Operators and Matrices
Volume 17, Number 3 (2023), 573–581
doi:10.7153/oam-2023-17-38

INVERTIBLE ELEMENTS AND COMMUTATIVITY IN BANACH ALGEBRAS

CHEIKH OULD HAMOUD

(*Communicated by J. Ball*)

Abstract. This paper investigates the commutativity of Banach algebras in which the group of invertible elements is subject to certain algebraic and metric conditions. Application to a particular class of Banach algebras is examined.

1. Introduction

Throughout this paper, A is a complex Banach algebra with norm $\|.\|$. If the algebra A is unital, with unit 1 and $\|1\| = 1$, the group of invertible elements is denoted $\mathscr{G}$. If the algebra is not assumed unital, its unitization A_1 is the Banach algebra $A_1 = A + \mathbb{C}$. An element $x \in A$ is quasi-invertible, with quasi-inverse $x^\circ \in A$, if $x + x^\circ - xx^\circ = 0$. The set of quasi-invertible elements of A is denoted $\mathscr{G}^\circ$. For an element x of the algebra A, the spectrum and the spectral radius of x are denoted $\sigma(x)$ and $r(x)$, respectively. By definition, the spectral radius is sub-additive on A if there exists a positive constant α such that $r(x+y) \leqslant \alpha(r(x)+r(y))$ for all x, y in A; similarly, the spectral radius is sub-multiplicative on A if $r(xy) \leqslant \beta(r(x)r(y))$ for some positive constant β and all x, y in A. The center of the algebra is $\mathscr{C}(A) = \{x \in A : xy = yx \text{ for all } y \in A\}$. The Jacobson radical of the algebra is denoted $\mathscr{R}(A)$. The algebra A is said to be almost commutative if the quotient algebra $A/\mathscr{R}(A)$ is commutative. An element $x \in A$ is said to be almost central if $\overline{x} = x + \mathscr{R}(A)$ is in the center of the quotient algebra $A/\mathscr{R}(A)$. An invertible element $x \in A$ is said to satisfy the condition (B) if $\overline{B}(x, \frac{1}{\|x^{-1}\|}) \cap (A \backslash \mathscr{G}) \neq \phi$, meaning that the boundary of the ball centered at x with radius $\frac{1}{\|x^{-1}\|}$ intersects the set of singular elements of A [8]. The Banach algebra A is said to satisfy condition (B) if every $x \in \mathscr{G}$ satisfies condition (B).

The investigation of algebraic and metric conditions leading to commutativity in Banach algebras dates back to C. Le Page's note [6]. The commutativity properties of Banach algebras have subsequently been examined in several works, including [2], [3], [5], [6], [10] and many others.

B. Aupetit [2] and V. Ptak and J. Zemanek [6], in particular, independently investigated the properties of the spectral radius and their relationship with commutativity

Mathematics subject classification (2020): Primary 46H05; Secondary 46H20.
Keywords and phrases: Banach algebra, invertible element, commutativity, spectral radius, radical.
The author is thankful to the reviewer for valuable comments and useful suggestions.

Paper OaM-17-38

in Banach algebras. Using subharmonic functions, B. Aupetit [2] established that in a complex Banach algebra, the following conditions are equivalent.

1. The spectral radius is sub-multiplicative.
2. The spectral radius is sub-additive.
3. The spectral radius is uniformly continuous.
4. The algebra A is almost commutative.

V. Ptak and J. Zemanek [6] and J. Zemánek [10], for their part, obtained similar results using elementary algebraic methods. Several other equivalent conditions involving both the norm and the spectral radius have also been established [3].

In a Banach algebra, the group of invertible elements is open and its linear envelope coincides with the algebra. It is therefore useful to determine to what extent the conditions on the elements of this group determine the properties of the algebra. This paper investigates the commutativity of Banach algebras in which the group of invertible elements is subject to certain algebraic and metric conditions related to the algebraic structure, the norm and the spectral radius. First, we obtain commutativity criteria from algebraic conditions on the group of invertible elements of a Banach algebra. A characterization of Banach algebras in which the spectral radius is a Banach algebra norm is obtained (Theorem 3) and the almost commutativity criteria for Banach algebras are established under weaker conditions. As an application, we examine the Banach algebras that satisfy condition (B) [8].

