(A^{-1}\sharp B)^{\alpha}A(A^{-1}\sharp B)^{\alpha}\}^h$$

and

$$\begin{aligned}&(xA)^{\frac{(1-t+r)\theta}{2}}\{(xA)^{-\frac{r}{2}}[(xA)^{\frac{t}{2}}\big((xA)^{-1}\sharp(yB)\big)^{2\alpha p}(xA)^{\frac{t}{2}}]^s(xA)^{-\frac{r}{2}}\}^{\frac{(1-t+r)\theta}{(p-t)s+r}}(xA)^{\frac{(1-t+r)\theta}{2}}\\&=x^{(1-\alpha)h}y^{\alpha h}\{A^{\frac{(1-t+r)\theta}{2}}\{A^{-\frac{r}{2}}[A^{\frac{t}{2}}(A^{-1}\sharp B)^{2\alpha p}A^{\frac{t}{2}}]^sA^{-\frac{r}{2}}\}^{\frac{(1-t+r)\theta}{(p-t)s+r}}A^{\frac{(1-t+r)\theta}{2}}\}\end{aligned}$$

hold for $x,y>0$, that is, $A^{\frac{(1-t+r)\theta}{2}}\{A^{-\frac{r}{2}}[A^{\frac{t}{2}}(A^{-1}\sharp B)^{2\alpha p}A^{\frac{t}{2}}]^sA^{-\frac{r}{2}}\}^{\frac{(1-t+r)\theta}{(p-t)s+r}}A^{\frac{(1-t+r)\theta}{2}}$ and $\{(A^{-1}\sharp B)^{\alpha}A(A^{-1}\sharp B)^{\alpha}\}^h$ have the same order of homogeneity for A,B.

Next, by Lemma 1.5, we shall prove that

$$\{(A^{-1}\sharp B)^{\alpha}A(A^{-1}\sharp B)^{\alpha}\}^h\leqslant I \tag{2.2}$$

ensures

$$A^{\frac{(1-t+r)\theta}{2}}\{A^{-\frac{r}{2}}[A^{\frac{t}{2}}(A^{-1}\sharp B)^{2\alpha p}A^{\frac{t}{2}}]^sA^{-\frac{r}{2}}\}^{\frac{(1-t+r)\theta}{(p-t)s+r}}A^{\frac{(1-t+r)\theta}{2}}\leqslant I. \tag{2.3}$$

Notice that (2.2) is equivalent to $A^{-1}\geqslant(A^{-1}\sharp B)^{2\alpha}$. Let $A_1=A^{-1}$ and $B_1=(A^{-1}\sharp B)^{2\alpha}$. Applying grand Furuta inequality to A_1 and B_1, then

$$\{A_1^{\frac{r}{2}}(A_1^{-\frac{t}{2}}B_1^pA_1^{-\frac{t}{2}})^sA_1^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}\leqslant A_1^{1-t+r} \tag{2.4}$$

holds for $0 \leqslant t \leqslant 1$, $s \geqslant 1$, $p \geqslant 1$ and $r \geqslant t$.

Applying Löwner-Heinz inequality to (2.4) for $0 \leqslant \theta \leqslant 1$, we have

$$\{A_1^{\frac{r}{2}}(A_1^{-\frac{t}{2}}B_1^pA_1^{-\frac{t}{2}})^sA_1^{\frac{r}{2}}\}^{\frac{(1-t+r)\theta}{(p-t)s+r}} \leqslant A_1^{(1-t+r)\theta}, \tag{2.5}$$

which is equivalent to

$$A_1^{-\frac{(1-t+r)\theta}{2}}\{A_1^{\frac{r}{2}}(A_1^{-\frac{t}{2}}B_1^pA_1^{-\frac{t}{2}})^sA_1^{\frac{r}{2}}\}^{\frac{(1-t+r)\theta}{(p-t)s+r}}A_1^{-\frac{(1-t+r)\theta}{2}} \leqslant I, \tag{2.6}$$

(2.6) is just (2.3), if A_1 and B_1 are replaced by A^{-1} and $(A^{-1}\sharp B)^{2\alpha}$, respectively.

