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THE WEIGHTED AND THE DAVIS–WIELANDT BEREZIN NUMBER

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Abstract. A functional Hilbert space is the Hilbert space of complex-valued functions on some set $\Theta \subseteq \mathbb{C}$ that the evaluation functionals $\varphi_\lambda(f) = f(\lambda)$, $\lambda \in \Theta$ are continuous on $\mathscr{H}$. The Berezin number of an operator T is defined by $\mathbf{ber}(T) = \sup_{\lambda\in\Theta} \left|\widetilde{T}(\lambda)\right| = \sup_{\lambda\in\Theta} \left|\langle T\hat{k}_\lambda, \hat{k}_\lambda\rangle\right|$, where the operator T acts on the reproducing kernel Hilbert space $\mathscr{H} = \mathscr{H}(\Theta)$ over some (non-empty) set Θ. In this paper, we defined the weighted Berezin radius and the weighted Berezin norms and then we obtain some related inequalities. It is shown, among other inequalities, that if $T \in \mathscr{L}(\mathscr{H})$ and $t \in [0,1]$, then

$$\mathbf{ber}_t^2(T) \leqslant (1-2t+2t^2)\|TT^* + T^*T\|_{\mathbf{ber},1} + (1-2t)\mathbf{ber}\left(T^2 + T^{*2}\right).$$

Moreover, we generalize the Davis-Wielandt Berezin number and present some inequalities involving this definition.

1. Introduction

Let $\mathscr{L}(\mathscr{H})$ be the Banach algebra of all bounded linear operators defined on a complex Hilbert space $(\mathscr{H}; \langle\cdot,\cdot\rangle)$ with the identity operator $1_{\mathscr{H}}$ in $\mathscr{L}(\mathscr{H})$. When $\mathscr{H} = \mathbb{C}^n$, we identify $\mathscr{L}(\mathscr{H})$ with the algebra $\mathscr{M}_n(\mathbb{C})$ of n-by-n complex matrices. Recall that the numerical range and the numerical radius of $T \in \mathscr{L}(\mathscr{H})$ are defined respectively, by

$$W(T) := \{\langle Tx, x\rangle : x \in \mathscr{H} \text{ and } \|x\| = 1\},$$

and

$$w(T) := \sup\{|\langle Tx, x\rangle| : \langle Tx, x\rangle \in W(T)\}.$$

For more facts about the numerical radius, we refer the reader to [7, 25, 26, 27] and references therein.

A functional Hilbert space is the Hilbert space of complex-valued functions on some set $\Theta \subseteq \mathbb{C}$ that the evaluation functionals $\varphi_\lambda(f) = f(\lambda)$, $\lambda \in \Theta$ are continuous on $\mathscr{H}$. Then, by the Riesz representation theorem there is a unique element $k_\lambda \in \mathscr{H}$

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such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathscr{H}$ and every $\lambda \in \Theta$. The function k on $\Theta \times \Theta$ defined by $k(z,\lambda) = k_\lambda(z)$ is called the reproducing kernel of $\mathscr{H}$, see [2]. It was shown that $k_\lambda(z)$ can be represented by

$$k_\lambda(z) = \sum_{n=1}^{\infty} \overline{e_n(\lambda)} e_n(z)$$

for any orthonormal basis $\{e_n\}_{n \geqslant 1}$ of $\mathscr{H}$, see [31]. For example, for the Hardy-Hilbert space $\mathscr{H}^2 = \mathscr{H}^2(\mathbb{D})$ over the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\{z^n\}_{n \geqslant 1}$ is an orthonormal basis, therefore the reproducing kernel of $\mathscr{H}^2$ is the function $k_\lambda(z) = \sum_{n=1}^{\infty} \overline{\lambda_n} z^n = \left(1 - \overline{\lambda} z\right)^{-1}$, $\lambda \in \mathbb{D}$. Let $\widehat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$ be the normalized reproducing kernel of the space $\mathscr{H}$. For a given a bounded linear operator T on $\mathscr{H}$, the Berezin symbol (or Berezin transform) of T is the bounded function $\widetilde{T}$ on Θ defined by

$$\widetilde{T}(\lambda) = \left\langle T\widehat{k}_\lambda(z), \widehat{k}_\lambda(z) \right\rangle, \ \lambda \in \Theta.$$

An important property of the Berezin symbol is that for all $T, S \in \mathscr{L}(\mathscr{H})$ if $\widetilde{T}(\lambda) = \widetilde{S}(\lambda)$ for all $\lambda \in \Theta$, then $T = S$(at least when $\mathscr{H}$ consists from analytic functions, see Zhu [35]). For more details, see [5, 6, 9, 10, 13, 14, 15, 20]–[24]. So, the map $T \to \widetilde{T}$ is injective [16]. The Berezin set and the Berezin number of an operator T are defined, respectively, by

$$\mathbf{Ber}(T) = \left\{ \widetilde{T}(\lambda) : \lambda \in \Theta \right\} = \text{Range}\left(\widetilde{T}\right),$$

and

$$\mathbf{ber}(T) = \sup\{|\gamma| : \gamma \in \text{Ber}(T)\} = \sup_{\lambda \in \Theta} \left| \widetilde{T}(\lambda) \right|.$$

The Crawford Berezin number of the operator T is defined by (see [21])

$$c_{\mathbf{ber}}(T) := \inf\{|\widetilde{T}(\lambda)| : \lambda \in \Theta\}.$$

The 1-Berezin norm of an operator $T \in \mathscr{L}(\mathscr{H})$ is defined by

$$\|T\|_{\text{ber},1} := \sup_{\lambda \in \Theta} \left\| T\widehat{k}_\lambda \right\|.$$

One can define also the 2-Berezin norm of T by the formula

$$\|T\|_{\text{ber},2} := \sup \left\{ \left| \langle T\widehat{k}_\lambda, \widehat{k}_\mu \rangle \right| : \lambda, \mu \in \Theta \right\}.$$

Clearly, $\|T\|_{\text{ber},2} \leqslant \|T\|_{\text{ber},1}$.