2. Algebraic conditions

We first consider algebraic conditions on the set of invertible elements of a Banach algebra.

THEOREM 1. *The unital Banach algebra A is commutative if and only if there exists a positive integer $j \geqslant 1$ such that $xy^j = y^j x$ for all invertible elements $x, y \in \mathscr{G}$.*

Proof. To show that the condition is sufficient for $j \geqslant 2$, observe that if x and y are elements in A, then:

$$e^x e^y = e^x (e^{\frac{1}{j}y})^j = (e^{\frac{1}{j}y})^j e^x = e^y e^x.$$

Therefore, the algebra is commutative by [4], Lemma 1.1. $\square$

COROLLARY 1. *Let $k \geqslant 2$ and $A^k = \{x_1 x_2 \cdots x_k; x_i \in \mathscr{G}, i = 1, 2. \cdots k\}$. Assume that A^k is contained in the center $\mathscr{C}(A)$ of the unital Banach algebra A. Then, the algebra A is commutative.*

Proof. Since by assumption $y^k \in \mathscr{C}(A)$ for all $y \in \mathscr{G}$, we have $xy^k = y^k x$ for all $x \in A$ and the algebra is commutative by Theorem 1. $\square$

A Banach algebra, without unit element, satisfying $xy^j = y^j x$ for all $x, y \in A$, and some positive integer $j \geqslant 2$ may not be commutative, as shown in the following example.

EXAMPLE 1. Consider the algebra $\mathbb{C}^3$ with product defined by

$$(x_1,x_2,x_3)(y_1,y_2,y_3) = (0,0,x_1y_2)$$

This algebra is not commutative and $xy^2 = y^2x = 0$ for all $x, y \in A$ as can be easily checked. □

THEOREM 2. *The unital Banach algebra A is commutative if and only if there exists a positive integer $k \geqslant 2$ such that $(xy)^k = x^k y^k$ for all invertible elements $x, y \in \mathcal{G}$.*

Proof. It is enough to show that the condition is sufficient.

First, note that multiplication of the relation $(xy)^k = x^k y^k$ by x^{-1} on the left and by y^{-1} on the right gives

$$(yx)^{k-1} = x^{k-1}y^{k-1}.$$

This ends the proof for $k = 2$. If $k > 2$, let $x \in A$, $y \in \mathcal{G}$ and λ a complex number. With $l = k-1$ for short, the above relation yields:

$$e^{\lambda x}y^l e^{-\lambda x} = (e^{\lambda x}ye^{-\lambda x})^l = (e^{-\lambda x})^l (e^{\lambda x}y)^l = e^{-l\lambda x}y^l e^{l\lambda x}.$$

Rearranging gives

$$e^{(l+1)\lambda x}y^l e^{-(l+1)\lambda x} = y^l.$$

Taking the derivative of both sides at $\lambda = 0$ we get $xy^l = y^l x$. Therefore, the algebra is commutative by Theorem 1. □

REMARK 1.

a) If $(xy)^k = y^k x^k$, then $(yx)^{k+1} = y^{k+1}x^{k+1}$ and the algebra is again commutative.

b) A Banach algebra, without unit element, which satisfies $(xy)^k = x^k y^k$ for all $x, y \in A$, and some positive integer $k \geqslant 2$ is almost commutative [4, Theorem 2.5]. Such algebra may not be commutative as shown in [4, Example 2.2].

3. Conditions on the norm and the spectral radius

Regarding the conditions on the norm and the spectral radius, we first establish a necessary and sufficient conditions for the spectral radius to be a Banach algebra norm on the Banach algebra $(A, \|.\|)$.

THEOREM 3. *The spectral radius is a Banach algebra norm on the unital complex Banach algebra $(A, \|.\|)$ if and only if $\|x\| \leqslant \kappa r(x)$ for some positive constant $\kappa \geqslant 1$ and all invertible elements $x \in \mathcal{G}$.*

Proof. If the spectral radius is a Banach algebra norm on A, the result follows from Banach's isomorphism theorem ([1], Corollary 3.42, p. 132) since $r(x) \leqslant \|x\|$ for all $x \in A$.