Next, we shall show that grand Furuta inequality can be derived from (2.1).

Let $\theta = 1$ in (2.1), we have

$$\{(A^{-1}\sharp B)^{\alpha}A(A^{-1}\sharp B)^{\alpha}\}^{\frac{(1-t+r)ps}{(p-t)s+r}} \leqslant I \tag{2.7}$$

ensures that

$$A^{\frac{1-t+r}{2}}\{A^{-\frac{r}{2}}[A^{\frac{t}{2}}(A^{-1}\sharp B)^{2\alpha p}A^{\frac{t}{2}}]^sA^{-\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{1-t+r}{2}} \leqslant I. \tag{2.8}$$

Notice that (2.7) is equivalent to $A^{-1} \geqslant (A^{-1}\sharp B)^{2\alpha}$. Let $A_1 = A^{-1}$ and $B_1 = (A^{-1}\sharp B)^{2\alpha}$, then $A = A_1^{-1}$, $B = B_1^{\frac{1}{2\alpha}}A_1^{-1}B_1^{\frac{1}{2\alpha}}$ and (2.7) is just that $A_1 \geqslant B_1$. Replacing A by A_1^{-1} and B by $B_1^{\frac{1}{2\alpha}}A_1^{-1}B_1^{\frac{1}{2\alpha}}$ in (2.8), we have

$$A_1^{-\frac{1-t+r}{2}}\{A_1^{\frac{r}{2}}[A_1^{-\frac{t}{2}}\left(A_1\sharp(B_1^{\frac{1}{2\alpha}}A_1^{-1}B_1^{\frac{1}{2\alpha}})\right)^{2\alpha p}A_1^{-\frac{t}{2}}]^sA_1^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A_1^{-\frac{1-t+r}{2}} \leqslant I, \tag{2.9}$$

which is equivalent to

$$\{A_1^{\frac{r}{2}}(A_1^{-\frac{t}{2}}B_1^pA_1^{-\frac{t}{2}})^sA_1^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \leqslant A_1^{1-t+r}. \tag{2.10}$$

(2.10) holds from $A_1 \geqslant B_1$, $0 \leqslant t \leqslant 1$, $s \geqslant 1$, $p \geqslant 1$ and $r \geqslant t$, which is just grand Furuta inequality.

Hence the proof of Theorem 2.1 is completed. □

If we put $p = \frac{1}{2\alpha}$ in Theorem 2.1, we have the following corollary.

COROLLARY 2.1. *If* $A,B > 0$,

$$A^{\frac{(1-t+r)\theta}{2}}\{A^{-\frac{r}{2}}[A^{\frac{t-1}{2}}(A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}A^{\frac{t-1}{2}}]^sA^{-\frac{r}{2}}\}^{\frac{2(1-t+r)\alpha\theta}{(1-2\alpha t)s+2\alpha r}}A^{\frac{(1-t+r)\theta}{2}}$$
$$\underset{(\log)}{\prec} \{(A^{-1}\sharp B)^{\alpha}A(A^{-1}\sharp B)^{\alpha}\}^{\frac{(1-t+r)\theta s}{(1-2\alpha t)s+2\alpha r}}$$

holds for $0 \leqslant \theta \leqslant 1$, $0 \leqslant t \leqslant 1$, $0 \leqslant \alpha \leqslant \frac{1}{2}$, $s \geqslant 1$ *and* $r \geqslant t$.

If we put $t = 1$ in Corollary 2.1, we have the following corollary.

COROLLARY 2.2. *If* $A,B>0$,

$$A^{\frac{\theta r}{2}}\{A^{-\frac{r}{2}}(A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{s}{2}}A^{-\frac{r}{2}}\}^{\frac{2\alpha\theta r}{(1-2\alpha)s+2\alpha r}}A^{\frac{\theta r}{2}} \underset{(\log)}{\prec} \{(A^{-1}\sharp B)^{\alpha}A(A^{-1}\sharp B)^{\alpha}\}^{\frac{\theta rs}{(1-2\alpha)s+2\alpha r}}$$

holds for $0\leqslant\theta\leqslant1$, $0\leqslant\alpha\leqslant\frac{1}{2}$, $s\geqslant1$ *and* $r\geqslant1$.