For $T, S \in \mathscr{L}(\mathscr{H})$, it is clear from the above definitions of the Berezin radius (or the Berezin number) and the Berezin norms that the following properties hold:

(1) $\mathbf{ber}(\alpha T) = |\alpha|\mathbf{ber}(T)$ for all $\alpha \in \mathbb{C}$,

(2) $\mathbf{ber}(T+S) \leqslant \mathbf{ber}(T) + \mathbf{ber}(S)$,

(3) $\mathbf{ber}(T) \leqslant \|T\|_{\text{ber},2} \leqslant \|T\|_{\text{ber},1}$,

(4) $\|\alpha T\|_{\text{ber},i} = |\alpha|\,\|T\|_{\text{ber},i}$ for all $\alpha \in \mathbb{C}$ and $i = 1,2$,

(5) $\|T+S\|_{\text{ber},i} \leqslant \|T\|_{\text{ber},i} + \|S\|_{\text{ber},i}$, $i = 1,2$,

(6) $\|T\|_{\text{ber},i} = \|T^*\|_{\text{ber},i}$ and $\mathbf{ber}(T) = \mathbf{ber}(T^*)$.

The Cartesian decomposition of an operator $T \in \mathscr{L}(\mathscr{H})$ can be written as $T = \Re(T) + i\Im(T)$, where $\Re(T) = \frac{T+T^*}{2}$ and $\Im(T) = \frac{T-T^*}{2i}$. A generalization of this decomposition was introduced in [29], called weighted real and imaginary part of T defined as:

$$\Re_t(T) = (1-t)T^* + tT \quad \text{and} \quad \Im_t(T) = \frac{(1-t)T - tT^*}{i} \quad \text{for all } t \in [0,1].$$

Obviously, for $t = \frac{1}{2}$, $\Re_t(T) = \Re(T)$ and $\Im_t(T) = \Im(T)$. It is easy to see that for every operator $T \in \mathscr{L}(\mathscr{H})$, $\Re_t(T) + i\Im_t(T) = (1-2t)T^* + T$. In [25], the authors defined the so-called weighted numerical radius by the formula:

$$\begin{aligned} w_t(T) &:= \sup_{\|x\|=1} |\langle \Re_t(T) + i\Im_t(T)x, x\rangle| \\ &= w((1-2t)T^* + T) \quad \text{for } T \in \mathscr{L}(\mathscr{H}) \text{ and } t \in [0,1]. \end{aligned}$$

Also, in [11], Conde et al. introduced the weighted numerical radius in the following way (see also Nayak [28]):

$$w_t(T) := \sup_{\theta \in \mathbb{R}} \left\| \Re_t(e^{i\theta}T) \right\|.$$

Similarly, if $\mathscr{H} = \mathscr{H}(\Theta)$ is a reproducing kernel Hilbert space, for $T \in \mathscr{L}(\mathscr{H})$ and $t \in [0,1]$, we define the weighed Berezin radius and the weighted Berezin norms by the following formulas, respectively:

$$\mathbf{ber}_t(T) := \sup_{\lambda \in \Theta} \left| \left\langle \Re_t(T) + i\Im_t(T)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| = \mathbf{ber}((1-2t)T^* + T),$$

$$\|T\|_{\mathbf{ber},1,t} := \sup_{\lambda \in \Theta} \left\| (\Re_t(T) + i\Im_t(T))\widehat{k}_\lambda \right\|$$

and

$$\|T\|_{\mathbf{ber},2,t} := \sup_{\lambda,\mu \in \Theta} \left| \left\langle (\Re_t(T) + i\Im_t(T))\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle \right|.$$

It is obvious that $\|T\|_{\mathbf{ber},i,t} = \|(1-2t)T^* + T\|_{\mathbf{ber},i}$ and $\mathbf{ber}_t(T) \leqslant \|T\|_{\mathbf{ber},2,t} \leqslant \|T\|_{\mathbf{ber},1,t}$. Similar to the Berezin radius inequality, the weighted Berezin radius also satisfies the triangle inequality

$$\mathbf{ber}_t(T+S) \leqslant \mathbf{ber}_t(T) + \mathbf{ber}_t(S) \quad \text{for } T,S \in \mathscr{L}(\mathscr{H}).$$

One can easily observe that for $t = \frac{1}{2}$, $\mathbf{ber}_t(T) = \mathbf{ber}(T)$ and $\|T\|_{\mathbf{ber},i,t} = \|T\|_{\mathbf{ber},i}$ for $i = 1,2$.

Moreover, one of the most less common celebrated generalization of the numerical range and the numerical radius is the Davis-Wielandt shell and its radius of $T \in \mathscr{L}(\mathscr{H})$, which are defined as:

$$DW(T) := \{(\langle Tx,x\rangle, \langle Tx,Tx\rangle), x \in \mathscr{H}, \|x\| = 1\},$$

and

$$dw_2(T) = \sup_{x \in \mathscr{H}, \|x\|=1} \left\{ \sqrt{|\langle Tx,x\rangle|^2 + \|Tx\|^4} \right\}. \tag{1}$$

It is easy to see that the Davis-Wielandt radius is not a norm. It has many properties that you can refer to reference [34]. The following inequality immediately comes from (1):

$$\max(w(T), \|T\|^2) \leqslant dw(T) \leqslant \sqrt{w^2(T) + \|T\|^4}$$

for any $T \in \mathscr{L}(\mathscr{H})$. Clearly, the projection of the set $DW(T)$ on the first coordinate is $W(T)$. One can easily check that $dw(T)$ is unitarily invariant but it does not define a norm on $\mathscr{L}(\mathscr{H})$. Several properties and generalizations of the Berezin number and the Davis-Wielandt radius have been given; see [3, 4, 17, 18, 32]

The following well known lemmas will let essential to prove our results. We start with the Buzano inequality.