Conversely, If $\|x\| \leqslant \kappa r(x)$ for every $x \in \mathscr{G}$, let $y \in A$ be any element of A and take λ such that $|\lambda| > r(y)$. Write $y = (y-\lambda)+\lambda$. Then, $\|y\| \leqslant \|y-\lambda\| + |\lambda| \leqslant \kappa r(y-\lambda)+|\lambda| < 3\kappa|\lambda|$. Therefore, $\|y\| \leqslant 3\kappa r(y)$ for all $y \in A$. Thus, the norms r and $\|.\|$ are equivalent. □

COROLLARY 2. *If $\|x^2\| = \|x\|^2$ for all invertible elements $x \in \mathscr{G}$ in the unital complex Banach algebra A, then $\|x\| \leqslant \alpha r(x)$ for all $x \in A$ where α is a positive constant.*

Proof. $\|x^2\| = \|x\|^2$ if and only if $\|x\| = r(x)$ ([7], Lemma 1.4.2, p. 11). Therefore, the given condition is equivalent to $\|x\| = r(x)$ for all invertible elements x. The conclusion follows from Theorem 3. □

Remark that the condition $r(x) = \|x\|$ on invertible elements may not extend to the whole algebra as shown by Example 2.1 in [9]. However, in that example, the spectral radius is a Banach algebra norm and there exists a positive constant α such that $\|x\| \leqslant \alpha r(x)$ for all $x \in A$.

Spectral radius sub-additivity and sub-multiplicativity are equivalent to almost commutativity in Banach algebra ([2], [5] and [10]). In fact, weaker mixed forms of sub-additivity and sub-multiplicativity are sufficient, as in the following theorem.

THEOREM 4. *Let $(A, \|.\|)$ be a unital Banach algebra. Then:*

1. The algebra A is almost commutative if and only if the following equivalent conditions are satisfied:

2. There exists $\alpha > 0$ such that $r(xy) \leqslant \alpha r(x)\|y\|$ for all $x \in \mathscr{G}, y \in A$.

3. There exists $\beta > 0$ such that $r(x+y) \leqslant r(x) + \beta\|y\|$ for all $x \in \mathscr{G}, y \in A$.

4. $|r(x) - r(y)| \leqslant \|x - y\|$ for all $x \in \mathscr{G}, y \in A$.

Proof. We show that $1. \Rightarrow 2. \Rightarrow 3. \Rightarrow 1. \Rightarrow 4. \Rightarrow 3.$

$1. \Rightarrow 2.$: Since the algebra is almost commutative, we have: $r(xy) = r(\overline{x}\,\overline{y}) \leqslant r(\overline{x})r(\overline{y}) = r(x)r(y) \leqslant r(x)\|y\|$ where $\overline{x} = x + \mathscr{R}(A)$. Therefore, condition 2 is satisfied with $\alpha = 1$.

$2. \Rightarrow 3.$: Let $x \in \mathscr{G}, y \in A$, and λ a complex number, $|\lambda| > r(x) + \alpha\|y\|$. Then, $\lambda - x$ is invertible and $r((\lambda-x)^{-1}y) \leqslant \alpha r((\lambda-x)^{-1})\|y\| \leqslant \frac{\alpha\|y\|}{|\lambda|-r(x)} < 1$. Therefore, $\lambda - x - y = (\lambda - x)[1-(\lambda-x)^{-1}y] \in \mathscr{G}$; hence, $r(x+y) \leqslant r(x) + \alpha\|y\|$, and condition 3 is satisfied with $\beta = \alpha$.

$3. \Rightarrow 1.$: Using the same arguments of [2], Theorem 2, p. 48, let $x \in \mathscr{G}, y \in A$ and $\lambda \in \mathbb{C}$. Set $f(\lambda) = \frac{e^{\lambda y}xe^{-\lambda y} - x}{\lambda} = [y,x] + \frac{\lambda}{2}[y,[y,x]] + \ldots$ where $[x,y] = xy - yx$. Then, $r(f(\lambda))$ is a subharmonic function such that $r(f(\lambda)) \leqslant \frac{r(x)+\beta\|x\|}{|\lambda|}$. This shows that the function $r(f(\lambda))$ is the zero function since it tends towards 0 at infinity. Therefore, $r(xy - yx) = 0$ for all $x \in \mathscr{G}, y \in A$; hence, $xy - yx \in \mathscr{R}(A)$ by [2], Corollary 8, p. 46. Thus, the open set $\mathscr{G}$ is contained in the closed subspace $L_y^{-1}(\mathscr{R}(A))$ where $L_y(x) = xy - yx$ and therefore $xy - yx \in \mathscr{R}(A)$ for all $x, y \in A$.