If we put $s=2$ in Corollary 2.2, we have the following corollary.

COROLLARY 2.3. *If* $A,B>0$,

$$A^{\frac{\theta r}{2}}(A^{\frac{1-r}{2}}BA^{\frac{1-r}{2}})^{\frac{\alpha\theta r}{1-2\alpha+\alpha r}}A^{\frac{\theta r}{2}} \underset{(\log)}{\prec} \{(A^{-1}\sharp B)^{\alpha}A(A^{-1}\sharp B)^{\alpha}\}^{\frac{\theta r}{1-2\alpha+\alpha r}}$$

holds for $0\leqslant\theta\leqslant1$, $0\leqslant\alpha\leqslant\frac{1}{2}$ *and* $r\geqslant1$.

If we put $r=2$ in Corollary 2.3, we have the following corollary.

COROLLARY 2.4. *If* $A,B>0$,

$$A^{\theta}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{2\alpha\theta}A^{\theta} \underset{(\log)}{\prec} \{(A^{-1}\sharp B)^{\alpha}A(A^{-1}\sharp B)^{\alpha}\}^{2\theta}$$

holds for $0\leqslant\theta\leqslant1$ *and* $0\leqslant\alpha\leqslant\frac{1}{2}$.

REMARK 2.1. If we put $\theta=\frac{1}{2}$ and replace α by t in Corollary 2.4, it is just Gan-Liu-Tam type log-majorization under the condition of $0\leqslant t\leqslant\frac{1}{2}$.

Next, we shall show an extension of Gan-Liu-Tam type log-majorization in the case of $\frac{1}{2}<t\leqslant1$.

THEOREM 2.2. (Gan-Liu-Tam type log-majorization in the case of $\frac{1}{2}<t\leqslant1$) *If* $A,B>0$, *then*

$$A\sharp_t B \underset{(\log)}{\prec} (A^{-1}\sharp_\alpha B)^{\frac{t}{2\alpha}}A^{1-2t+\frac{t}{\alpha}}(A^{-1}\sharp_\alpha B)^{\frac{t}{2\alpha}} \tag{2.11}$$

holds for $0<\alpha\leqslant\frac{1}{2}<t\leqslant1$.

Proof. First, it is easy to obtain $det(A\sharp_t B)=det((A^{-1}\sharp_\alpha B)^{\frac{t}{2\alpha}}A^{1-2t+\frac{t}{\alpha}}(A^{-1}\sharp_\alpha B)^{\frac{t}{2\alpha}})$ because $det(A\sharp_t B)=(detA)^{1-t}(detB)^t$ and

$$\begin{aligned}
&det((A^{-1}\sharp_\alpha B)^{\frac{t}{2\alpha}}A^{1-2t+\frac{t}{\alpha}}(A^{-1}\sharp_\alpha B)^{\frac{t}{2\alpha}})\\
=&det((A^{-1}\sharp_\alpha B)^{\frac{t}{\alpha}})det(A^{1-2t+\frac{t}{\alpha}})\\
=&[(detA)^{\alpha-1}(detB)^{\alpha}]^{\frac{t}{\alpha}}det(A^{1-2t+\frac{t}{\alpha}})\\
=&(detA)^{1-t}(detB)^t.
\end{aligned}$$

Thus, by Lemma 1.5, we only need to prove that

$$(A^{-1}\sharp_\alpha B)^{\frac{t}{2\alpha}}A^{1-2t+\frac{t}{\alpha}}(A^{-1}\sharp_\alpha B)^{\frac{t}{2\alpha}}\leqslant I \tag{2.12}$$

ensures that

$$A\sharp_t B \leqslant I. \tag{2.13}$$

(2.12) is equivalent to $A^{1-2t+\frac{t}{\alpha}} \leqslant (A^{-1}\sharp_\alpha B)^{-\frac{t}{\alpha}}$. Let $A_1 = (A^{-1}\sharp_\alpha B)^{-\frac{t}{\alpha}}$, $B_1 = A^{1-2t+\frac{t}{\alpha}}$, and $q = \frac{\alpha}{t}$, $c = (1-2t+\frac{t}{\alpha})^{-1}$. Then we have

