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ANALYTICITY AND STABILITY OF SEMIGROUP RELATED TO AN ABSTRACT INITIAL–BOUNDARY VALUE PROBLEM

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Abstract. This paper is concerned with an abstract initial-boundary value problem with dynamical boundary conditions. The analyticity and stability of semigroup generated by the associated operator are obtained, by the spectral properties of one-sided coupled operator matrices. As applications, the well-posedness of a heat equation with dynamical boundary conditions and the stability of a diffusion-transport system with dynamical boundary conditions are further presented.

1. Introduction

In this article, we discuss the following abstract initial-boundary value problem with dynamical boundary conditions

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & t \geqslant 0, \\ \dot{u}(t) = Du(t), & t \geqslant 0, \\ \Gamma u(t) = x(t), & t \geqslant 0, \\ u(0) = u_0, \quad x(0) = x_0 \end{cases} \tag{1.1}$$

and use semigroup method to examine the existence and stability of its solutions. In the problem (1.1),

X, ∂X are the state and boundary Banach spaces, respectively;
$A : D(A) \subset \partial X \to \partial X$; $D : D(D) \subset X \to X$;
$B : D(B) \subset X \to \partial X$ is called a feedback operator; and
$\Gamma : D(\Gamma) \subset X \to \partial X$ is called a boundary operator.

It is well-known that the abstract Cauchy problem is well-posed if and only if its govern operator generates a C_0-semigroup on the underlying space, and the analyticity of the semigroup will be helpful in improving the regularity and asymptotic properties of solutions of the corresponding abstract Cauchy problem (cf. [12] and references

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therein). In recent years, the problem (1.1) frequently appeared in the literature. Authors discussed its well-posedness by studying the generator property of operators with generalized Wentzell boundary conditions on X (cf. [11, 2, 3]). On the other hand, under some assumptions, in a similar way as in the proof of [21, Section 1.2] one can show that the well-posedness of the problem (1.1) is equivalent to that of the abstract Cauchy problem associated to the operator matrix

$$\mathscr{A} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \quad D(\mathscr{A}) = \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in D(A) \times \Big(D(B) \cap D(D) \Big) : \Gamma u = x \right\}$$

in the product space $\partial X \times X$, and by the factorization of $\lambda - \mathscr{A}$ [23, formula (3.2)]

$$\lambda - \mathscr{A} = \begin{pmatrix} \lambda - A + BL_\lambda & -B \\ 0 & \lambda - D_0 \end{pmatrix} \begin{pmatrix} I & 0 \\ L_\lambda & I \end{pmatrix} \tag{1.2}$$

with Dirichlet operator $-L_\lambda = (\Gamma|_{N(\lambda - D)})^{-1}$ and $D_0 = D|_{N(\Gamma)}$ for $\lambda \in \rho(D_0)$. Based on this factorization, many authors studied the generation of analytic semigroups by $\mathscr{A}$ on $\partial X \times X$ by means of similarity transformations and perturbation theory (cf. [4, 21, 23]). The paper [17] used the theory of one-sided coupled operator matrices to consider the positivity and exponential stability of the semigroup generated by $\mathscr{A}$. Note that the operator matrix of the form (1.2) is one-sided coupled, which has been extensively studied, see [8, 9, 10, 17].

Many evolution equations like wave equations, heat equations or diffusion-transport equations with dynamical boundary conditions have been discussed systematically by A. Favini, G. R. Goldstein, J. A. Goldstein, et al [13, 14, 15]. On the other hand, one knows that such equations can be reformulated as the problem (1.1) by considering suitable spaces and operators, see e.g. [5, 22, 4, 17]. In the present paper, we use the resolvent estimate (2.1) and the involved spectral properties to study the analyticity and stability of the associated semigroups generated by one-sided coupled operator matrices, and apply these abstract results to $\mathscr{A}$ arising from the problem (1.1). It is also worth mentioning that for the generation of analytic semigroups by $\mathscr{A}$ we extend the condition in [23] to more general settings. As applications, the well-posedness of a heat equation with dynamical boundary conditions and the stability of a diffusion-transport equation with dynamical boundary conditions are given.

2. Preliminaries

Unless stated specially, X, Y and Z are always Banach spaces and the operators involved are always linear in the whole paper. $X \hookrightarrow Y$ indicates that X is continuously imbedded in Y. For $\omega \in \mathbb{R}$ and $r \geqslant 0$, write

$$H_{\omega,r} := \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > \omega, |\lambda - \omega| \geqslant r\}.$$

For an operator $A : X \to Y$, the notations $D(A)$, $N(A)$, $R(A)$ and A^* are reserved for the domain, kernel, range and adjoint of A, respectively; if A is closed, by $[D(A)]$ we denote $(D(A), \|\cdot\|_A)$ equipped with the graph norm

$$\|x\|_A = \|Ax\| + \|x\|, \quad x \in D(A).$$

Now let A be an operator in X. The point spectrum, residual spectrum, spectrum and resolvent set of A are respectively defined as

$$\begin{aligned}
&\sigma_p(A) = \{\lambda \in \mathbb{C} : \lambda - A \text{ is not injective}\},\\
&\sigma_r(A) = \{\lambda \in \mathbb{C} : R(\lambda - A) \text{ is not dense in } X\},\\
&\sigma(A) = \{\lambda \in \mathbb{C} : \lambda - A \text{ is not bijective or else } (\lambda - A)^{-1} \text{ is unbounded}\}
\end{aligned}$$

and $\rho(A) = \mathbb{C}\backslash\sigma(A)$. For $\lambda \in \rho(A)$, the inverse $R(\lambda; A) = (\lambda - A)^{-1}$ is called the resolvent of A at the point λ. In particular, if A is densely defined closed, the MP spectrum of A is defined as

$$\sigma_{mp}(A) = \{\lambda \in \mathbb{C} : R(\lambda - A) \text{ is not closed in } X\}.$$

DEFINITION 2.1. (see [17]) Assume that A and D are closed operators in X and Y, respectively, and that $B : [D(D)] \to X$ and $L : [D(A)] \to Y$ are bounded operators. Then the operator

$$\mathscr{A} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ L & I \end{pmatrix}$$

with domain

$$D(\mathscr{A}) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in D(A) \times Y : Lx + y \in D(D) \right\}$$

is called an one-sided coupled operator matrix in the product space $X \times Y$.

DEFINITION 2.2. (see [12]) Let A be the generator of a C_0-semigroup $T(\cdot)$ on X. The semigroup $T(\cdot)$ is said to be strongly stable if $\lim_{t\to\infty} \|T(t)x\| = 0$ for every $x \in X$.

DEFINITION 2.3. (see [1]) Let X and Y be Hilbert spaces. An operator $B : Y \to X$ is called a Tseng inverse of the operator $A : X \to Y$, if $R(A) \subset D(B)$, $R(B) \subset D(A)$ and the following relations are fulfilled:

$$BA = P_{\overline{R(B)}}|_{D(A)}, \quad AB = P_{\overline{R(A)}}|_{D(B)},$$

where $P_{\overline{R(B)}}$ and $P_{\overline{R(A)}}$ are orthogonal projections onto $\overline{R(B)}$ and onto $\overline{R(A)}$, respectively.

Note that A has a Tseng inverse if and only if

$$D(A) = N(A) \oplus (D(A) \cap N(A)^{\perp}),$$

in which case $R(B) = D(A) \cap N(A)^{\perp}$ and $N(B)$ is an arbitrary subspace of $R(A)^{\perp}$. Such decomposition of the domain is clearly true for bounded and closed operator classes, since their kernels are closed subspaces of the whole Hilbert space.

DEFINITION 2.4. (see [1]) Let X and Y be Hilbert spaces. The maximal Tseng inverse $A^{\dagger}$ of an operator $A : X \to Y$ is the Tseng inverse of A with $D(A^{\dagger}) = R(A) \oplus R(A)^{\perp}$ and $N(A^{\dagger}) = R(A)^{\perp}$. In particular, $A^{\dagger} = A^{-1}$ if A is invertible.