1. $\Rightarrow$ 4.: Since the algebra A is almost commutative, we have $|r(x)-r(y)| = |r(\overline{x})-r(\overline{y})| \leqslant r(\overline{x}-\overline{y}) = r(x-y) \leqslant \|x-y\|$ where $\overline{x} = x+\mathscr{R}(A)$.

4. $\Rightarrow$ 3.: Let $x \in \mathscr{G}$, $y \in A$. By 4. we have $r(x+y)-r(x) \leqslant |r(x+y)-r(x)| \leqslant \|y\|$; hence, $r(x+y) \leqslant r(x)+\|y\|$, and condition 3 is satisfied with $\beta = 1$. $\square$

Conditions 2 and 3 of Theorem 4 are equivalent to the almost commutativity of the algebra. Therefore, these conditions are actually satisfied with $\alpha = 1$ and $\beta = 1$, respectively, as long as they are satisfied in the apparently less restrictive form stated in the theorem, with arbitrary positive constants α and β.

The conditions in Theorem 4 are also equivalent to the following ones:

THEOREM 5. *The unital Banach algebra $(A, \|.\|)$ is almost commutative if and only if the following equivalent conditions are satisfied:*

5. For every $x \in \mathscr{G}$, the open ball $B_x = \left\{x-y : y \in A, \|y\| < \frac{1}{r(x^{-1})}\right\}$ is contained in $\mathscr{G}$.

6. For every $x \in \mathscr{G}$, the set $B_{x,r} = \left\{x-y : y \in A, r(y) < \frac{1}{r(x^{-1})}\right\}$ is contained in $\mathscr{G}$.

Proof. The theorem is proven if we show that:

Almost commutativity $\Rightarrow$ Condition 6 $\Rightarrow$ Condition 5 $\Rightarrow$ Almost commutativity.

Since condition 5 is a consequence of condition 6, it is enough to prove that (1) almost commutativity implies condition 6, and (2) condition 5 implies almost commutativity.

(1) If the algebra A is almost commutative, let $x \in \mathscr{G}$ and $y \in A$ such that $r(y) < \frac{1}{r(x^{-1})}$ then $r(x^{-1}y) \leqslant r(x^{-1})r(y) < 1$ and therefore $x-y = x(1-x^{-1}y)$ is invertible; hence, $\mathrm{B}_{x,r} \subseteq \mathscr{G}$ and condition 6 is satisfied.

(2) If condition 5 is satisfied, $\mathrm{B}_z \subseteq \mathscr{G}$ for every $z \in \mathscr{G}$. Let $x \in \mathscr{G}, y \in A$ and $\lambda \in \mathbb{C}, |\lambda| > r(x)+\|y\|$. Then, $\lambda - x$ is invertible and $r((\lambda-x)^{-1})\|y\| \leqslant \frac{\|y\|}{|\lambda|-r(x)} < 1$. Therefore, $\lambda - x - y \in \mathrm{B}_{\lambda-x} \subseteq \mathscr{G}$. Hence, $r(x+y) \leqslant r(x)+\|y\|$ and the algebra is almost commutative by Theorem 4.(3). $\square$

Notice that if the unital Banach algebra A is almost commutative, then, by condition 6 of Theorem 5, the set $\mathscr{G}$ of invertible elements in A is open in the topology induced on A by the spectral radius semi-norm r.

The properties of the spectral radius and the norm, which are equivalent to the almost commutativity in unital Banach algebra, are summarized in the following corollary.

COROLLARY 3. *The unital Banach algebra $(A, \|.\|)$ is almost commutative if and only if the following equivalent conditions are satisfied for all $x \in \mathscr{G}$ and $y \in A$.*

1. $r(xy) \leqslant r(x)r(y)$.
2. $r(xy) \leqslant r(x)\|y\|$.
3. $r(x+y) \leqslant r(x)+\|y\|$.
4. $r(x+y) \leqslant r(x)+r(y)$.
5. $|r(x)-r(y)| \leqslant \|x-y\|$.