LEMMA 1. [8] *Let $x,y,e \in \mathscr{H}$ with $\|e\| = 1$. Then*

$$|\langle x,e\rangle\langle e,x\rangle| \leqslant \frac{1}{2}\left(|\langle x,y\rangle| + \|x\|\|y\|\right).$$

Next lemma is the McCarthy inequality for positive operators.

LEMMA 2. (McCarthy inequality) *Let $T \in \mathscr{L}(\mathscr{H})$ be a positive operator. Then for all unit vector $x \in \mathscr{H}$ we have*

$$\langle Tx,x\rangle^r \leqslant \langle T^r x,x\rangle,$$

where $r \geqslant 1$. This inequality is reversed if $0 < r \leqslant 1$.

The generalized mixed Schwarz inequality was introduced in [19], as follows:

LEMMA 3. [25, Theorem 1] *Let $T \in \mathscr{L}(\mathscr{H})$ and $x, y \in \mathscr{H}$ be any vectors. If $0 \leqslant r \leqslant 1$, then*

$$|\langle Tx, y\rangle|^2 \leqslant \left\langle |T|^{2r} x, x\right\rangle \left\langle |T^*|^{2(1-r)} y, y\right\rangle.$$

In this paper, we study the weighted Berezin radius and the weighted Berezin norms and then we obtain some related inequalities. Further, we generalize the Davis-Wielandt Berezin number and present some inequalities involving this definition.

2. Some inequalities for the weighted Berezin radius and weighted Berezin norms

In the present section, we prove some new inequalities for the weighted Berezin radius and the weighted Berezin norms of operators on the reproducing kernel Hilbert space $\mathscr{H} = \mathscr{H}(\Theta)$. Our first result of this section is the following.

THEOREM 1. *Let $T \in \mathscr{L}(\mathscr{H})$ and $t \in [0,1]$. Then*

$$\boldsymbol{ber}_t^2(T) \leqslant (1-2t)^2 \boldsymbol{ber}^2(T) + (1-2t)\boldsymbol{ber}(T^2) + (1-t)\|T^*T + TT^*\|_{\boldsymbol{ber},1}.$$

Proof. Let $\lambda \in \Theta$ be arbitrary. Then we have

$$\left|\left\langle ((1-2t)T^* + T)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| \leqslant (1-2t)\left|\left\langle T^*\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| + \left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|,$$

whence

$$\begin{aligned}
&\left|\left\langle ((1-2t)T^* + T)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^2 \\
&\leqslant (1-2t)^2\left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^2 + \left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^2 \\
&\quad + 2(1-2t)\left|\left\langle T^*\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| \\
&\leqslant (1-2t)^2\left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^2 + \left|\left\langle |T|\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\left|\left\langle |T^*|\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| \\
&\quad + (1-2t)\left[\left|\left\langle T\widehat{k}_\lambda, T^*\widehat{k}_\lambda\right\rangle\right| + \left\|T\widehat{k}_\lambda\right\|\left\|T^*\widehat{k}_\lambda\right\|\right] \\
&\qquad\qquad \text{(using Lemmas 1 and 3)} \\
&\leqslant (1-2t)^2\left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^2 + \frac{1}{2}\left|\left\langle (|T|^2 + |T^*|^2)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| \\
&\quad + (1-2t)\left|\left\langle T^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| + \frac{1}{2}(1-2t)\left|\left\langle (|T|^2 + |T^*|^2)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| \\
&\qquad\qquad \text{(using Lemma 2)} \\
&\leqslant (1-2t)^2\mathbf{ber}^2(T) + (1-2t)\mathbf{ber}(T^2) + (1-t)\|T^*T + TT^*\|_{\mathbf{ber},1}.
\end{aligned}$$

Now, taking the supremum over $\lambda \in \Theta$ we get the required bound. This proves the theorem. □

REMARK 1. If we take $t = \frac{1}{2}$ in Theorem 1, then we get (see [4])

$$\mathbf{ber}^2(T) \leqslant \frac{1}{2}\|T^*T + TT^*\|_{\mathbf{ber},1}.$$

Next, we prove an inequality involving the weighted norm.

PROPOSITION 2. *Let $T,S \in \mathscr{L}(\mathscr{H})$ and $t \in [0,1]$. Then*

$$\|TS\|^2_{\boldsymbol{ber},1,t} \leqslant (2-4t+4t^2)\|TS\|^2_{\boldsymbol{ber},1} + (1-2t)\boldsymbol{ber}\left((TS)^2\right) + (1-2t)\boldsymbol{ber}\left((S^*T^*)^2\right).$$

Proof. Let $\lambda \in \Theta$. By a simple calculation we get

$$\begin{aligned}
&\left\|((1-2t)(TS)^* + TS)\widehat{k}_\lambda\right\|^2 \\
&= \left\langle((1-2t)(TS)^* + TS)\widehat{k}_\lambda, ((1-2t)(TS)^* + TS)\widehat{k}_\lambda\right\rangle \\
&= (1-2t)^2\|(TS)^*\widehat{k}_\lambda\|^2 + (1-2t)\left\langle(TS)^*\widehat{k}_\lambda, TS\widehat{k}_\lambda\right\rangle \\
&\quad + (1-2t)\left\langle TS\widehat{k}_\lambda, (TS)^*\widehat{k}_\lambda\right\rangle + \|TS\widehat{k}_\lambda\|^2 \\
&\leqslant (2-4t+4t^2)\|TS\|^2_{\mathbf{ber},1} + (1-2t)\mathbf{ber}\left((TS)^2\right) + (1-2t)\mathbf{ber}\left((S^*T^*)^2\right).
\end{aligned}$$

Taking the supremum over $\lambda \in \Theta$, we get our required inequality. □

In the following theorem, we get an upper bound for the weighted Berezin radius which improves the inequality $\mathbf{ber}(T) \leqslant \|T\|_{\mathbf{ber},1}$.