DEFINITION 2.5. (see [7]) Let A be the generator of a C_0-semigroup $T(\cdot)$ on X. Then a Banach space $(Z, \|\cdot\|_Z)$ satisfies the (Z)-condition with respect to A, if

(i) $Z \hookrightarrow X$,

(ii) for all $t > 0$ and continuous functions $\phi \in C([0,t], Z)$, we have $\int_0^t T(t-s)\phi(s)\,\mathrm{d}s \in D(A)$, and

(iii) there is an increasing continuous function $\gamma(t) : [0,\infty) \to [0,\infty)$ with $\gamma(0) = 0$, such that

$$\left\| A \int_0^t T(t-s)\phi(s)\,\mathrm{d}s \right\| \leqslant \gamma(t) \sup_{0 \leqslant s \leqslant t} \|\phi(s)\|_Z.$$

REMARK 2.6. Let A be the generator of a C_0-semigroup $T(\cdot)$ on X. Then Z can be $[D(A)]$ and the Favard class of $T(\cdot)$, i.e.

$$Z = \left\{ x \in X : \limsup_{t \to 0^+} \frac{1}{t} \|T(t)x - x\| < \infty \right\}$$

endowed with the norm

$$\|x\|_Z = \|x\| + \limsup_{t \to 0^+} \frac{1}{t} \|T(t)x - x\|.$$

In particular, if $T(\cdot)$ is analytic, then we can take $Z = ([D(A)], X)_{\theta}$, the real interpolation space of order θ between $[D(A)]$ and X, where $\theta \in (0,1)$, cf. [6].

LEMMA 2.7. (see [12]) *A densely defined operator A is the generator of an analytic semigroup on X if and only if there exist $\omega \in \mathbb{R}$, $M > 0$ and $r \geqslant 0$ such that $\lambda \in \rho(A)$ and*

$$\|R(\lambda; A)\| \leqslant \frac{M}{|\lambda - \omega|} \tag{2.1}$$

whenever $\lambda \in H_{\omega,r}$.

LEMMA 2.8. (see [12]) *Let A be the generator of a bounded analytic semigroup $T(\cdot)$ on a reflexive space X. Then the following statements are equivalent:*

(i) $0 \notin \sigma_r(A)$;

(ii) $T(\cdot)$ is strongly stable.

We collect the following propositions whose simple proofs are omitted.

PROPOSITION 2.9. *Let $\mathscr{A} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ L & I \end{pmatrix}$ be one-sided coupled in $X \times Y$. If A and D are densely defined, then $\mathscr{A}$ is densely defined.*

PROPOSITION 2.10. *Let* $M = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ *be a bounded operator matrix on* $X \times Y$. *Then*

$$\max\left\{\|A_{11}\|, \|A_{12}\|, \|A_{21}\|, \|A_{22}\|\right\} \leqslant \|M\| \leqslant \|A_{11}\| + \|A_{12}\| + \|A_{21}\| + \|A_{22}\|.$$

PROPOSITION 2.11. *Let* $B : X \to Y$ *and* $A : Y \to Z$ *be linear operators. If* B *is injective and* $D(A) \subset R(B|_{D(AB)})$, *then* $R(AB) = R(A)$ *and* $\dim N(AB) = \dim N(A)$.

3. Analytic semigroups

This section is devoted to the analyticity of semigroups generated by one-sided coupled operator matrices.

THEOREM 3.1. *Let* $\mathscr{A} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\begin{pmatrix} I & 0 \\ L & I \end{pmatrix}$ *be one-sided coupled in* $X \times Y$, *and let* D *be invertible and generate an analytic semigroup* $T(\cdot)$ *on* Y. *Assume that* Z *satisfies the* (Z)*-condition with respect to* D *and* $L : X \to Z$ *is bounded. Then the following statements are equivalent:*

(i) A *generates an analytic semigroup on* X;

(ii) $\mathscr{A}$ *generates an analytic semigroup on* $X \times Y$.

Proof. Let $\mathscr{A}_0 = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} I & 0 \\ L & I \end{pmatrix}$. Since D generates an analytic semigroup, by Lemma 2.7 there exist $\omega \in \mathbb{R}$, $M > 0$ and $r \geqslant 0$ such that $\lambda \in \rho(D)$ and $\|R(\lambda; D)\| \leqslant \frac{M}{|\lambda - \omega|}$ for all $\lambda \in H_{\omega,r}$. Thus $\|T(t)\| \leqslant Me^{\omega t}$ for all $t \geqslant 0$. Obviously, there exist a bounded and invertible operator $U : Y \to X$ and constants $a, b \geqslant 0$ such that the operator $LU : Y \to Z$ is bounded and $\|By\| \leqslant a\|y\| + b\|Dy\|$ for all $y \in D(D)$. Hence there exists $\widetilde{M} > \frac{aM + bM(r + |\omega|) + br}{r}$ such that

$$\|LUy\|_Z \leqslant \widetilde{M}\|y\| \ \text{ and } \ \|BR(\lambda; D)\| \leqslant \widetilde{M}$$

for all $y \in Y$ and all $\lambda \in H_{\omega,r}$. Since Z satisfies the (Z)-condition with respect to D and $T(\cdot)$ is analytic, we have for all sufficiently large $\mathrm{Re}\lambda$ and all $y \in Y$ that

$$\begin{aligned}
\|R(\lambda; D)LUy\|_D &= \|\int_0^\infty e^{-\lambda s} T(s) LUy \, \mathrm{d}s\|_D \\
&\leqslant \|\int_0^t e^{-\lambda s} T(s) LUy \, \mathrm{d}s\|_D + \|\int_t^\infty e^{-\lambda s} T(s) LUy \, \mathrm{d}s\|_D \\
&= \|\int_0^t T(t-s)(e^{-\lambda(t-s)} LUy) \, \mathrm{d}s\|_D + \|e^{-\lambda t} T(t) \int_0^\infty e^{-\lambda s} T(s) LUy \, \mathrm{d}s\|_D \\
&\leqslant \left(\gamma(t) + \frac{M}{\omega}(e^{\omega t} - 1)\right) \sup_{0 \leqslant s \leqslant t} \|e^{-\lambda(t-s)} LUy\|_Z \\
&\quad + Me^{-(\mathrm{Re}\lambda - \omega)t} \|R(\lambda; D) LUy\|_D.
\end{aligned}$$

Putting

$$t(\lambda) = \frac{\ln 2M}{\mathrm{Re}\lambda - \omega}.$$

Then

$$\|DR(\lambda;D)LUy\| \leqslant \|R(\lambda;D)LUy\|_D \leqslant 2\widetilde{M}\Big(\gamma\big(t(\lambda)\big) + \frac{M}{\omega}(e^{\omega t(\lambda)}-1)\Big)\|y\|.$$

Since $\lim_{\mathrm{Re}\lambda\to\infty} t(\lambda) = 0$, we have $\lim_{\mathrm{Re}\lambda\to\infty}\Big(\gamma\big(t(\lambda)\big) + \frac{M}{\omega}(e^{\omega t(\lambda)}-1)\Big) = 0$. Let $\varepsilon > 0$. Then

$$\|DR(\lambda;D)LU\| \leqslant \varepsilon \tag{3.1}$$

for $\mathrm{Re}\lambda$ sufficiently large. We point out that the estimate when $\varepsilon = \frac{1}{2}$ can be found in the proof of [16, Theorem 2.1].