6. $|r(x)-r(y)| \leqslant r(x-y)$.

7. For every $x \in \mathscr{G}$*, the open ball* $B_x = \left\{x-y : y \in A,\, \|y\| < \frac{1}{r(x^{-1})}\right\}$ *is contained in* $\mathscr{G}$.

8. For every $x \in \mathscr{G}$*, the set* $B_{x,r} = \left\{x-y : y \in A,\, r(y) < \frac{1}{r(x^{-1})}\right\}$ *is contained in* $\mathscr{G}$.

If the Banach algebra is not assumed unital, we have:

THEOREM 6. *The complex Banach algebra* $(A, \|.\|)$ *is*

1. almost commutative if and only if the following equivalent conditions are satisfied for all $x \in \mathscr{G}^\circ$ *and* $y \in A$.

2. $r(xy) \leqslant r(x)r(y)$.

3. $r(xy) \leqslant r(x)\|y\|$.

4. $r(x+y) \leqslant r(x) + \|y\|$.

5. $r(x+y) \leqslant r(x) + r(y)$.

6. $|r(x)-r(y)| \leqslant \|x-y\|$.

7. $|r(x)-r(y)| \leqslant r(x-y)$.

Proof. The theorem is proven if we show that:

$$1. \Rightarrow 2. \Rightarrow 3. \Rightarrow 4. \Rightarrow 1. \Rightarrow 5. \Rightarrow 7. \Rightarrow 4. \text{ and } 4. \Leftrightarrow 6.$$

It will be enough to prove the two implications $3. \Rightarrow 4.$ and $4. \Rightarrow 1.$ as the other implications are obvious.

$3. \Rightarrow 4.$: Let $x \in \mathscr{G}^\circ$, $y \in A$ and λ a complex number, $|\lambda| > r(x) + \|y\|$. Then $\frac{1}{\lambda}x$ and $\frac{1}{\lambda}y$ are both quasi-invertible and

$$1 - \tfrac{x+y}{\lambda} = (1-\tfrac{x}{\lambda})[1-(1-\tfrac{x}{\lambda})^{-1}\tfrac{x}{\lambda}\tfrac{y}{\lambda}(1-\tfrac{y}{\lambda})^{-1}](1-\tfrac{y}{\lambda}).$$

The term in the middle is invertible since

$$r((1-\tfrac{x}{\lambda})^{-1}\tfrac{x}{\lambda}\tfrac{y}{\lambda}(1-\tfrac{y}{\lambda})^{-1}) \leqslant r((1-\tfrac{x}{\lambda})^{-1}\tfrac{x}{\lambda})\|\tfrac{y}{\lambda}(1-\tfrac{y}{\lambda})^{-1}\| \leqslant \tfrac{r(x)}{|\lambda|-r(x)}\tfrac{\|y\|}{|\lambda|-\|y\|} < 1.$$

Therefore $1 - \frac{x+y}{\lambda}$ is invertible, hence $r(x+y) \leqslant r(x) + \|y\|$.

$4. \Rightarrow 1.$: If $x \in \mathscr{G}^\circ$ then $e^{\lambda y}xe^{-\lambda y} \in \mathscr{G}^\circ$ for all $y \in A$ and $\lambda \in \mathbb{C}$; the proof is similar to $3. \Rightarrow 1.$ in Theorem 4. $\square$

The conditions in Theorem 6 are also equivalent to the following ones:

THEOREM 7. *The complex Banach algebra* $(A, \|.\|)$ *is almost commutative if and only if the following equivalent conditions are satisfied:*

8. For every $x \in \mathscr{G}^\circ$ *the open ball* $B_x = \left\{x-y : y \in A, \|y\| < \frac{1}{1+r(x^\circ)}\right\}$ *is contained in* $\mathscr{G}^\circ$.

9. For every $x \in \mathscr{G}^\circ$ *the set* $B_{x,r} = \left\{x-y : y \in A, r(y) < \frac{1}{1+r(x^\circ)}\right\}$ *is contained in* $\mathscr{G}^\circ$.