$$B_1 \leqslant A_1, \tag{2.14}$$

$$A = B_1^c, \tag{2.15}$$

and

$$B = B_1^{-\frac{c}{2}}(B_1^{\frac{c}{2}}A_1^{-q}B_1^{\frac{c}{2}})^{\frac{1}{\alpha}}B_1^{-\frac{c}{2}}. \tag{2.16}$$

Next, we shall prove (2.13) holds in two cases.

Case I. If $t \geqslant 2\alpha$, let $\beta = 2-\frac{1}{t}$. Then $0 \leqslant \beta \leqslant 1$ and $\frac{1}{\alpha} = \frac{c\beta}{c-q}$. Hence we have

$$B_1^{\frac{c-q}{\alpha}} = B_1^{c\beta} \geqslant (B_1^{\frac{c}{2}}A_1^{-q}B_1^{\frac{c}{2}})^{\frac{c\beta}{c-q}} = (B_1^{\frac{c}{2}}A_1^{-q}B_1^{\frac{c}{2}})^{\frac{1}{\alpha}}$$

by Lemma 1.1 and Lemma 1.2.

Moreover, it implies that

$$B = B_1^{-\frac{c}{2}}(B_1^{\frac{c}{2}}A_1^{-q}B_1^{\frac{c}{2}})^{\frac{1}{\alpha}}B_1^{-\frac{c}{2}} \leqslant B_1^{-\frac{c}{2}}B_1^{\frac{c-q}{\alpha}}B_1^{-\frac{c}{2}} = B_1^{\frac{c-q}{\alpha}-c}.$$

Consequently, we have

$$A\sharp_t B \leqslant B_1^c \sharp_t B_1^{\frac{c-q}{\alpha}-c} = I.$$

Case II. If $t \leqslant 2\alpha$, let $\gamma = \frac{c-q}{\alpha(1+c-2q)}$. Then $0 \leqslant \gamma \leqslant 1$ holds for $(1-2\alpha)(\alpha+t^2-2\alpha t) \geqslant 0$. Hence we have

$$B_1^{\gamma(1+c-2q)} \geqslant (B_1^{\frac{c}{2}}A_1^{-q}B_1^{\frac{c}{2}})^{\gamma\frac{2q-c-1}{q-c}} = (B_1^{\frac{c}{2}}A_1^{-q}B_1^{\frac{c}{2}})^{\frac{1}{\alpha}}$$

by Lemma 1.1 and Lemma 1.2.

Moreover, it implies that

$$B = B_1^{-\frac{c}{2}}(B_1^{\frac{c}{2}}A_1^{-q}B_1^{\frac{c}{2}})^{\frac{1}{\alpha}}B_1^{-\frac{c}{2}} \leqslant B_1^{-\frac{c}{2}}B_1^{\gamma(1+c-2q)}B_1^{-\frac{c}{2}} = B_1^{\frac{c-q}{\alpha}-c}.$$

Consequently, we have

$$A\sharp_t B \leqslant B_1^c \sharp_t B_1^{\frac{c-q}{\alpha}-c} = I.$$

Hence the proof of Theorem 2.2 is completed. □

REMARK 2.2. If we put $\alpha = \frac{1}{2}$, Theorem 2.2 is just Gan-Liu-Tam type log-majorization in the case of $\frac{1}{2} < t \leqslant 1$.

3. A generalization of Gan-Liu-Tam type log-majorization

In this section, we shall show a generalization of Gan-Liu-Tam type log-majorization for any $t \in [0,1]$.

THEOREM 3.1. *If* $A,B>0$, *then*

$$A^{t(2\alpha-1)}(A\sharp_{2\alpha t}B)A^{t(2\alpha-1)} \underset{(\log)}{\prec} (A^{-1}\sharp_{\alpha}B)^t A (A^{-1}\sharp_{\alpha}B)^t \tag{3.1}$$

holds for $\frac{1}{2} \leqslant \alpha \leqslant 1,\ 0 \leqslant 2\alpha t \leqslant 1$.