"(i)⇒(ii)" Let $\lambda \in \rho(D)$. Then

$$\lambda \in \rho(\mathscr{A}_0) \Leftrightarrow \lambda \in \rho(A).$$

In this case

$$R(\lambda;\mathscr{A}_0) = \begin{pmatrix} R(\lambda;A) & 0 \\ DR(\lambda;D)LR(\lambda;A) & R(\lambda;D) \end{pmatrix} \tag{3.2}$$

and

$$\lambda - \mathscr{A} = Q(\lambda - \mathscr{A}_0), \tag{3.3}$$

where $Q = \begin{pmatrix} I-\lambda BR(\lambda;D)LR(\lambda;A) & -BR(\lambda;D) \\ 0 & I \end{pmatrix}$. Since A generates an analytic semigroup, $D(\mathscr{A}) = D(\mathscr{A}_0)$ is dense in $X\times Y$ by Proposition 2.9. And by Lemma 2.7, there exist $\omega_1 \geqslant \omega$, $M_1 > 0$ and $r_1 \geqslant r$ such that $\lambda \in \rho(A)$ and $\|R(\lambda;A)\| \leqslant \frac{M_1}{|\lambda-\omega_1|}$ for all $\lambda \in H_{\omega_1,r_1}$. Let $M_2 = \frac{M_1(r_1+|\omega_1|)}{r_1}$. Then

$$\|\lambda R(\lambda;A)\| \leqslant M_1\left(1 + \frac{|\omega_1|}{|\lambda-\omega_1|}\right) \leqslant M_2.$$

Let ε be a positive number satisfying $\varepsilon \leqslant \frac{1}{2M_2\|BD^{-1}\|\|U^{-1}\|}$. Then by (3.1) there exists $\omega_2 \geqslant \omega_1$ such that

$$\|DR(\lambda;D)L\| \leqslant \frac{1}{2M_2\|BD^{-1}\|} \quad \text{and} \quad \|BR(\lambda;D)L\| \leqslant \frac{1}{2M_2}$$

for $\mathrm{Re}\lambda > \omega_2$. This implies

$$\|DR(\lambda;D)LR(\lambda;A)\| \leqslant \frac{M_1}{2M_2\|BD^{-1}\|}\frac{1}{|\lambda-\omega_2|} \tag{3.4}$$

and

$$\|\lambda BR(\lambda;D)LR(\lambda;A)\| \leqslant \frac{1}{2} \tag{3.5}$$

for all $\lambda \in H_{\omega_2,r_1}$. Thus we have from (3.2), (3.4) and Proposition 2.10 that $\lambda \in \rho(\mathscr{A}_0)$ and

$$\|R(\lambda;\mathscr{A}_0)\| \leqslant \left(M + M_1 + \frac{M_1}{2M_2\|BD^{-1}\|}\right)\frac{1}{|\lambda-\omega_2|}$$

for all $\lambda \in H_{\omega_2,r_1}$. Hence $\mathscr{A}_0$ generates an analytic semigroup.

From (3.5), by the Banach Lemma, the operator $I-\lambda BR(\lambda;D)LR(\lambda;A)$ is invertible and the norm of its inverse is not greater than 2 for all $\lambda \in H_{\omega_2,r_1}$. Then Q is invertible and

$$Q^{-1} = \begin{pmatrix} (I-\lambda BR(\lambda;D)LR(\lambda;A))^{-1} & (I-\lambda BR(\lambda;D)LR(\lambda;A))^{-1}BR(\lambda;D) \\ 0 & I \end{pmatrix}.$$

An easy computation shows that $\|Q^{-1}\| \leqslant 3+2\widetilde{M}$. By (3.3) we have $\lambda \in \rho(\mathscr{A})$,

$$R(\lambda;\mathscr{A}) = R(\lambda;\mathscr{A}_0)Q^{-1},$$

and hence

$$\|R(\lambda;\mathscr{A})\| \leqslant \|R(\lambda;\mathscr{A}_0)\|\|Q^{-1}\| \leqslant (3+2\widetilde{M})\left(M+M_1+\frac{M_1}{2M_2\|BD^{-1}\|}\right)\frac{1}{|\lambda-\omega_2|}$$

for all $\lambda \in H_{\omega_2,r_1}$. Therefore $\mathscr{A}$ generates an analytic semigroup, namely (ii) holds.

"(ii)$\Rightarrow$(i)" Let $\lambda \in \rho(D)$. Then

$$\lambda \in \rho(\mathscr{A}) \Leftrightarrow \lambda \in \rho(A_\lambda).$$

In this case

$$R(\lambda;\mathscr{A}) = \begin{pmatrix} R(\lambda;A_\lambda) & R(\lambda;A_\lambda)BR(\lambda;D) \\ -L_\lambda R(\lambda;A_\lambda) & R(\lambda;D)-L_\lambda R(\lambda;A_\lambda)BR(\lambda;D) \end{pmatrix} \tag{3.6}$$

and

$$\lambda-\mathscr{A}_0 = Q(\lambda-\mathscr{A}), \tag{3.7}$$

where $Q = \begin{pmatrix} I+\lambda BR(\lambda;D)LR(\lambda;A_\lambda) & (I+\lambda BR(\lambda;D)LR(\lambda;A_\lambda))BR(\lambda;D) \\ 0 & I \end{pmatrix}$, $A_\lambda = A+\lambda BR(\lambda;D)L$ and $L_\lambda = -DR(\lambda;D)L$. Since $\mathscr{A}$ generates an analytic semigroup, there exist $\omega_1 \geqslant \omega, M_1 > 0$ and $r_1 \geqslant r$ such that $\lambda \in \rho(\mathscr{A})$ and $\|R(\lambda;\mathscr{A})\| \leqslant \frac{M_1}{|\lambda-\omega_1|}$ for all $\lambda \in H_{\omega_1,r_1}$. Using (3.6) and Proposition 2.10 we conclude that $\lambda \in \rho(A_\lambda)$ and

$$\|R(\lambda;A_\lambda)\| \leqslant \frac{M_1}{|\lambda-\omega_1|}$$

for all $\lambda \in H_{\omega_1,r_1}$. In a similar way as in the proof of "(i)$\Rightarrow$(ii)", we have that there exists $\omega_2 \geqslant \omega_1$ such that

$$\|\lambda BR(\lambda;D)LR(\lambda;A_\lambda)\| \leqslant \frac{1}{2}$$

for all $\lambda \in H_{\omega_2,r_1}$. Thus Q is invertible and $\|Q^{-1}\| \leqslant 3+\widetilde{M}$, where

$$Q^{-1} = \begin{pmatrix} (I+\lambda BR(\lambda;D)LR(\lambda;A_\lambda))^{-1} & -BR(\lambda;D) \\ 0 & I \end{pmatrix}.$$

By (3.7) we have $\lambda \in \rho(\mathscr{A}_0)$ and

$$\|R(\lambda;\mathscr{A}_0)\| \leqslant \|R(\lambda;\mathscr{A})\|\|Q^{-1}\| \leqslant \frac{M_1(3+\widetilde{M})}{|\lambda-\omega_2|}$$

for all $\lambda \in H_{\omega_2,r_1}$. Hence $\mathscr{A}_0$ generates an analytic semigroup.

From (3.2) and Proposition 2.10, $\lambda \in \rho(A)$ and

$$\|R(\lambda;A)\| \leqslant \frac{M_1(3+\widetilde{M})}{|\lambda-\omega_2|}$$

for all $\lambda \in H_{\omega_2,r_1}$. Since $D(\mathscr{A}) \subset D(A)\times Y$, $D(A)$ is dense in X. Therefore A generates an analytic semigroup, which proves the assertion (i). □

COROLLARY 3.2. *Let $\mathscr{A} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\begin{pmatrix} I & 0 \\ L & I \end{pmatrix}$ be one-sided coupled in $X\times Y$, and let D be invertible and generate an analytic semigroup $T(\cdot)$ on Y. Assume that Z satisfies the (Z)-condition with respect to D, $L: X \to Y$ is bounded and $L(X) \hookrightarrow Z$. Then the following statements are equivalent:*

(i) A generates an analytic semigroup on X;

(ii) $\mathscr{A}$ generates an analytic semigroup on $X\times Y$.