THEOREM 3. *Let $T \in \mathscr{L}(\mathscr{H})$ and $t \in [0,1]$. Then*

$$\boldsymbol{ber}_t^2(T) \leqslant (1-2t+2t^2)\|TT^* + T^*T\|_{\boldsymbol{ber},1} + (1-2t)\boldsymbol{ber}\left(T^2 + T^{*2}\right).$$

Proof. Let $\lambda \in \Theta$ be arbitrary. By applying Lemma 3, we have:

$$\begin{aligned}
&\left|\left\langle[(1-2t)T^* + T]\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^2 \\
&\leqslant \left\langle|(1-2t)T^* + T|\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\left\langle|(1-2t)T^* + T|\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \\
&\leqslant \frac{1}{2}\left[\left\langle|(1-2t)T^* + T|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle + \left\langle|(1-2t)T^* + T|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right] \\
&\qquad\qquad\qquad\qquad \text{(using Lemma 2)} \\
&= \frac{1}{2}\left[(1-2t)^2\left\langle(TT^* + T^*T)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle + \left\langle(TT^* + T^*T)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right] \\
&\quad + (1-2t)\left\langle(T^2 + T^{*2})\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \\
&\leqslant (1-2t+2t^2)\,\|TT^* + T^*T\|_{\mathbf{ber},1} + (1-2t)\mathbf{ber}\left(T^2 + T^{*2}\right).
\end{aligned}$$

Taking the supremum over $\lambda \in \Theta$, we get the desired inequality. $\square$

The following well-known Generalized Polarization Identity will be used in the sequel.

LEMMA 4. *Let $T \in \mathscr{L}(\mathscr{H})$ and $x, y \in \mathscr{H}$. Then*

$$\langle Tx, y\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \left\langle T(x+i^k y), x+i^k y\right\rangle.$$

For $T \in \mathscr{L}(\mathscr{H})$ its so-called the Aluthge transformation $\widehat{T}$ is defined by $\widehat{T} := |T|^{1/2}U|T|^{1/2}$, where $|T| := (T^*T)^{1/2}$ and U is the partial isometry associated with the polar decomposition $T = U|T|$ and $\mathbf{ker}(T) = \mathbf{ker}(U)$.

THEOREM 4. *Let $T \in \mathscr{L}(\mathscr{H})$ and $t \in [0,1]$. Then*

$$\boldsymbol{ber}_t(T) \leqslant (1-t)\left(\|T\|_{\boldsymbol{ber},2} + \boldsymbol{ber}\left(\widehat{T}\right)\right).$$

Proof. Let $\lambda, \mu \in \Theta$ be arbitrary points. Assume $T = U|T|$ be the polar decomposition of T. Then for every $\theta \in \mathbb{R}$ we have

$$\begin{aligned}
&\Re\left\langle e^{i\theta}\left[(1-2t)T^* + T\right]\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle \\
&= (1-2t)\Re\left\langle e^{i\theta}T^*\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle + \Re\left\langle e^{i\theta}T\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle \\
&= (1-2t)\Re\left\langle e^{i\theta}|T|U^*\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle + \Re\left\langle e^{i\theta}U|T|\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle \\
&= (1-2t)\Re\left\langle e^{-i\theta}|T|\widehat{k}_\lambda, U^*\widehat{k}_\mu\right\rangle + \Re\left\langle e^{i\theta}|T|\widehat{k}_\lambda, U^*\widehat{k}_\mu\right\rangle \\
&\qquad\qquad\qquad\qquad (\text{since } \Re z = \Re \bar{z}) \\
&= \frac{1-2t}{4}\left\langle |T|(e^{-i\theta}+U^*)\widehat{k}_\lambda, (e^{-i\theta}+U^*)\widehat{k}_\mu\right\rangle \\
&\quad - \frac{1-2t}{4}\left\langle |T|(e^{-i\theta}-U^*)\widehat{k}_\lambda, (e^{-i\theta}-U^*)\widehat{k}_\mu\right\rangle \\
&\quad + \frac{1}{4}\left\langle |T|(e^{i\theta}+U^*)\widehat{k}_\lambda, (e^{i\theta}+U^*)\widehat{k}_\mu\right\rangle - \frac{1}{4}\left\langle |T|(e^{i\theta}+U^*)\widehat{k}_\lambda, (e^{i\theta}+U^*)\widehat{k}_\mu\right\rangle \\
&\qquad\qquad\qquad\qquad (\text{using Lemma } 4) \\
&= \frac{1-2t}{4}\left\langle (e^{i\theta}+U)|T|(e^{-i\theta}+U^*)\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle \\
&\quad - \frac{1-2t}{4}\left\langle (e^{i\theta}-U)|T|(e^{-i\theta}-U^*)\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle \\
&\quad + \frac{1}{4}\left\langle (e^{-i\theta}+U)|T|(e^{i\theta}+U^*)\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle - \frac{1}{4}\left\langle (e^{-i\theta}+U)|T|(e^{i\theta}+U^*)\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle \\
&\leqslant \frac{1-2t}{4}\left\|(e^{i\theta}-U)|T|(e^{-i\theta}-U^*)\right\|_{\mathbf{ber},2} + \frac{1}{2}\left\|(e^{-i\theta}+U)|T|(e^{i\theta}+U^*)\right\|_{\mathbf{ber},2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1-2t}{4}\left\| |T|^{1/2}(e^{-i\theta}-U^*)(e^{i\theta}-U)|T|^{1/2}\right\|_{\mathbf{ber},2} \\
&\quad + \frac{1}{2}\left\| |T|^{1/2}(e^{i\theta}+U^*)(e^{-i\theta}+U)|T|^{1/2}\right\|_{\mathbf{ber},2} \\
&\qquad\qquad\qquad\qquad (\text{using the fact that } \|S^*S\|_{\mathbf{ber},2} = \|SS^*\|_{\mathbf{ber},2}) \\
&= \frac{1-2t}{4}\left\| 2|T| + e^{-i\theta}|T|^{1/2}U|T|^{1/2} + e^{i\theta}|T|^{1/2}U^*|T|^{1/2}\right\|_{\mathbf{ber},2} \\
&\quad + \frac{1}{4}\left\| 2|T| + e^{i\theta}|T|^{1/2}U|T|^{1/2} + e^{-i\theta}|T|^{1/2}U^*|T|^{1/2}\right\|_{\mathbf{ber},2} \\
&= \frac{1-2t}{2}\left\| |T| + \Re\left(e^{-i\theta}\widehat{T}\right)\right\|_{\mathbf{ber},2} + \frac{1}{2}\left\| |T| + \Re\left(e^{i\theta}\widehat{T}\right)\right\|_{\mathbf{ber},2} \\
&\leqslant \frac{1-2t}{2}\left(\|T\|_{\mathbf{ber},2} + \mathbf{ber}\left(\widehat{T}\right)\right) + \frac{1}{2}\left(\|T\|_{\mathbf{ber},2} + \mathbf{ber}\left(\widehat{T}\right)\right) \\
&= (1-t)\left(\|T\|_{\mathbf{ber},2} + \mathbf{ber}\left(\widehat{T}\right)\right). \quad \square
\end{aligned}
$$

