Operators and Matrices
Volume 17, Number 2 (2023), 517–524 doi:10.7153/oam-2023-17-33

A NON–INJECTIVE VERSION OF WIGNER'S THEOREM

MARK PANKOV AND LUCIJAN PLEVNIK

(*Communicated by L. Molnár*)

Abstract. Let H be a complex Hilbert space and let $\mathscr{F}_s(H)$ be the real vector space of all self-adjoint finite rank operators on H. We prove the following non-injective version of Wigner's theorem: every linear operator on $\mathscr{F}_s(H)$ sending rank one projections to rank one projections (without any additional assumption) is either induced by a linear or conjugate-linear isometry or constant on the set of rank one projections.

1. Introduction

Wigner's theorem plays an important role in mathematical foundations of quantum mechanics. Pure states of a quantum mechanical system are identified with rank one projections (see, for example, [21]) and Wigner's theorem [22] characterizes all symmetries of the space of pure states as unitary and anti-unitary operators. We present a non-injective version of this result in terms of linear operators on the real vector space of self-adjoint finite rank operators which send rank one projections to rank one projections.

Let H be a complex Hilbert space. For every natural $k < \dim H$ we denote by $\mathscr{P}_k(H)$ the set of all rank k projections, i.e. bounded self-adjoint idempotent operators of rank k. Let $\mathscr{F}_s(H)$ be the real vector space of all self-adjoint finite rank operators on H. This vector space is spanned by $\mathscr{P}_k(H)$, see e.g. [10, Lemma 2.1.5].

Classical Wigner's theorem says that every bijective transformation of $\mathscr{P}_1(H)$ preserving the angle between the images of any two projections, or equivalently, preserving the trace of the composition of any two projections, is induced by a unitary or anti-unitary operator. The first rigorous proof of this statement was given in [8], see also [20] for the case when $\dim H \geqslant 3$. By the non-bijective version of this result [2, 3, 4], arbitrary (not necessarily bijective) transformation of $\mathscr{P}_1(H)$ preserving the angles between the images of projections (it is clear that such a transformation is injective) is induced by a linear or conjugate-linear isometry.

Various analogues of Wigner's theorem for $\mathscr{P}_k(H)$ can be found in [5, 6, 7, 9, 10, 11, 12, 13, 15, 17]. In particular, transformations of $\mathscr{P}_k(H)$ preserving principal angles between the images of any two projections and transformations preserving the trace of the composition of any two projections are determined in [9, 11] and [5], respectively. All such transformations are induced by linear or conjugate-linear isometries, except

Mathematics subject classification (2020): 47B49, 15A86, 81P16.
Keywords and phrases: Projection, self-adjoint operator of finite rank, Wigner's theorem.

Paper OaM-17-33

in the case $\dim H = 2k \geqslant 4$ when there is an additional class of transformations. The description of transformations preserving the trace of the composition given in [5] is based on the following fact from [9]: every transformation of $\mathscr{P}_k(H)$ preserving the trace of the composition of two projections can be extended to an injective linear operator on $\mathscr{F}_s(H)$. So, there is an intimate relation between Wigner's type theorems mentioned above and results concerning linear operators sending projections to projections [1, 14, 16, 19].

Consider a linear operator L on $\mathscr{F}_s(H)$ such that

$$L(\mathscr{P}_k(H)) \subset \mathscr{P}_k(H) \tag{1}$$

such that the restriction of L to $\mathscr{P}_k(H)$ is injective. We also assume that $\dim H \geqslant 3$. By [14], this operator is induced by a linear or conjugate-linear isometry if $\dim H \neq 2k$. In the case when $\dim H = 2k$, it can be also a composition of an operator induced by a linear or conjugate-linear isometry and an operator which sends any projection on a k-dimensional subspace X to the projection on the orthogonal complement $X^{\perp}$. This statement is a small generalization of the result obtained in [1]. The main result of [14] concerns linear operators sending $\mathscr{P}_k(H)$ to $\mathscr{P}_m(H)$, as above, whose restrictions to $\mathscr{P}_k(H)$ are injective.

In this paper, we determine all possibilities for a linear operator L on $\mathscr{F}_s(H)$ satisfying (1) for $k = 1$ without any additional assumption. Such an operator is either induced by a linear or conjugate-linear isometry or its restriction to $\mathscr{P}_1(H)$ is constant. We mention that this result could be easily obtained from [18, Theorem 2.1], as such a map L clearly preserves the adjacency relation on the set $\mathscr{F}_s(H)$. However, we will present an elementary approach by only using the Wigner's theorem.