Proof. If $L: X\to Y$ is bounded and $L(X) \hookrightarrow Z$, then $L: X \to Z$ is bounded. By virtue of Theorem 3.1, the conclusion is clear. □

REMARK 3.3. By the matrix $\lambda - \mathscr{A} = \begin{pmatrix} \lambda - A_0 & 0 \\ -B & \lambda-\widetilde{B}-BD_\lambda^{A,L} \end{pmatrix}\begin{pmatrix} I & -D_\lambda^{A,L} \\ 0 & I \end{pmatrix}$ in $X\times\partial X$, [21, Theorem 2.2.8.(iii)] investigated the generation of analytic semigroups by $\mathscr{A}$ and yielded the following result:

Let A_0 and $\widetilde{B}$ generate anayltic semigroups on X and ∂X, respectively. If $[D(A)]_L \hookrightarrow ([D(A_0)],X)_\theta$ for some $\theta \in (0,1)$, and if $B: [D(A)]_L \to \partial X$ and $B: [D(A_0)] \to ([D(\widetilde{B})],\partial X)_\theta$ are bounded, then $\mathscr{A}$ generates an analytic semigroup on $X\times\partial X$, where $[D(A)]_L$ is a Banach space obtained by endowing $D(A)$ with the graph norm of $\begin{pmatrix} A \\ L \end{pmatrix}$.

In fact, the boundedness of $B: [D(A)]_L \to \partial X$ implies that $B: [D(A_0)] \to \partial X$ is bounded, and hence $\lambda - \mathscr{A}$ is a one-sided coupled matrix. Observe that $\lambda - \widetilde{B} - BD_\lambda^{A,L}$ is a bounded perturbation of $\widetilde{B}$. Therefore, the (analytic) generation property of $\widetilde{B}$ implies the same property of $\lambda - \widetilde{B} - BD_\lambda^{A,L}$. Combining the assumption $[D(A)]_L \hookrightarrow ([D(A_0)],X)_\theta$ with the boundedness of $D_\lambda^{A,L}: \partial X \to [D(A)]_L$, we have that $D_\lambda^{A,L}: \partial X \to ([D(A_0)],X)_\theta$ is bounded. Note that $Z = ([D(A_0)],X)_\theta$ is one of the spaces satisfying the (Z)-condition with respect to A_0. By Theorem 3.1 we conclude that $\mathscr{A}$ generates an analytic semigroup on $X\times\partial X$. Therefore, compared with [21, Theorem 2.2.8.(iii)], this paper can discuss the (analytic) generation property of $\mathscr{A}$ by more general conditions. It is also worth mentioning that we extend the work of [21, Theorem 2.2.8.(iii)] to the more general case of the operator matrix under more general conditions.

4. Stability of analytic semigroups

In this section, X and Y are Hilbert spaces. The purpose of this section is to consider the stability of analytic semigroups associated with one-sided coupled operator matrices.

THEOREM 4.1. *Let* $\mathscr{A}_0 = \left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)\left(\begin{smallmatrix} I & 0 \\ L & I \end{smallmatrix}\right)$ *be one-sided coupled in* $X \times Y$. *Assume that there exist* $\omega_0 \in \mathbb{R}$ *and* $r_0 \geqslant 0$ *such that* $H_{\omega_0, r_0} \subset \rho(D) \cup \rho(A)$, $D(D^*) \subset D(L^*)$, *and* $\mathscr{A}_0$ *generates an analytic semigroup* $\mathscr{T}(\cdot)$ *on* $X \times Y$. *Then there exists* $\omega \in \mathbb{R}$ *such that the following statements are equivalent:*

(*i*) $(e^{-\omega t}\mathscr{T}(t))_{t \geqslant 0}$ *is strongly stable;*

(*ii*) $\omega \notin \sigma_r(A) \cup \sigma_r(D)$.

Proof. From (3.2) and Proposition 2.10, both A and D generate analytic semigroups. Let A and D be the generators of $T(\cdot)$ and $S(\cdot)$, respectively. Obviously, there exists $\omega \in \mathbb{R}$ such that $\mathscr{A}_0 - \omega$ generates a bounded analytic semigroup $(e^{-\omega t}\mathscr{T}(t))_{t \geqslant 0}$. By Lemma 2.8,

$$(e^{-\omega t}\mathscr{T}(t))_{t \geqslant 0} \text{ is strongly stable} \Leftrightarrow 0 \notin \sigma_r(\mathscr{A}_0 - \omega).$$

Observe that $\mathscr{A}_0 - \omega = \mathscr{Q}\mathscr{V}$, where $\mathscr{Q} = \left(\begin{smallmatrix} A-\omega & 0 \\ \omega L & D-\omega \end{smallmatrix}\right)$ and $\mathscr{V} = \left(\begin{smallmatrix} I & 0 \\ L & I \end{smallmatrix}\right)$. Since $\mathscr{V}$ is injective and $D(\mathscr{Q}) \subset R(\mathscr{V}|_{D(\mathscr{Q}\mathscr{V})})$, we have from Proposition 2.11 that $R(\mathscr{Q}\mathscr{V}) = R(\mathscr{Q})$ and $\dim N(\mathscr{Q}\mathscr{V}) = \dim N(\mathscr{Q})$. Hence

$$(e^{-\omega t}\mathscr{T}(t))_{t \geqslant 0} \text{ is strongly stable} \Leftrightarrow 0 \notin \sigma_r(\mathscr{Q}).$$

"(ii)$\Rightarrow$(i)" We are going to show that $R(\mathscr{Q})$ is dense in $X \times Y$. In fact, if $R(\mathscr{Q})$ is not dense, then $R(\mathscr{Q})^{\perp} = N(\mathscr{Q}^*) \neq \{0\}$ and hence $\mathscr{Q}^*$ is not injective. Since $D(D^*) \subset D(L^*)$, we have

$$\mathscr{Q}^* = \begin{pmatrix} A^* - \omega & \omega L^* \\ 0 & D^* - \omega \end{pmatrix}.$$

Then

$$R(A-\omega)^{\perp} = N(A^* - \omega) \neq \{0\} \text{ or } R(D-\omega)^{\perp} = N(D^* - \omega) \neq \{0\}.$$

This is a contradiction. Hence we have $0 \notin \sigma_r(\mathscr{Q})$, i.e., $(e^{-\omega t}\mathscr{T}(t))_{t \geqslant 0}$ is strongly stable.

"(i)$\Rightarrow$(ii)" Write $\widetilde{G}(t)x := D\int_0^t S(t-s)LT(s)x\,\mathrm{d}s$ for all $x \in D(A)$. Assume $G(t)$ to be the continuous extension of $\widetilde{G}(t)$ to the whole space X. The convolution theorem for the Laplace transform implies that the Laplace transform $\widehat{G}(\lambda)$ of $G(t)$ exists and

$$\widehat{G}(\lambda) := DR(\lambda; D)LR(\lambda; A)$$

for $\mathrm{Re}\lambda$ sufficiently large. Since $R(\lambda; \mathscr{A}_0)_{21} = DR(\lambda; D)LR(\lambda; A)$ is the Laplace transform of $\mathscr{T}(t)_{21}$ for $\mathrm{Re}\lambda$ large, we have $G(t) = \mathscr{T}(t)_{21}$. Similarly, $T(t) = \mathscr{T}(t)_{11}$ and $S(t) = \mathscr{T}(t)_{22}$. Hence

$$\mathscr{T}(t) = \begin{pmatrix} T(t) & 0 \\ G(t) & S(t) \end{pmatrix}. \tag{4.1}$$

Obviously, $(e^{-\omega t}T(t))_{t\geqslant 0}$ and $(e^{-\omega t}S(t))_{t\geqslant 0}$ are also bounded analytic semigroups.