COROLLARY 1. *Let $T \in \mathscr{L}(\mathscr{H})$. Then*

$$
\boldsymbol{ber}(T) \leqslant \frac{1}{2}\left(\|T\|_{\boldsymbol{ber},2} + \boldsymbol{ber}\left(\widehat{T}\right)\right).
$$

Proof. If we take $t = \frac{1}{2}$ in Theorem 4, then we will get the desired result. $\square$

REMARK 2. We remark that Corollary 1 is an analog of the famous Yamazaki inequality [33]

$$
w(T) \leqslant \frac{1}{2}\left(\|T\| + w\left(\widehat{T}\right)\right).
$$

LEMMA 5. *Let $T \in \mathscr{L}(\mathscr{H})$. Then the function $g(t) = \boldsymbol{ber}_t(T)$ is convex on the interval* $[0,1]$.

Proof. Assume $\nu, \lambda, \mu \in [0,1]$. Then by the definition of the weighted Berezin number we have

$$
\begin{aligned}
&g(\lambda\nu + (1-\lambda)\mu) \\
&= \mathbf{ber}_{\lambda\nu+(1-\lambda)\mu}(T) \\
&= \mathbf{ber}\big(1 - 2(\lambda\nu + (1-\lambda)\mu)T^* + T\big) \\
&= \mathbf{ber}\big((\lambda - 2\lambda\nu)T^* + \lambda T + [(1-\lambda) - 2(1-\lambda)\mu]T^* + (1-\lambda)T\big) \\
&\leqslant \mathbf{ber}\big((\lambda - 2\lambda\nu)T^* + \lambda T\big) + \mathbf{ber}\big([(1-\lambda) - 2(1-\lambda)\mu]T^* + (1-\lambda)T\big) \\
&= \lambda\,\mathbf{ber}\big((1-2\nu)T^* + T\big) + (1-\lambda)\mathbf{ber}\big((1-2\mu)T^* + T\big) \\
&= \lambda g(\nu) + (1-\lambda)g(\mu).
\end{aligned}
$$

Hence, the function $g(t) = \mathbf{ber}_t(T)$ is convex on the interval $[0,1]$. $\square$

Applying the celebrated Hermite-Hadamard inequality, we have the next result.

PROPOSITION 5. *Let $T \in \mathscr{L}(\mathscr{H})$ and $s,t,\lambda \in [0,1]$. Then*

$$\begin{aligned}
\boldsymbol{ber}_{\frac{s+t}{2}}(T) &\leqslant (1-\lambda)\boldsymbol{ber}_{\frac{(1-\lambda)s+(1+\lambda)t}{2}}(T)+\lambda \boldsymbol{ber}_{\frac{(2-\lambda)s+\lambda t}{2}}(T) \\
&\leqslant \frac{1}{s-t}\int_s^t \boldsymbol{ber}_x(T)dx \\
&\leqslant \frac{\boldsymbol{ber}_{(1-\lambda)s+\lambda t}(T)+(1-\lambda)\boldsymbol{ber}_s(T)+\lambda \boldsymbol{ber}_t(T)}{2} \\
&\leqslant \frac{\boldsymbol{ber}_s(T)+\boldsymbol{ber}_t(T)}{2}.
\end{aligned}$$

Proof. The refined Hermite-Hadamard inequality for a convex function g on the interval $[0,1]$ asserts that

$$g\left(\frac{s+t}{2}\right) \leqslant l(\lambda,s,t) \leqslant \frac{1}{t-s}\int_s^t g(t)\,dt \leqslant L(\lambda,s,t) \leqslant \frac{g(s)+g(t)}{2},$$

where $s,t \in [0,1]$,

$$l(\lambda,s,t) = (1-\lambda)g\left(\frac{(1-\lambda)s+(1+\lambda)t}{2}\right)+\lambda g\left(\frac{(2-\lambda)s+\lambda t}{2}\right)$$

and

$$L(\lambda,s,t) = \frac{1}{2}\left(g((1-\lambda)s+\lambda t)+(1-\lambda)g(s)+\lambda g(t)\right).$$

Now, utilizing this inequality for the function $g(t) = \mathbf{ber}_t(T)$ we get the desired result. $\square$

LEMMA 6. *If $T \in \mathscr{L}(\mathscr{H})$ and $t \in [0,1]$, then*

(1) $\boldsymbol{ber}_t(T) = \boldsymbol{ber}_t(T^*)$;

(2) $\boldsymbol{ber}(\Re(T)) \leqslant \frac{\boldsymbol{ber}_{\gamma(t)}(T)}{2\Gamma(t)}$,

where $\gamma(t) = \min\{t,1-t\}$ and $\Gamma(t) = \max\{t,1-t\}$.