Some remarks concerning the case when $k > 1$ will be given in the last section.

2. The main result

We investigate linear maps on $\mathscr{F}_s(H)$ preserving the set of projections of rank one. Our main result is the following.

THEOREM 1. *Let H be a complex Hilbert space,* $\dim H \geqslant 2$, *and* $L\colon \mathscr{F}_s(H) \to \mathscr{F}_s(H)$ *a linear map. Then we have*

$$L(\mathscr{P}_1(H)) \subset \mathscr{P}_1(H) \tag{2}$$

if and only if either there exists $P_0 \in \mathscr{P}_1(H)$ such that

$$L(A) = (\operatorname{tr} A)P_0, \quad A \in \mathscr{F}_s(H)$$

or there exists a linear or conjugate-linear isometry $U: H \to H$ such that

$$L(A) = UAU^*, \quad A \in \mathscr{F}_s(H).$$

3. Preliminaries

Denote by P_X the projection whose image is a closed subspace $X \subset H$. Since P_X belongs to $\mathscr{P}_k(H)$ if and only if X is k-dimensional, $\mathscr{P}_k(H)$ will be identified with the Grassmannian $\mathscr{G}_k(H)$. For any subspace $Z \subset H$, denote

$$\langle Z]_1 = \{X \in \mathscr{G}_1(H) \mid X \subset Z\}.$$

If $\dim H \geqslant 2$, then $\mathscr{G}_1(H)$ is a projective space, whose projective lines are exactly sets of the form $\langle S]_1$, $S \in \mathscr{G}_2(H)$.

We will show that the maps f, satisfying (2), behave nicely on projective lines in $\mathscr{G}_1(H)$. In order to do that, we will need the following concept, which is a modification of the concept, introduced in [5]. For any $X, Y \in \mathscr{G}_1(H)$ and $t \in \left(\frac{1}{2}, \infty\right)$, define the set

$$\chi_t(X,Y) = \{Z \in \mathscr{G}_1(H) : t(P_X + P_Y) + (1-2t)P_Z \in \mathscr{P}_1(H)\}.$$

The following lemma describes this set.

LEMMA 1. *Let $X, Y \in \mathscr{G}_1(H)$ and $t \in \left(\frac{1}{2}, \infty\right)$. Then the following statements hold.*

- $\chi_t(X,Y) \subset \langle X+Y]_1$
- $\chi_t(X,Y) \neq \emptyset \iff \operatorname{tr}(P_X P_Y) \geqslant \left(1-\frac{1}{t}\right)^2$
- *If X and Y are orthogonal, then $\chi_1(X,Y) = \langle X+Y]_1$.*
- *If $X \neq Y$ and $\operatorname{tr}(P_X P_Y) > \left(1-\frac{1}{t}\right)^2$, then $\chi_t(X,Y)$ is homeomorphic to a circle.*
- *If $X = Y$ or $\operatorname{tr}(P_X P_Y) = \left(1-\frac{1}{t}\right)^2 \neq 0$, then $\chi_t(X,Y)$ is a singleton.*

Proof. It is easy to show that $\chi_t(X,X) = \{X\}$.

Assume now that $X \neq Y$ and denote $S = X+Y$ and $A = P_X + P_Y$. Then A is a positive semidefinite operator with trace 2. Its kernel equals $X^\perp \cap Y^\perp$, so its range equals S. Therefore, if $Z \in \chi_t(X,Y)$, then $tA + (1-2t)P_Z$ is positive semidefinite, implying that $Z \in \langle S]_1$.