If $(e^{-\omega t}\mathscr{T}(t))_{t\geqslant 0}$ is strongly stable, from (4.1) we see that

$$\lim_{t\to\infty}\left\|e^{-\omega t}\mathscr{T}(t)\begin{pmatrix}x\\ y\end{pmatrix}\right\| = \lim_{t\to\infty}\left\|\begin{pmatrix}e^{-\omega t}T(t)x\\ e^{-\omega t}G(t)x+e^{-\omega t}S(t)y\end{pmatrix}\right\| = 0$$

for all $\binom{x}{y}\in X\times Y$. Taking $x=0$ yields $\lim_{t\to\infty}\|e^{-\omega t}S(t)y\|=0$ for all $y\in Y$, which means that $(e^{-\omega t}S(t))_{t\geqslant 0}$ is strongly stable. Letting $y=0$ gives $\lim_{t\to\infty}\left\|\begin{pmatrix}e^{-\omega t}T(t)x\\ e^{-\omega t}G(t)x\end{pmatrix}\right\|$ $=0$ for all $x\in X$, which implies that $(e^{-\omega t}T(t))_{t\geqslant 0}$ is strongly stable. By Lemma 2.8, $\omega\notin\sigma_r(A)\cup\sigma_r(D)$. $\square$

THEOREM 4.2. *Let $\mathscr{A}=\begin{pmatrix}A&B\\0&D\end{pmatrix}\begin{pmatrix}I&0\\L&I\end{pmatrix}$ be one-sided coupled in $X\times Y$. Assume that $\mathscr{A}$ generates an analytic semigroup $\mathscr{T}(\cdot)$ on $X\times Y$. Then there exists $\omega\in\mathbb{R}$ such that the following statements hold:*

(i) If $\omega\in\rho(D)$, then $(e^{-\omega t}\mathscr{T}(t))_{t\geqslant 0}$ is strongly stable if and only if $0\notin\sigma_r(\Delta)$;

(ii) If $\omega\notin\sigma_{mp}(D)$, then $(e^{-\omega t}\mathscr{T}(t))_{t\geqslant 0}$ is strongly stable provided that $R(\Gamma_1)=R(D-\omega)^{\perp}$ and $\overline{R(\Gamma_2)}=X$.

In particular, if L is bounded on $D(A)$, then $(e^{-\omega t}\mathscr{T}(t))_{t\geqslant 0}$ is strongly stable for $\omega\notin\sigma_{mp}(D)$, under the conditions that $\overline{R(\Gamma_1)}=R(D-\omega)^{\perp}$ and $\overline{R(\Gamma_2)}=X$. Here,

$$\Delta = A-\omega-\omega B(D-\omega)^{\dagger}L,$$

$$\Gamma_1=\omega(I-(D-\omega)(D-\omega)^{\dagger})L|_{D(A)} \text{ and } \Gamma_2=B|_{N(D-\omega)}.$$

Proof. Obviously, there exists $\omega\in\mathbb{R}$ such that $\mathscr{A}-\omega$ generates a bounded analytic semigroup $(e^{-\omega t}\mathscr{T}(t))_{t\geqslant 0}$. In a similar way as in the proof of Theorem 4.1, we find that

$$(e^{-\omega t}\mathscr{T}(t))_{t\geqslant 0} \text{ is strongly stable} \Leftrightarrow 0\notin\sigma_r(\mathscr{Q}),$$

where $\mathscr{Q}=\begin{pmatrix}A-\omega & B\\ \omega L & D-\omega\end{pmatrix}$. Let $\omega\notin\sigma_{mp}(D)$. Then

$$\mathscr{Q}=\begin{pmatrix}I & B(D-\omega)^{\dagger}\\ 0 & I\end{pmatrix}\begin{pmatrix}\Delta & B(I-(D-\omega)^{\dagger}(D-\omega))\\ \omega(I-(D-\omega)(D-\omega)^{\dagger})L & D-\omega\end{pmatrix}$$
$$\times\begin{pmatrix}I & 0\\ \omega(D-\omega)^{\dagger}L & I\end{pmatrix}.$$

Write

$$\mathscr{U}=\begin{pmatrix}I & B(D-\omega)^{\dagger}\\ 0 & I\end{pmatrix},\quad \mathscr{V}=\begin{pmatrix}I & 0\\ \omega(D-\omega)^{\dagger}L & I\end{pmatrix},$$

and

$$\mathscr{U}^{-1}\mathscr{Q}\mathscr{V}^{-1}=\begin{pmatrix}\Delta & B(I-(D-\omega)^{\dagger}(D-\omega))\\ \omega(I-(D-\omega)(D-\omega)^{\dagger})L & D-\omega\end{pmatrix}.$$

Note that, since $(D-\omega)^{\dagger}$ and $B(D-\omega)^{\dagger}$ are bounded, $\mathscr{U}$ is bounded and has a bounded inverse on $X\times Y$. Then $0\notin\sigma_r(\mathscr{Q})$ if and only if $0\notin\sigma_r(\mathscr{U}^{-1}\mathscr{Q})$. Since

$\mathscr{V}$ is injective in $X \times Y$ and $D(\mathscr{U}^{-1}\mathscr{Q}\mathscr{V}^{-1}) = D(A) \times D(D) = R(\mathscr{V}|_{D(\mathscr{U}^{-1}\mathscr{Q})})$, we have from Proposition 2.11 that $R(\mathscr{U}^{-1}\mathscr{Q}) = R(\mathscr{U}^{-1}\mathscr{Q}\mathscr{V}^{-1})$ and $\dim N(\mathscr{U}^{-1}\mathscr{Q}) = \dim N(\mathscr{U}^{-1}\mathscr{Q}\mathscr{V}^{-1})$. Hence $0 \notin \sigma_r(\mathscr{Q})$ if and only if $0 \notin \sigma_r(\mathscr{U}^{-1}\mathscr{Q}\mathscr{V}^{-1})$. Since

$$\mathscr{U}^{-1}\mathscr{Q}\mathscr{V}^{-1} = \begin{pmatrix} \Delta & \Gamma_2 & 0 \\ \Gamma_1 & 0 & 0 \\ 0 & 0 & (D-\omega)|_{D(D-\omega)\cap N(D-\omega)^{\perp}} \end{pmatrix}$$
$$: \begin{pmatrix} D(A) \\ N(D-\omega) \\ D(D-\omega)\cap N(D-\omega)^{\perp} \end{pmatrix} \to \begin{pmatrix} X \\ R(D-\omega)^{\perp} \\ R(D-\omega) \end{pmatrix}$$

and $(D-\omega)|_{D(D-\omega)\cap N(D-\omega)^{\perp}} : D(D-\omega)\cap N(D-\omega)^{\perp} \to R(D-\omega)$ is invertible, we conclude

$$0 \notin \sigma_r(\mathscr{Q}) \Leftrightarrow \overline{R(\mathscr{S})} = X \times R(D-\omega)^{\perp},$$

where $\mathscr{S} = \begin{pmatrix} \Delta & \Gamma_2 \\ \Gamma_1 & 0 \end{pmatrix} : D(A) \times N(D-\omega) \to X \times R(D-\omega)^{\perp}$.

(i) If $\omega \in \rho(D)$, then $\mathscr{S} = \Delta$. Hence $0 \notin \sigma_r(\Delta)$ if and only if $\overline{R(\mathscr{S})} = X \times R(D-\omega)^{\perp}$. The assertion (i) is clear.

(ii) Since $R(\Gamma_1) = R(D-\omega)^{\perp}$, there exists $x \in D(A)$ such that $\Gamma_1 x = v$ for all $v \in R(D-\omega)^{\perp}$. Since $R(\Gamma_2)$ is dense in X, there exists a sequence $\{y_n\} \subset N(D-\omega)$ such that $\Gamma_2 y_n \to u - \Delta x$ for all $u \in X$. Hence

$$\mathscr{S}\begin{pmatrix} x \\ y_n \end{pmatrix} = \begin{pmatrix} \Delta x + \Gamma_2 y_n \\ \Gamma_1 x \end{pmatrix} \to \begin{pmatrix} u \\ v \end{pmatrix},$$

which shows that $\overline{R(\mathscr{S})} = X \times R(D-\omega)^{\perp}$.