Proof. First, it is easy to obtain

$$det(A^{t(2\alpha-1)}(A\sharp_{2\alpha t}B)A^{t(2\alpha-1)}) = det((A^{-1}\sharp_{\alpha}B)^t A (A^{-1}\sharp_{\alpha}B)^t)$$

because

$$\begin{aligned} & det(A^{t(2\alpha-1)}(A\sharp_{2\alpha t}B)A^{t(2\alpha-1)}) \\ = & (detA)^{2t(2\alpha-1)}[(detA)^{1-2\alpha t}(detB)^{2\alpha t}] \\ = & (detA)^{2\alpha t-2t+1}(detB)^{2\alpha t} \end{aligned}$$

and

$$\begin{aligned} & det((A^{-1}\sharp_{\alpha}B)^t A (A^{-1}\sharp_{\alpha}B)^t) \\ = & det(A^{-1}\sharp_{\alpha}B)^{2t} detA \\ = & [(detA)^{\alpha-1}(detB)^{\alpha}]^{2t} detA \\ = & (detA)^{2\alpha t-2t+1}(detB)^{2\alpha t}. \end{aligned}$$

Thus, by Lemma 1.5, we only need to prove that

$$(A^{-1}\sharp_{\alpha}B)^t A (A^{-1}\sharp_{\alpha}B)^t \leqslant I \tag{3.2}$$

ensures that

$$A^{t(2\alpha-1)}(A\sharp_{2\alpha t}B)A^{t(2\alpha-1)} \leqslant I. \tag{3.3}$$

Notice that (3.2) is equivalent to $A \leqslant (A^{-1}\sharp_{\alpha}B)^{-2t}$. Let $A_1 = (A^{-1}\sharp_{\alpha}B)^{-2t}$.

Lemmas 1.3 and 1.1 imply that, since $0 \leqslant \frac{1}{\alpha}-1 \leqslant 1$ and $A \leqslant A_1$,

$$B = A^{-\frac{1}{2}}(A^{\frac{1}{2}}A_1^{-\frac{1}{2t}}A^{\frac{1}{2}})^{\frac{1}{\alpha}}A^{-\frac{1}{2}} = A_1^{-\frac{1}{4t}}(A_1^{-\frac{1}{4t}}AA_1^{-\frac{1}{4t}})^{\frac{1}{\alpha}-1}A_1^{-\frac{1}{4t}} \leqslant A_1^{(1-\frac{1}{2t})(\frac{1}{\alpha}-1)-\frac{1}{2t}}.$$

Therefore, we have

$$A\sharp_{2\alpha t}B \leqslant A_1\sharp_{2\alpha t}A_1^{(1-\frac{1}{2t})(\frac{1}{\alpha}-1)-\frac{1}{2t}} = A_1^{2t-4\alpha t} \leqslant A^{2t-4\alpha t}$$

because $0 \leqslant 4\alpha t-2t \leqslant 1$ and so $-1 \leqslant 2t-4\alpha t \leqslant 0$. Finally it follows that

$$A^{t(2\alpha-1)}(A\sharp_{2\alpha t}B)A^{t(2\alpha-1)} \leqslant A^{t(2\alpha-1)}A^{2t-4\alpha t}A^{t(2\alpha-1)} = I.$$

Hence the proof of Theorem 3.1 is completed. □

REMARK 3.1. If we put $\alpha = \frac{1}{2}$, Theorem 3.1 is just Gan-Liu-Tam type log-majorization.

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Jian Shi
Hebei Key Laboratory of Machine Learning and Computational Intelligence
College of Mathematics and Information Science, Hebei University
Baoding, 071002, P.R. China
e-mail: mathematic@126.com
shijianmath@qq.com

Ying Dai
Hebei Key Laboratory of Machine Learning and Computational Intelligence
College of Mathematics and Information Science, Hebei University
Baoding, 071002, P.R. China
e-mail: eightdaiying@qq.com