Moreover, there exist $c \in [0,1)$ and an orthonormal base $\mathscr{B}$ of S, according to which we have the matrix representation

$$A|_S = \begin{bmatrix} 1+c & 0 \\ 0 & 1-c \end{bmatrix}.$$

Note that

$$\operatorname{tr}(P_X P_Y) = \tfrac{1}{2}\operatorname{tr}(A^2 - A) = c^2.$$

If Z is any element of $\langle S]_1$, then, according to $\mathscr{B}$,

$$P_Z|_S = \begin{bmatrix} s & w \\ \overline{w} & 1-s \end{bmatrix}$$

for some $s \in [0,1]$ and $w \in \mathbb{C}$, $|w| = \sqrt{s(1-s)}$. Any such Z belongs to $\chi_t(X,Y)$ if and only if

$$\det\left(t\begin{bmatrix} 1+c & 0 \\ 0 & 1-c \end{bmatrix} + (1-2t)\begin{bmatrix} s & w \\ \overline{w} & 1-s \end{bmatrix}\right) = 0.$$

A straightforward calculation shows that the latter holds if and only if we have either $c = 0$ and $t = 1$ or $c \neq 0$ and s equals

$$\frac{(1+c)(t(1+c)-1)}{2c(2t-1)}. \tag{3}$$

Thus, if X and Y are orthogonal, then $\chi_t(X,Y)$ is non-empty if and only if $t = 1$ and in this case, it equals $\langle X+Y]_1$. In the case when they are not orthogonal, $\chi_t(X,Y)$ is non-empty if and only if (3) belongs to $[0,1]$, which is equivalent to $c \geqslant \left|1-\frac{1}{t}\right|$. Next, if (3) belongs to $\{0,1\}$, which is equivalent to $c = \left|1-\frac{1}{t}\right|$, then $\chi_t(X,Y)$ is a singleton. Finally, if (3) belongs to $(0,1)$ and equals s, then any $Z \in \chi_t(X,Y)$ can be identified with an element w of the circle with origin 0 and radius $\sqrt{s(1-s)}$. $\square$

4. Proof of Theorem 1

Recall that L is a linear map $\mathscr{F}_s(H) \to \mathscr{F}_s(H)$ satisfying (2). Denote by f the transformation $\mathscr{G}_1(H) \to \mathscr{G}_1(H)$, induced by L, i.e. $L(P_X) = P_{f(X)}$, $X \in \mathscr{G}_1(H)$.

LEMMA 2. *The following assertions are fulfilled:*

1. *For any $t \in \mathbb{R} \setminus \left\{0, \frac{1}{2}\right\}$ and $X, Y \in \mathscr{G}_1(H)$ we have*

$$f(\chi_t(X,Y)) \subset \chi_t(f(X), f(Y)).$$

If $f(X) = f(Y)$, then f is constant on $\chi_t(X,Y)$.

2. *f transfers any projective line to a subset of a projective line.*

Proof.

1. Easy verification.

2. If $S \in \mathscr{G}_2(H)$ and $X, Y \in \langle S]_1$ are orthogonal, then $\chi_1(X,Y)$ coincides with $\langle S]_1$ by Lemma 1 and we have

$$f(\langle S]_1) \subset \chi_1(f(X), f(Y)) \subset \langle S']_1$$

with $S' = f(X) + f(Y)$ if $f(X) \neq f(Y)$, otherwise we take any 2-dimensional subspace S' containing $f(X) = f(Y)$. $\square$

LEMMA 3. *The restriction of f to any projective line is either injective or constant.*

Proof. Let $S \in \mathscr{G}_2(H)$. Suppose that the restriction of f to $\langle S]_1$ is not injective. Then there exist distinct $X, Y \in \langle S]_1$ such that $f(X) = f(Y)$. For every $t \in \mathbb{R}$ define

$$g(t) = \det((t(P_X + P_Y) + (1-2t)P_{S\cap X^\perp})|_S).$$

Then $g\colon \mathbb{R} \to \mathbb{R}$ is a continuous function such that $g(\frac{1}{2}) > 0$. Let $A = P_X + P_Y - P_{S\cap X^\perp}$. Let $\langle\cdot,\cdot\rangle$ denote the scalar product on H. Since $\langle Ax, x\rangle > 0$ for $x \in X$ and $\langle Az, z\rangle \leqslant 0$ for $z \in S\cap X^\perp$, we have $g(1) \leqslant 0$. Therefore, there exists $t \in \left(\frac{1}{2}, 1\right]$ such that $g(t) = 0$. For such t we have $S\cap X^\perp \in \chi_t(X,Y)$. By Lemma 2, $f(S\cap X^\perp) = f(X) = f(Y)$. Another application of Lemma 2 yields that f is constant on $\chi_1\left(X, S\cap X^\perp\right)$, which equals $\langle S]_1$ by Lemma 1. $\square$

LEMMA 4. *The restriction of f to any projective line is continuous.*

Proof. Let $S \in \mathscr{G}_2(H)$. Then the linear span of $\{P_X \mid X \in \langle S]_1\}$ is finite-dimensional, which implies that the restriction of L to this linear span is bounded. Hence, the restriction of f to $\langle S]_1$ is continuous. $\square$

Proof of Theorem 1. The two examples in the conclusion of the theorem clearly satisfy (2). Assume now that (2) holds.