In particular, if L is bounded on $D(A)$, then $\mathscr{S}$ is closed and $\mathscr{S}^* = \begin{pmatrix} \Delta^* & \Gamma_1^* \\ \Gamma_2^* & 0 \end{pmatrix}$. Since $\overline{R(\Gamma_1)} = R(D-\omega)^{\perp}$ and $\overline{R(\Gamma_2)} = X$, it is clear that $\mathscr{S}^*$ is injective. The desired proof follows immediately. □

5. Application to abstract initial-boundary value problem

We apply the results of Section 3 and Section 4 to the problem (1.1) that satisfies the following assumptions.

ASSUMPTIONS 5.1. (A1) ∂Y is a Banach space and $\partial Y \hookrightarrow \partial X$;
(A2) A is closed in ∂X;
(A3) $D(A) \cap \partial Y$ is dense in ∂X;
(A4) $\Gamma : D(D) \subset X \to \partial X$ and $\Gamma|_{D(B)} : D(B) \to \partial Y$ are surjective;
(A5) $\binom{\Gamma}{D} : D(D) \to \partial X \times X$ and $\binom{\Gamma}{D}|_{D(B)} : D(B) \to \partial Y \times X$ are closed;
(A6) $D_0 = D|_{N(\Gamma)}$ is densely defined and has nonempty resolvent set;
(A7) B is bounded from $[D(D_0)]$ to ∂X.

The above assumptions ensure that we can convert (1.1) into an abstract Cauchy problem

$$\begin{cases} \dot{U}(t) = \mathscr{A}U(t), & t \geqslant 0, \\ U(0) = \binom{x_0}{u_0} \end{cases} \tag{ACP}$$

in $\partial X \times X$ with $U(t) = \begin{pmatrix} \Gamma u(t) \\ u(t) \end{pmatrix}$, $t \geqslant 0$, $\mathscr{A}$ is a densely defined closed operator of the form

$$\mathscr{A} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \quad D(\mathscr{A}) = \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in D(A) \times \Big(D(B) \cap D(D) \Big) : \Gamma u = x \right\},$$

and $\lambda - \mathscr{A}$ can be represented as an one-sided coupled operator matrix

$$\lambda - \mathscr{A} = \begin{pmatrix} \lambda - A + BL_\lambda & -B \\ 0 & \lambda - D_0 \end{pmatrix} \begin{pmatrix} I & 0 \\ L_\lambda & I \end{pmatrix}$$

for $\lambda \in \rho(D_0)$. Here the Dirichlet operator $-L_\lambda = (\Gamma|_{N(\lambda - D)})^{-1}$ is bounded from ∂X to Z for all Banach spaces Z satisfying $D(D^k) \subset Z \hookrightarrow X$ for some $k \in \mathbb{N}$ (see [21,23] for details). Note that if $0 \in \rho(D_0)$, then $L_\lambda = (I - \lambda R(\lambda; D_0))L_0$.

Theorem 3.1 in combination with Corollary 3.2 yields the following fact.

COROLLARY 5.2. *Under the Assumptions 5.1, let D_0 be the generator of an analytic semigroup $T(\cdot)$ on X, $k \in \mathbb{N}$ and $\theta \in (0,1)$. If Z satisfies the (Z)-condition with respect to D_0 and if $D(D^k) \subset Z$ or $L_\lambda(\partial X) \hookrightarrow Z$, then the following statements are equivalent:*

(i) $A - BL_\lambda$ generates an analytic semigroup on ∂X;

(ii) $\mathscr{A}$ generates an analytic semigroup on $\partial X \times X$.

In particular, if $L_\lambda(\partial X) \hookrightarrow ([D(D_0)], X)_\theta$, then the equivalence of (i) and (ii) remains true, which has been proved in [23].

Recall that a closed operator $A : X \to Y$ is said to be Fredholm if $R(A)$ is closed, $\dim N(A) < \infty$ and $\dim Y/R(A) < \infty$. We obtain from Theorem 4.2 the following fact for the case $\dim \partial X < \infty$.

COROLLARY 5.3. *Under the Assumptions 5.1, let X and ∂X be Hilbert spaces, and let D_0 be invertible. Assume that $\mathscr{A}$ generates an analytic semigroup $\mathscr{T}(\cdot)$ on $\partial X \times X$. If $\dim \partial X < \infty$, D_0 is self-adjoint and $\lambda - D_0$ is Fredholm for all $\lambda \in \mathbb{R}$, then there exists $\omega \in \mathbb{R}$ such that the following statements hold:*

(i) If $\omega \in \rho(D_0)$, then $(e^{-\omega t}\mathscr{T}(t))_{t \geqslant 0}$ is strongly stable if and only if $A - BL_0 - \omega + \omega BR(\omega, D_0)L_0$ is injective;

(ii) If $\omega \in \sigma(D_0)$, then $(e^{-\omega t}\mathscr{T}(t))_{t \geqslant 0}$ is strongly stable provided that Γ_1 and Γ_2 are injective, where $\Gamma_1 = \omega(I - (D_0 - \omega)(D_0 - \omega)^\dagger)L_0$ and $\Gamma_2 = B|_{N(D_0 - \omega)}$.

Two typical examples are presented to illustrate our results.

EXAMPLE 5.4. We consider the following heat equation:

$$\begin{cases} \dot{u}(t,x) = \Delta u(t,x), & t \geqslant 0,\ x \in \Omega, \\ \dot{u}(t,z) = q\Delta_{\partial\Omega}u(t,z) - \beta\dfrac{\partial u}{\partial n}(t,z) + \gamma u(t,z), & t \geqslant 0,\ z \in \partial\Omega, \\ u(0,x) = g(x), & x \in \Omega, \\ u(0,z) = f(z), & z \in \partial\Omega, \end{cases} \tag{5.1}$$

where $\Omega \subset \mathbb{R}^m$ is a open domain whose nonempty boundary $\partial\Omega$ to be a $(m-1)$-dimensional smooth manifold, with Ω locally on one side of $\partial\Omega$, $\frac{\partial}{\partial n}$ denotes the outward normal derivative in the trace sense on $\partial\Omega$, and $q,\beta,\gamma \in \mathbb{R}$.

In order to satisfy the Assumptions 5.1 we consider $X = L^2(\Omega)$, $\partial X = L^2(\partial\Omega)$ and the following operators

$$D = \Delta, \quad D(D) = \{u \in H^{\frac{1}{2}}(\Omega) : \Delta u \in L^2(\Omega)\},$$
$$Bu = -\beta\frac{\partial u}{\partial n}, \quad D(B) = \left\{u \in D(D) : \frac{\partial u}{\partial n} \in L^2(\partial\Omega)\right\},$$

the trace operator $\Gamma : u \mapsto u|_{\partial\Omega}$, and $A = q\Delta_{\partial\Omega} + \gamma$ with domain

$$D(A) = \begin{cases} H^2(\partial\Omega) & \text{if } q \neq 0, \\ L^2(\partial\Omega) & \text{if } q = 0. \end{cases}$$

Thus the problem (5.1) can be rewritten as (ACP) in $\partial X \times X$ with $U(t) = \binom{\Gamma u(t)}{u(t)}$, $t \geqslant 0$, $U(0) = \binom{f}{g}$, and the governing operator

$$\mathscr{A} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \quad D(\mathscr{A}) = \left\{ \begin{pmatrix} v \\ u \end{pmatrix} \in D(A) \times D(B) : \Gamma u = v \right\}.$$

Let $D_0 = D|_{N(\Gamma)}$, then

$$D(D_0) = H^2(\Omega) \cap H_0^1(\Omega).$$

Since $0 \in \rho(D_0)$, $\mathscr{A}$ can be represented by an one-sided coupled matrix

$$\mathscr{A} = \begin{pmatrix} A - BL_0 & B \\ 0 & D_0 \end{pmatrix} \begin{pmatrix} I & 0 \\ L_0 & I \end{pmatrix},$$

where $L_0 = -(\Gamma|_{N(D)})^{-1}$.