Proof. Assume $T \in \mathscr{L}(\mathscr{H})$ and $t \in [0,1]$. Then

$$\begin{aligned}
\mathbf{ber}_t(T) &= \mathbf{ber}((1-2t)T^*+T) \\
&= \mathbf{ber}((1-2t)(\Re(T)-i\Im(T))+\Re(T)+i\Im(T)) \\
&= 2\mathbf{ber}((1-t)\Re(T)+it\Im(T)) \\
&= 2\mathbf{ber}((1-t)\Re(T)-it\Im(T)) \quad (\text{since } \mathbf{ber}(T)=\mathbf{ber}(T^*)) \\
&= \mathbf{ber}((1-2t)(\Re(T)+i\Im(T))+\Re(T)-i\Im(T)) \\
&= \mathbf{ber}((1-2t)T^*+T) \\
&= \mathbf{ber}_t(T^*).
\end{aligned}$$

Hence, we get the first result. For the second result we have

$$\begin{aligned}4(1-t)\mathbf{ber}(\Re(T)) &= 2(1-t)\mathbf{ber}(T+T^*)\\ &= \mathbf{ber}_t(T+T^*)\\ &\leqslant \mathbf{ber}_t(T)+\mathbf{ber}_t(T^*)\\ &= 2\mathbf{ber}_t(T) \quad \text{(by part (1))},\end{aligned}$$

whence $\mathbf{ber}(\Re(T)) \leqslant \frac{\mathbf{ber}_{\gamma(t)}(T)}{2\Gamma(t)}$, where $\gamma(t) = \min\{t,1-t\}$ and $\Gamma(t) = \max\{t,1-t\}$. $\square$

THEOREM 6. *Let $T \in \mathscr{L}(\mathscr{H})$ and $t \in [0,1]$. Then*

$$2\gamma(t)\boldsymbol{ber}(T) \leqslant \boldsymbol{ber}_t(T) \leqslant 2\Gamma(t)\boldsymbol{ber}(T), \tag{2}$$

where $\gamma(t) = \min\{t,1-t\}$ and $\Gamma(t) = \max\{t,1-t\}$.

Proof. Assume $0 \leqslant t \leqslant \frac{1}{2}$. Then by the definition of the weighted Berezin number and this fact $\mathbf{ber}_t(T) = \mathbf{ber}_t(T^*)$ we have

$$\begin{aligned}\mathbf{ber}_t(T) &= \mathbf{ber}((1-2t)T^*+T)\\ &\leqslant (1-2t)\mathbf{ber}(T^*)+\mathbf{ber}(T)\\ &= 2(1-t)\mathbf{ber}(T)\end{aligned}$$

and for $\frac{1}{2} \leqslant t \leqslant 1$ we have

$$\begin{aligned}\mathbf{ber}_t(T) &= \mathbf{ber}((1-2t)T^*+T)\\ &\leqslant (2t-1)\mathbf{ber}(T^*)+\mathbf{ber}(T)\\ &= 2t\mathbf{ber}(T).\end{aligned}$$

Hence $\mathbf{ber}_t(T) \leqslant 2\Gamma(t)\mathbf{ber}(T)$, where $\Gamma(t) = \max\{t,1-t\}$. Similarly, if $0 \leqslant t \leqslant \frac{1}{2}$, then

$$\begin{aligned}\mathbf{ber}_t(T) &= \mathbf{ber}((1-2t)T^*+T)\\ &\geqslant \mathbf{ber}(T)-(2t-1)\mathbf{ber}(T^*)\\ &= 2(1-t)\mathbf{ber}(T).\end{aligned}$$

and for $\frac{1}{2} \leqslant t \leqslant 1$ we have

$$\begin{aligned}\mathbf{ber}_t(T) &= \mathbf{ber}((1-2t)T^*+T)\\ &= \mathbf{ber}(T-(1-2t)T^*)\\ &\geqslant \mathbf{ber}(T)-(1-2t)\mathbf{ber}(T^*)\\ &= 2t\mathbf{ber}(T).\end{aligned}$$

Therefore by combining the above two recent inequalities, we get $2\gamma(t)\mathbf{ber}(T) \leqslant \mathbf{ber}_t(T)$, where $\gamma(t) = \min\{t,1-t\}$. $\square$

COROLLARY 2. *Let $T \in \mathscr{L}(\mathscr{H})$ and $t \in [0,1]$. Then*

$$\begin{aligned}(1-|1-2t|)\boldsymbol{ber}(T) &\leqslant \boldsymbol{ber}_{1-|1-2t|}(T) \leqslant \boldsymbol{ber}_t(T) \\ &\leqslant \boldsymbol{ber}_{1+|1-2t|}(T) \leqslant (1+|1-2t|)\boldsymbol{ber}(T).\end{aligned}$$

Proof. The proof follows form Theorem 2, $\gamma(t) = \min\{t, 1-t\} = \frac{1-|1-2t|}{2}$ and $\Gamma(t) = \max\{t, 1-t\} = \frac{1+|1-2t|}{2}$. □

Integration over inequalities (2) we have the following result.