If f is constant, then $\phi(A) = (\operatorname{tr} A)P_0$, $A \in \mathscr{F}_s(H)$, for some $P_0 \in \mathscr{P}_1(H)$.

Assume now that f is not constant. Then there exist $X, Y \in \mathscr{G}_1(H)$ such that $f(X) \neq f(Y)$. Denote $S = X + Y \in \mathscr{G}_2(H)$. We will first show that

$$f(\langle S]_1) = \langle f(X) + f(Y)]_1. \tag{4}$$

By Lemma 2, Lemma 3, and Lemma 4, f is an injective continuous map from $\langle S]_1$ to $\langle f(X) + f(Y)]_1$, which are both homeomorphic to the 2-dimensional sphere $\mathbb{S}^2$. Thus, f induces an injective continuous map $\widetilde{f}\colon \mathbb{S}^2 \to \mathbb{S}^2$. If $\widetilde{f}$ was not surjective, then it would map into $\mathbb{S}^2 \setminus \{p\}$ for some $p \in \mathbb{S}^2$, which is homeomorphic to $\mathbb{R}^2$, but this would contradict the Borsuk-Ulam theorem. Therefore, we deduce (4).

We next assert that

$$\operatorname{tr}(P_{f(X)}P_{f(Y)}) = \operatorname{tr}(P_X P_Y). \tag{5}$$

Assume first that X and Y are orthogonal. Lemma 1 implies that $\chi_1(X,Y) = \langle S]_1$, so it follows from (4) that $\chi_1(f(X), f(Y)) = \langle f(X) + f(Y)]_1$. Another application of Lemma 1 yields that $f(X)$ and $f(Y)$ are orthogonal, as desired. Suppose now that X and Y are not orthogonal and denote $t = \frac{1}{1+\sqrt{\operatorname{tr}(P_X P_Y)}} \in \left(\frac{1}{2}, 1\right)$. By Lemma 1, $\chi_t(X,Y)$ is a singleton. We claim that

$$f(\chi_t(X,Y)) = \chi_t(f(X), f(Y)). \tag{6}$$

Indeed, the left-hand side is contained in the right-hand side by Lemma 2. Let now $W \in \chi_t(f(X), f(Y))$. Then $W \in \langle f(X) + f(Y)]_1$ by Lemma 1, so (4) yields that $W = f(W')$ for some $W' \in \langle S]_1$. Hence,

$$t(P_{f(X)} + P_{f(Y)}) + (1-2t)P_{f(W')} = P_{W''} \tag{7}$$

for some $W'' \in \mathscr{G}_1(H)$. Then we have $W'' \in \chi_{\frac{t}{2t-1}}(f(X), f(Y))$, hence another application of (4) implies that $W'' = f(W''')$ for some $W''' \in \langle S]_1$. Denote

$$A = t(P_X + P_Y) + (1-2t)P_{W'} - P_{W'''}.$$

By (7), $L(A) = 0$. We assert that $A = 0$. Indeed, $A = aP_Z + bP_{S\cap Z^\perp}$ for some $Z \in \langle S]_1$ and $a, b \in \mathbb{R}$. Since $f|_{\langle S]_1}$ is injective, $f(Z) \neq f(S\cap Z^\perp)$, so $P_{f(Z)}$ and $P_{f(S\cap Z^\perp)}$ are linearly independent. Now $0 = L(A) = aP_{f(Z)} + bP_{f(S\cap Z^\perp)}$ implies that $a = b = 0$ and $A = 0$, which completes the proof of (6).

By (6), $\chi_t(f(X), f(Y))$ is a singleton. Another application of Lemma 1 yields that $t = \frac{1}{1+\sqrt{\operatorname{tr}(P_{f(X)}P_{f(Y)})}}$, so (5) holds.

We have shown that (5) holds whenever $X, Y \in \mathscr{G}_1(H)$ are such that $f(X) \neq f(Y)$. By Lemma 3, the same holds for any pair from $\langle X+Y]_1$.