From [20, Equation (14.32)], we have $H^{\frac{1}{2}}(\Omega) = ([D(D_0)],X)_{\frac{3}{4}}$. Since $D(D) \subset H^{\frac{1}{2}}(\Omega) \hookrightarrow L^2(\Omega)$, $L_0 : \partial X \to ([D(D_0)],X)_{\frac{3}{4}}$ is bounded. We see that D_0 generates an analytic semigroup on X, and $Z = ([D(D_0)],X)_{\frac{3}{4}}$ satisfies the (Z)-condition with respect to D_0. Since $-BL_0 = \beta\mathscr{N}$, we have

$$A - BL_0 = q\Delta_{\partial\Omega} + \beta\mathscr{N} + \gamma.$$

Here $\mathscr{N}$ is the Dirichlet-to-Neumann operator and bounded from $H^1(\partial\Omega)$ to $L^2(\partial\Omega)$. By Corollary 5.2, $\mathscr{A}$ generates an analytic semigroup on $\partial X \times X$ if and only if $q\Delta_{\partial\Omega} + \beta\mathscr{N}$ generates an analytic semigroup on ∂X.

Note that $(q\Delta_{\partial\Omega}, H^2(\partial\Omega))$ generates an analytic semigroup on $L^2(\partial\Omega)$ for $q > 0$. Again by the unboundedness of $\Delta_{\partial\Omega}$ one has that $(q\Delta_{\partial\Omega}, H^2(\partial\Omega))$ generates an analytic semigroup on $L^2(\partial\Omega)$ if and only if $q > 0$. Similarly, $\beta\mathscr{N}$ generates an analytic semigroup on ∂X if and only if $\beta > 0$ (see [18, Theorem 4]). Since $\mathscr{N}$ is bounded from $H^1(\partial\Omega)$ to $L^2(\partial\Omega)$, there exists $M_1 > 0$ such that

$$\|\mathscr{N}u\|_{L^2(\partial\Omega)} \leqslant M_1\|u\|_{H^1(\partial\Omega)}, \quad u \in H^1(\partial\Omega). \tag{5.2}$$

By [19, Proposition I.2.3]) and [19, Theorem I.7.7]), we have

$$\|u\|_{H^1(\partial\Omega)} \leqslant \|u\|_{L^2(\partial\Omega)}^{\frac{1}{2}}\|u\|_{H^2(\partial\Omega)}^{\frac{1}{2}}, \quad u \in H^2(\partial\Omega).$$

Combining this with Young inequality, we obtain that for every $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that

$$\|u\|_{H^1(\partial\Omega)} \leqslant \varepsilon\|u\|_{H^2(\partial\Omega)} + M_\varepsilon\|u\|_{L^2(\partial\Omega)}, \quad u \in H^2(\partial\Omega). \tag{5.3}$$

From [19, p. 37], there exists $M_2 > 0$ such that

$$\|u\|_{H^2(\partial\Omega)} \leqslant M_2(\|u\|_{L^2(\partial\Omega)} + \|\Delta_{\partial\Omega}u\|_{L^2(\partial\Omega)}), \quad u \in H^2(\partial\Omega),$$

which together with the estimates (5.2) and (5.3) implies

$$\|\mathscr{N}u\|_{L^2(\partial\Omega)} \leqslant M_1M_2\varepsilon\|\Delta_{\partial\Omega}u\|_{L^2(\partial\Omega)} + M_1(M_2\varepsilon + M_\varepsilon)\|u\|_{L^2(\partial\Omega)}, \quad u \in H^2(\partial\Omega).$$

Hence $\mathscr{A}$ generates an analytic semigroup on $\partial X \times X$ if and only if either of the following holds:

(i) $q = 0$ and $\beta > 0$,

(ii) $q > 0$.

EXAMPLE 5.5. Consider the following diffusion-transport system with dynamical boundary conditions:

$$\begin{cases} \dot{u}(t,x) = u''(t,x), & t \geqslant 0,\ x \in [0,1], \\ \dot{u}(t,0) = u'(t,0) + \alpha u(t,0), & t \geqslant 0, \\ \dot{u}(t,1) = -u'(t,1) + \beta u(t,1), & t \geqslant 0, \\ u(0,x) = f(x), & x \in [0,1], \\ u(0,0) = u_0, \quad u(0,1) = u_1, \end{cases} \tag{5.4}$$

where $\alpha, \beta, u_0, u_1 \in \mathbb{C}$, $f : [0,1] \to \mathbb{C}$.

We consider $X = L^2(0,1)$, $\partial X = \mathbb{C}^2$ and the following operators

$$
\begin{aligned}
A &= \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in \mathbb{C}^{2\times 2}, \\
Du &= u'', \quad D(D) = H^2(0,1), \\
Bu &= \begin{pmatrix} u'(0) \\ -u'(1) \end{pmatrix}, \quad D(B) = D(D), \\
\Gamma u &= \begin{pmatrix} u(0) \\ u(1) \end{pmatrix}, \quad D(\Gamma) = D(D).
\end{aligned}
$$

Let $D_0 = D|_{N(\Gamma)}$ with $D(D_0) = H^2(0,1) \cap H_0^1(0,1)$. Note that the problem (5.4) can be reformulated as (1.1), and satisfies the Assumptions 5.1. Then it can be written as (ACP) in $\partial X \times X$ with $U(t) = \begin{pmatrix} \Gamma u(t) \\ u(t) \end{pmatrix}$, $t \geqslant 0$, $U(0) = \begin{pmatrix} \binom{u_0}{u_1} \\ f \end{pmatrix}$, and

$$
\mathscr{A} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \quad D(\mathscr{A}) = \left\{ \begin{pmatrix} v \\ u \end{pmatrix} \in \partial X \times D(D) : \Gamma u = v \right\}.
$$

Since $0 \in \rho(D_0)$, $\mathscr{A}$ can be represented by

$$
\mathscr{A} = \begin{pmatrix} A - BL_0 & B \\ 0 & D_0 \end{pmatrix} \begin{pmatrix} I & 0 \\ L_0 & I \end{pmatrix},
$$

where $L_0 = -(\Gamma|_{N(D)})^{-1}$ is the Dirichlet operator. From [17, Theorem 9.4] it follows that $\mathscr{A}$ generates an analytic semigroup $\mathscr{T}(\cdot)$ on $\partial X \times X$.