COROLLARY 3. *Let $T \in \mathscr{L}(\mathscr{H})$. Then*

$$\frac{1}{2}\boldsymbol{ber}(T)\,dt \leqslant \int_0^1 \boldsymbol{ber}_t(T)\,dt \leqslant \frac{3}{2}\boldsymbol{ber}(T).$$

Proof. The result obtains from inequalities (2) and these facts $\int_0^1 \Gamma(t)\,dt = \frac{3}{4}$ and $\int_0^1 \gamma(t)\,dt = \frac{1}{4}$, where $\gamma(t) = \min\{t, 1-t\}$ and $\Gamma(t) = \max\{t, 1-t\}$. □

Using the definition of the weighted Berezin number we have $\mathbf{ber}_t(T) \leqslant \mathbf{ber}(\Re(T)) + \mathbf{ber}(\Im(T))$. Employing the Jensen inequality we get lower and upper bounds for this expression.

COROLLARY 4. *Let $T \in \mathscr{L}(\mathscr{H})$ and $t \in [0,1]$. Then*

$$\begin{aligned}&\frac{2(1-t)\boldsymbol{ber}(\Re(T)) + 2t\boldsymbol{ber}(\Im(T)) - \boldsymbol{ber}_t(T)}{1+|1-2t|} \\ &\qquad \leqslant \boldsymbol{ber}(\Re(T)) + \boldsymbol{ber}(\Im(T)) - \boldsymbol{ber}_t(T) \\ &\qquad \leqslant \frac{2(1-t)\boldsymbol{ber}(\Re(T)) + 2t\boldsymbol{ber}(\Im(T)) - \boldsymbol{ber}_t(T)}{1-|1-2t|}.\end{aligned}$$

Proof. If $g : [0,1] \to \mathbb{R}$ is a convex function, then the Jensen inequality [12] asserts that

$$\begin{aligned}\frac{(1-t)g(0) + tg(1) - g(t)}{1+|1-2t|} &\leqslant \frac{g(0)+g(1)}{2} - g\left(\frac{1}{2}\right) \\ &\leqslant \frac{(1-t)g(0) + tg(1) - g(t)}{1-|1-2t|},\end{aligned}$$

where $t \in [0,1]$. Applying this inequality for the convex function $g(t) = \mathbf{ber}_t(T)$ we get the desired result. □

Applying Lemma 6 we obtain the following theorem.

THEOREM 7. *Suppose that $T \in \mathscr{L}(\mathscr{H})$ and $t \in [0,1]$. Then*

$$\boldsymbol{ber}(T) = \sup_{\theta \in \mathbb{R}} \frac{\boldsymbol{ber}_{\gamma(t)}(e^{i\theta}T)}{2\Gamma(t)},$$

where $\gamma(t) = \min\{t, 1-t\}$ and $\Gamma(t) = \max\{t, 1-t\}$.

Proof. If $T \in \mathscr{L}(\mathscr{H})$ and $\theta \in \mathbb{R}$, then

$$\begin{aligned}\mathbf{ber}(\Re(e^{i\theta}T)) &\leqslant \frac{\mathbf{ber}_{\gamma(t)}(e^{i\theta}T)}{2\Gamma(t)} \quad \text{(by Lemma 6(2))} \\ &\leqslant \mathbf{ber}(e^{i\theta}T) \quad \text{(by inequality (2))} \\ &= \mathbf{ber}(T).\end{aligned}$$

Note that, $\mathbf{ber}(T) = \sup_{\theta \in \mathbb{R}} \mathbf{ber}(\Re(e^{i\theta}T))$; see [4]. Therefore, by taking supremum over $\theta \in \mathbb{R}$ we get

$$\mathbf{ber}(T) = \sup_{\theta \in \mathbb{R}} \frac{\mathbf{ber}_{\gamma(t)}(e^{i\theta}T)}{2\Gamma(t)}. \quad \square$$

REMARK 3. Note that, Theorem 7 is an analog of a result in [11], i.e. $w(T) = \sup_{\theta \in \mathbb{R}} \left\{ \frac{w_{\gamma(t)}(e^{i\theta}T)}{2\Gamma(t)} \right\}$, where $T \in \mathscr{L}(\mathscr{H})$, $t \in [0,1]$, $\gamma(t) = \min\{t, 1-t\}$ and $\Gamma(t) = \max\{t, 1-t\}$.

3. Some the Davis-Wielandt Berezin number inequalities

In this section, we show a generalization of the Davis-Wielandt Berezin number and then we obtain some results. The concepts of the Davis-Wielandt Berezin set and the Davis-Wielandt Berezin number were introduced in [1] and [30] as follows:

$$DW_{\mathbf{ber}}(T) = \left\{ (\langle T\widehat{k}_\lambda, \widehat{k}_\lambda \rangle, \langle T\widehat{k}_\lambda, T\widehat{k}_\lambda \rangle), \lambda \in \Theta \right\},$$

and

$$dw_{\mathbf{ber}}(T) = \sup_{\lambda \in \Theta} \left\{ \sqrt{|\langle T\widehat{k}_\lambda, \widehat{k}_\lambda \rangle|^2 + \|T\widehat{k}_\lambda\|^4} \right\}.$$

Now, we can clearly see that $dw_{\mathbf{ber}}(T)$ is an generalization of $\mathbf{ber}(T)$, moreover $dw_{\mathbf{ber}}(T) \leqslant dw(T)$. It is easy to see that the Davis-Wielandt Berezin number of $T \in \mathscr{L}(\mathscr{H})$ satisfying the following inequality:

$$\max\left(\mathbf{ber}(T), \|T\|_{\mathbf{ber}}^2\right) \leqslant dw_{\mathbf{ber}}(T) \leqslant \sqrt{\mathbf{ber}^2(T) + \|T\|_{\mathbf{ber}}^4}. \tag{3}$$

We define a f-generalization of the Davis-Wielandt Berezin number as following:

DEFINITION 1. Assume $T \in \mathcal{L}(\mathcal{H})$ and $f : [0,\infty) \to [0,\infty)$ be a continuous increasing function. We define the f-Davis-Wielandt Berezin number of the operator T by

$$dw_{\mathbf{ber}_f}(T) = \sup_{\lambda \in \Theta} f^{-1}\left(f\left(|\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|\right) + f\left(\langle T^*T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle\right)\right).$$

In this section, we present some inequalities involving the Davis-Wielandt Berezin number. First, we obtain a lower bound for the Davis-Wielandt Berezin number in $\mathcal{L}(\mathcal{H})$.