We will next show f is injective. If $\dim H = 2$, there is nothing more to do, so assume that $\dim H \geqslant 3$. Seeking a contradiction, suppose that there exist pairwise distinct $X, Y, Z \in \mathscr{G}_1(H)$ such that $f(X) \neq f(Y)$ and $f(Z) = f(X)$. Denote $S = X+Y$ and let $Z' = (S+Z)\cap S^\perp$. By Lemma 3, $Z \not\subset S$, thus $Z' \in \mathscr{G}_1(H)$. Let $Y' \in \langle S]_1 \setminus \{X\}$ be non-orthogonal to X. By the previous paragraph, $f(X)$ and $f(Y')$ are distinct and non-orthogonal. Since Z' is orthogonal Y', $f(Z')$ is either equal or orthogonal to $f(Y')$, so $f(Z') \neq f(X)$. Because Z' is orthogonal to X, $f(Z')$ is orthogonal to $f(X)$, which equals $f(Z)$. By the previous paragraph, Z' is orthogonal to Z. Hence,

$$\{0\} = (S+Z)\cap(S+Z)^\perp = Z'\cap Z^\perp = Z',$$

a contradiction. This contradiction shows that, since f is not constant, it must be injective. Thus, (5) holds for all $X, Y \in \mathscr{G}_1(H)$. The conclusion of the theorem now follows from Wigner's theorem, see e.g. [4]. $\square$

5. Final remarks

Consider a linear map L on $\mathscr{F}_s(H)$ satisfying

$$L(\mathscr{P}_k(H)) \subset \mathscr{P}_k(H)$$

for a certain $k \in \mathbb{N}$, $k < \dim H$. As above, L induces a transformation f of $\mathscr{G}_k(H)$ which is not necessarily injective. The general case can be reduced to the case when $\dim H \geqslant 2k$.

For subspaces M and N satisfying $\dim M < k < \dim N$ and $M \subset N$ we denote by $[M,N]_k$ the set of all $X \in \mathscr{G}_k(H)$ such that $M \subset X \subset N$. For any $X, Y \in \mathscr{G}_k(H)$ we have

$$\chi_1(X,Y) = \{Z \in \mathscr{G}_k(H) : P_X + P_Y - P_Z \in \mathscr{P}_k(H)\} \subset [X\cap Y, X+Y]_k$$

and the inverse inclusion holds if and only if X, Y are compatible, i.e. there is an orthonormal basis of H such that X and Y are spanned by subsets of this basis. If X and Y are orthogonal, then $\chi_1(X,Y) = \langle X+Y]_k$ and

$$f(\langle X+Y]_k) \subset \chi_1(f(X), f(Y)) \subset \langle f(X)+f(Y)]_k.$$

As in the proof of Lemma 4, we show that for any $(2k)$-dimensional subspace $S \subset H$ the restriction of f to $\langle S]_k$ is continuous. In the case when $k=1$, the restriction of f to any projective line is a continuous map to a projective line.

In the general case, a line of $\mathscr{G}_k(H)$ is a subset of type $[M,N]_k$, where M is a $(k-1)$-dimensional subspace contained a $(k+1)$-dimensional subspace N. This line can be identified with the line of $\langle M^{\perp}]_1$ associated to the 2-dimensional subspace $N \cap M^{\perp}$. Two distinct k-dimensional subspaces are contained in a common line if and only if they are adjacent, i.e. their intersection is $(k-1)$-dimensional. If $X,Y \in \mathscr{G}_k(H)$ are adjacent, then the line containing them is $[X \cap Y, X+Y]_k$. It was noted above that this line coincides with $\chi_1(X,Y)$ only in the case when X and Y are compatible. If X and Y are non-compatible, then $\chi_1(X,Y)$ is a subset of the line $[X \cap Y, X+Y]_k$ homeomorphic to a circle.

For every line there is a $(2k)$-dimensional subspace S such that $\langle S]_k$ contains this line, i.e. the restriction of f to each line is continuous. Using analogous arguments as in the proof of Lemma 3, we establish that the restriction of f to every line is either injective or constant; but we are not be able to show that f sends lines to subsets of lines.

On the other hand, if f is injective, then it is adjacency and orthogonality preserving (see [1, 5, 14] for the details). By [13], this immediately implies that f is induced by a linear or conjugate-linear isometry if $\dim H > 2k$ and there is one other option for f if $\dim H = 2k$.