For $\lambda \in \mathbb{R}$ and $U = \binom{\Gamma u}{u} \in D(\mathscr{A})$,

$$
\begin{aligned}
\mathrm{Re}\langle(\mathscr{A} - \lambda)U, U\rangle = &- \int_0^1 |u'(x)|^2 \mathrm{d}x - \lambda \int_0^1 |u(x)|^2 \mathrm{d}x \\
&+ (\mathrm{Re}\beta - \lambda)|u(1)|^2 + (\mathrm{Re}\alpha - \lambda)|u(0)|^2.
\end{aligned}
$$

This implies that $\mathscr{A} - \lambda$ is dissipative for $\lambda \geqslant \max\{0, \mathrm{Re}\alpha, \mathrm{Re}\beta\}$. By [17, Proposition 9.8], $\mathscr{A}$ is self-adjoint if and only if $\alpha, \beta \in \mathbb{R}$. Combining these facts and taking $\alpha, \beta \in \mathbb{R}$, for $\omega \geqslant \max\{0, \alpha, \beta\}$ we obtain that $\mathscr{A} - \omega$ generates a bounded analytic semigroup $(e^{-\omega t}\mathscr{T}(t))_{t\geqslant 0}$. Now apply Corollary 5.3 to the strong stability of $(e^{-\omega t}\mathscr{T}(t))_{t\geqslant 0}$. The spectrum of D_0 is given by

$$
\sigma_p(D_0) = \sigma(D_0) = \{-k^2\pi^2 : k = 1, 2, \cdots\}.
$$

Then $\omega \in \rho(D_0)$. Hence, $(e^{-\omega t}\mathscr{T}(t))_{t\geqslant 0}$ is strongly stable if and only if $A - BL_0 - \omega + \omega BR(\omega, D_0)L_0 = A - \omega - BL_\omega$ is injective, where

$$
-BL_\omega = \begin{cases} \dfrac{1}{e^{\mu_2} - e^{\mu_1}} \begin{pmatrix} \mu_2 e^{\mu_1} - \mu_1 e^{\mu_2} & \mu_1 - \mu_2 \\ (\mu_1 - \mu_2)e^{\mu_1 + \mu_2} & \mu_2 e^{\mu_2} - \mu_1 e^{\mu_1} \end{pmatrix} & \text{if } \omega \neq 0, \\ \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} & \text{if } \omega = 0, \end{cases} \tag{5.5}
$$

and $\mu_{1,2} = \pm\sqrt{\omega}$. We conclude by (5.5) that

$$(e^{-\omega t}\mathscr{T}(t))_{t\geqslant 0} \text{ is strongly stable} \Leftrightarrow \begin{cases} h(\omega) \neq 0 & \text{if } \omega \neq 0, \\ \alpha\beta - \alpha - \beta \neq 0 & \text{if } \omega = 0, \end{cases}$$

where $h(\omega) = \omega^2 - \omega[\alpha+\beta-1-\frac{(\mu_1-\mu_2)(e^{\mu_1}+e^{\mu_2})}{e^{\mu_1}-e^{\mu_2}}] + \alpha\beta + \alpha\frac{\mu_2 e^{\mu_2}-\mu_1 e^{\mu_1}}{e^{\mu_1}-e^{\mu_2}} + \beta\frac{\mu_2 e^{\mu_1}-\mu_1 e^{\mu_2}}{e^{\mu_1}-e^{\mu_2}}$.

Therefore, for all $\omega \geqslant \max\{0,\alpha,\beta\}$ we obtain that $(e^{-\omega t}\mathscr{T}(t))_{t\geqslant 0}$ is strongly stable if and only if either of the following conditions holds:

(i) $h(\omega) \neq 0$ for $\omega \neq 0$,

(ii) $\alpha\beta - \alpha - \beta \neq 0$ for $\omega = 0$.

REMARK 5.6. Let $\alpha, \beta < 0$. Then $\mathscr{T}(\cdot)$ is a bounded analytic semigroup and $\alpha\beta - \alpha - \beta \neq 0$. By the above results, $(\mathscr{T}(t))_{t\geqslant 0}$ is strongly stable. On the other hand, since $\alpha + \beta < \min\{2, \alpha\beta\}$, by [17, Proposition 9.12] we have that $(\mathscr{T}(t))_{t\geqslant 0}$ is uniformly exponentially stable, which also implies that $(\mathscr{T}(t))_{t\geqslant 0}$ is strongly stable.

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REFERENCES

[1] A. BEN-ISRAEL, T. N. E. GREVILLE, *Generalized Inverses: Theory and Applications*, 2nd edition, Springer-Verlag, New York, 2003.

[2] T. BINZ, *Analytic semigroups generated by Dirichlet-to-Neumann operators on manifolds*, Semigroup Forum 103 (2021) 38–61.

[3] T. BINZ, K.-J. ENGEL, *Operators with Wentzell boundary conditions and the Dirichlet-to-Neumann operator*, Math. Nachr. 292 (2019) 733–746.

[4] V. CASARINO, K.-J. ENGEL, R. NAGEL, G. NICKEL, *A semigroup approach to boundary feedback systems*, Int. Equ. Oper. Theory 47 (2003) 289–306.

[5] V. CASARINO, K.-J. ENGEL, G. NICKEL, S. PIAZZERA, *Decoupling techniques for wave equations with dynamic boundary conditions*, Discrete Contin. Dyn. Syst. 12 (2005) 761–772.

[6] W. DESCH, W. SCHAPPACHER, *On relatively bounded perturbations of linear C_0-semigroups*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 11 (1984) 327–341.

[7] W. DESCH, W. SCHAPPACHER, *Some Generation Results for Perturbed Semigroups, in: Semigroup theory and applications*, Lect. Notes in Pure Appl. Math., Vol. 116, Marcel Dekker, New York (1989), 125–152.

[8] K.-J. ENGEL, *Positivity and stability for one-sided coupled operator matrices*, Positivity 1 (1997) 103–124.

[9] K.-J. ENGEL, *Matrix representation of linear operators on product spaces*, Rend. Circ. Mat. Palermo (2) Suppl. 56 (1998) 219–224.

[10] K.-J. ENGEL, *Spectral theory and generator property for one-sided coupled operator matrices*, Semigroup Forum 58 (1999) 267–295.

[11] K.-J. ENGEL, G. FRAGNELLI, *Analyticity of semigroups generated by operators with generalized Wentzell boundary conditions*, Adv. Diff. Equ. 10 (2005) 1301–1320.

[12] K.-J. ENGEL, R. NAGEL, *One-parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Math., vol. 194, Springer-Verlag, New York, 2000.

[13] A. FAVINI, G. R. GOLDSTEIN, J. A. GOLDSTEIN, S. ROMANELLI, *C_0-semigroups generated by second order differential operators with general Wentzell boundary conditions*, Proc. Amer. Math. Soc. 128 (2000) 1981–1989.

[14] A. FAVINI, G. R. GOLDSTEIN, J. A. GOLDSTEIN, S. ROMANELLI, *The one dimensional wave equation with Wentzell boundary conditions*, in: Differential Equations and Control Theory, S. Aicovici and N. Pavel, (eds), Lect. Notes in Pure Appl. Math., Vol. 225, Marcel Dekker, New York (2001), 139–145.

[15] A. FAVINI, G. R. GOLDSTEIN, J. A. GOLDSTEIN, S. ROMANELLI, *The heat equation with generalized Wentzell boundary conditions*, J. Evol. Equ. 2 (2002) 1–19.

[16] M. JUNG, *Multiplicative perturbations in semigroup theory with the (Z)-condition*, Semigroup Forum 52 (1996) 197–211.

[17] M. KRAMAR, D. MUGNOLO, R. NAGEL, *Theory and applications of one-sided coupled operator matrices*, Conf. Semin. Mat. Univ. Bari 283 (2003) 1–29.

[18] H. KUNITA, *General boundary conditions for multi-dimensional diffusion processes*, Kyoto J. Math. 10 (1970) 273–335.

[19] J. L. LIONS, E. MAGENES, *Non-Homogeneous Boundary Value Problems and Applications*, Grundlehren der mathematischen Wissenschaften, vol. 1, Springer-Verlag, Berlin, 1972.

[20] J. L. LIONS, E. MAGENES, *Non-Homogeneous Boundary Value Problems and Applications*, Grundlehren der mathematischen Wissenschaften, vol. 2, Springer-Verlag, Berlin, 1972.

[21] D. MUGNOLO, *Second order abstract initial-boundary value problems*, Ph.D. Thesis, Eberhard-Karls-Universität Tübingen, Tübingen, 2004.

[22] D. MUGNOLO, *Abstract wave equations with acoustic boundary conditions*, Math. Nachr. 279 (2006) 299–318.

[23] D. MUGNOLO, *Asymptotics of semigroups generated by operator matrices*, Arab. J. Math. 3 (2014) 419–435.

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