THEOREM 8. *Let $T \in \mathcal{L}(\mathcal{H})$ and $f : [0,\infty) \to [0,\infty)$ be a continuous increasing function. Then*

$$\begin{aligned} &dw_{\mathbf{ber}_f}(T) \\ &\geqslant \max\left\{f^{-1}\left(f\left(\mathbf{ber}(T)\right) + f\left(c_{\mathbf{ber}}(T^*T)\right)\right), f^{-1}\left(f\left(c_{\mathbf{ber}}(T)\right) + f\left(\mathbf{ber}(T^*T)\right)\right)\right\}. \end{aligned}$$

Proof. Let $\widehat{k}_\lambda \in \mathcal{H}(\Theta)$ be a normalized reproducing kernel. Applying the monotonicity of f and f^{-1} we have

$$f^{-1}\left(f\left(|\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|\right) + f\left(|\langle T^*T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|\right)\right) \geqslant f^{-1}\left(f\left(|\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|\right) + f\left(c_{\mathbf{ber}}(T^*T)\right)\right),$$

whence by taking the supremum over all $\lambda \in \Theta$, we get

$$dw_{\mathbf{ber}_f}(T) \geqslant f^{-1}\left(f\left(\mathbf{ber}(T)\right) + f\left(c_{\mathbf{ber}}(T^*T)\right)\right). \tag{4}$$

Moreover, we have

$$f^{-1}\left(f\left(|\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|\right) + f\left(|\langle T^*T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|\right)\right) \geqslant f^{-1}\left(f\left(c_{\mathbf{ber}}(T)\right) + f\left(|\langle T^*T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|\right)\right),$$

and so

$$dw_{\mathbf{ber}_f}(T) \geqslant f^{-1}\left(f\left(c_{\mathbf{ber}}(T)\right) + f\left(\mathbf{ber}(T^*T)\right)\right). \tag{5}$$

From (4) and (5), the desired inequality holds. □

REMARK 4. If $T \in \mathcal{L}(\mathcal{H})$ and $f : [0,\infty) \to [0,\infty)$ is a continuous increasing function, then using arithmetic-geometric inequality and inequalities (4) and (5) we have

$$\begin{aligned} dw_{\mathbf{ber}_f}(T) &\geqslant f^{-1}\left(f\left(\mathbf{ber}(T)\right) + f\left(c_{\mathbf{ber}}(T^*T)\right)\right) \\ &\geqslant f^{-1}\left(2\sqrt{f\left(\mathbf{ber}(T)\right) f\left(c_{\mathbf{ber}}(T^*T)\right)}\right). \end{aligned}$$

and

$$dw_{\mathbf{ber}_f}(T) \geqslant f^{-1}\left(f\left(c_{\mathbf{ber}}(T)\right) + f\left(\mathbf{ber}(T^*T)\right)\right)$$

$$\geqslant f^{-1}\left(2\sqrt{f\left(c_{\mathbf{ber}}(T)\right)f\left(\mathbf{ber}(T^*T)\right)}\right).$$

These inequalities imply that

$$\begin{aligned}&dw_{\mathbf{ber}_f}(T)\\&\geqslant \max\left\{f^{-1}\left(2\sqrt{f\left(\mathbf{ber}(T)\right)f\left(c_{\mathbf{ber}}(T^*T)\right)}\right), f^{-1}\left(2\sqrt{f\left(c_{\mathbf{ber}}(T)\right)f\left(\mathbf{ber}(T^*T)\right)}\right)\right\}.\end{aligned}$$

REMARK 5. Assume $T\in\mathscr{L}(\mathscr{H})$ and $f:[0,\infty)\to[0,\infty)$ be a continuous increasing function. It follows from Theorem 8 that

$$\begin{aligned}&\max\left\{\mathbf{ber}(T),\|T\|^2_{\mathbf{ber}}\right\}\\&=\max\left\{\mathbf{ber}(T),\mathbf{ber}^2(|T|)\right\} \qquad \text{(since } |T| \text{ is positive)}\\&\leqslant \max\left\{\mathbf{ber}(T),\mathbf{ber}(|T|^2)\right\} \qquad \text{(by Lemma 2)}\\&=\max\left\{f^{-1}\left(f\left(\mathbf{ber}(T)\right)\right), f^{-1}\left(f\left(\mathbf{ber}(|T|^2)\right)\right)\right\}\\&\leqslant \max\left\{f^{-1}\left(f\left(\mathbf{ber}(T)\right)+f\left(c_{\mathbf{ber}}(|T|^2)\right)\right), f^{-1}\left(f\left(c_{\mathbf{ber}}(T)\right)+f\left(\mathbf{ber}(|T|^2)\right)\right)\right\}\\&\qquad\qquad\qquad\qquad \text{(since } f^{-1} \text{ is monotone)}\\&\leqslant dw_{\mathbf{ber}_f}(T).\end{aligned}$$

Therefore, the inequality obtained in Theorem 8 is sharper than the lower bound obtained in (3).

COROLLARY 5. *Let $T\in\mathscr{L}(\mathscr{H})$ and $p>0$. Then*

$$dw^p_{\boldsymbol{ber}_p}(T)\geqslant \max\left\{(\boldsymbol{ber}(T))^p+(c_{\boldsymbol{ber}}(T^*T))^p, (c_{\boldsymbol{ber}}(T))^p+(\boldsymbol{ber}(T^*T))^p\right\}.$$

Proof. Utilizing Theorem 8 for the function $f(t)=t^p$ $(p>0)$ we get the request result. □

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