REFERENCES

[1] P. ANIELLO, D. CHRUŚCIŃSKI, *Symmetry witnesses*, J. Phys. A: Math. Theor. **50** (2017), no. 28, 16 p.

[2] V. BARGMANN, *Note on Wigner's theorem on symmetry operations*, J. Math. Phys. **5** (1964), 862–868.

[3] J. BARVÍNEK, J. HAMHALTER, *Linear algebraic proof of Wigner theorem and its consequences*, Math. Slovaca **67** (2017), 371–386.

[4] G. P. GEHÉR, *An elementary proof for the non-bijective version of Wigner's theorem*, Phys. Lett. A **378** (2014), 2054–2057.

[5] G. P. GEHÉR, *Wigner's theorem on Grassmann spaces*, J. Funct. Anal. **273** (2017), 2994–3001.

[6] G. P. GEHÉR, P. ŠEMRL, *Isometries of Grassmann spaces*, J. Funct. Anal. **270** (2016), 1585–1601.

[7] M. GYÖRY, *Transformations on the set of all n-dimensional subspaces of a Hilbert space preserving orthogonality*, Publ. Math. Debrecen **65** (2004), 233–242.

[8] J. S. LOMONT AND P. MENDELSON, *The Wigner unitary-antiunitary theorem*, Ann. Math. **78** (1963), 548–559.

[9] L. MOLNÁR, *Transformations on the set of all n-dimensional subspaces of a Hilbert space preserving principal angles*, Comm. Math. Phys. **217** (2001), 409–421.

[10] L. MOLNÁR, *Selected Preserver Problems on Algebraic Structures of Linear Operators and on Function Spaces*, Lecture Notes in Mathematics **1895**, Springer, 2007.

[11] L. MOLNÁR, *Maps on the n-dimensional subspaces of a Hilbert space preserving principal angles*, Proc. Amer. Math. Soc. **136** (2008), 3205–3209.

[12] M. PANKOV, *Orthogonal apartments in Hilbert Grassmanians*, Linear Algebra and its Applications **506** (2016), 168–182.

[13] M. PANKOV, *Geometric version of Wigner's theorem for Hilbert Grassmannians*, J. Math. Anal. Appl. **459** (2018), 135–144.

[14] M. PANKOV, *Wigner's type theorem in terms of linear operators which send projections of a fixed rank to projections of other fixed rank*, J. Math. Anal. Appl. **474** (2019), 1238–1249.

[15] M. PANKOV, *Wigner Type Theorems for Hilbert Grassmannians*, London Mathematical Society Lecture Note Series, vol. 460, Cambridge University Press, 2020.
[16] G. SARBICKI, D. CHRUŚCIŃSKI, M. MOZRZYMAS, *Generalising Wigner's theorem*, J. Phys. A: Math. Theor. **49** (2016), no. 30, 7 p.
[17] P. ŠEMRL, *Orthogonality preserving transformations on the set of n-dimensional subspaces of a Hilbert space*, Illinois J. Math. **48** (2004), 567–573.
[18] P. ŠEMRL, *Symmetries on bounded observables: a unified approach based on adjacency preserving maps*, Integral Equations Operator Theory **72** (2012), 7–66.
[19] E. STØRMER, *Positive maps which map the set of rank k projections onto itself*, Positivity **21** (2017), 509–511.
[20] U. UHLHORN, *Representation of symmetry transformations in quantum mechanics*, Ark. Fys. **23** (1963), 307–340.
[21] V. S. VARADARAJAN, *Geometry of Quantum Theory* (2nd edition), Springer, 2000.
[22] E. P. WIGNER, *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektrum*, Fredrik Vieweg und Sohn **1931** (english translation *Group Theory and its Applications to the Quantum Mechanics of Atomic Spectra*, Academic Press 1959).

(Received September 28, 2022)

Mark Pankov
University of Warmia and Mazury
Faculty of Mathematics and Computer Science
Słoneczna 54, Olsztyn, Poland
e-mail: pankov@matman.uwm.edu.pl

Lucijan Plevnik
University of Ljubljana
Faculty for Mathematics and Physics
Jadranska 21, Ljubljana, Slovenia
e-mail: lucijan.plevnik@fmf.uni-lj.si