Proof. Since condition 8 is a consequence of condition 9, it is enough to prove that (i) almost commutativity implies condition 9, and (ii) condition 8 implies almost commutativity.

(i) Note first that if the algebra A (not assumed unital) is almost commutative, then its unitization A_1 is too.

Let $x \in \mathscr{G}^\circ$ and $y \in A$ such that $r(y) < \frac{1}{1+r(x^\circ)}$. Then, $1-(x-y) = (1-x)[1-(1-x)^{-1}(-y)]$ and, by the sub-additivity of the spectral radius, we have

$$r((1-x)^{-1}(-y)) = r((1-x^\circ)y) \leqslant r(1-x^\circ)r(y) \leqslant (1+r(x^\circ))r(y) < 1.$$

Therefore, $x-y$ is quasi-invertible and the set $\mathrm{B}_{x,r}$ is contained in the set $\mathscr{G}^\circ$ of quasi-invertible elements.

(ii) If condition 8 is satisfied, let $x \in \mathscr{G}^\circ$, $y \in A$ and λ a complex number, $|\lambda| > r(x) + \|y\|$. Then, $\frac{1}{\lambda}x \in \mathscr{G}^\circ$ and

$$\begin{aligned}[1+r((\tfrac{1}{\lambda}x)^\circ)]\left\|-\frac{1}{\lambda}y\right\| &= [1+r(-x(\lambda-x)^{-1})]\left\|\frac{1}{\lambda}y\right\| \\ &\leqslant [1+\tfrac{r(x)}{|\lambda|-r(x)}]\left\|\frac{1}{\lambda}y\right\| = \tfrac{\|y\|}{|\lambda|-r(x)} < 1.\end{aligned}$$

Therefore, $\frac{1}{\lambda}x + \frac{1}{\lambda}y \in \mathrm{B}_{\frac{1}{\lambda}x} \subseteq \mathscr{G}^\circ$. Hence, $r(x+y) \leqslant r(x) + \|y\|$ and the algebra is almost commutative by Theorem 6. $\square$

4. Banach algebras satisfying condition (B)

Banach algebras satisfying condition (B) have been investigated by G. Sebastian and S. Daniel [8]. As the authors point out, condition (B) was initially introduced as part of the study of the ε-condition spectrum. In the present paper, however, algebras that satisfy condition (B) are considered from the perspective of commutativity criteria in Banach algebras.

Banach algebras satisfying condition (B) include unital C^*-algebras ([8], Theorem 5). Examples of commutative unital Banach algebras that do not satisfy condition (B) are also given in the same paper.

The following theorem gives a necessary and sufficient condition for a unital Banach algebra to be commutative and satisfy condition (B).

THEOREM 8. *The unital Banach algebra A is commutative and satisfies condition (B) if and only if $\|x\| = r(x)$ for all $x \in \mathscr{G}$.*

Moreover, the spectral radius r is then a Banach algebra norm on the algebra A.

Proof. If the algebra is commutative and $x \in \mathscr{G}$, then $r(x^{-1}y) \leqslant r(x^{-1})r(y) \leqslant r(x^{-1})\|y\|$ for all $y \in A$. Therefore, $B(x, \frac{1}{r(x^{-1})}) \subset \mathscr{G}$. If, in addition, A satisfies condition (B), there exists a singular element $s \in \overline{B}(x, \frac{1}{\|x^{-1}\|}) \cap (A \backslash \mathscr{G})$. Then, $s \in \overline{B}(x, \frac{1}{r(x^{-1})}) \cap (A \backslash \mathscr{G})$ since $B(x, \frac{1}{\|x^{-1}\|}) \subset B(x, \frac{1}{r(x^{-1})}) \subset \mathscr{G}$. Hence, $\|x^{-1}\| = r(x^{-1})$.

Conversely, If $\|x\| = r(x)$ for all $x \in \mathscr{G}$, the algebra satisfies condition (B) and $r(xy) \leqslant \|xy\| \leqslant \|x\|\|y\| = r(x)\|y\|$ for all $y \in A$. Therefore, the algebra is almost commutative by Theorem 4.(2). However, the algebra is semi-simple: if $x \in \mathscr{R}(A)$, then $x - \frac{1}{n} \in \mathscr{G}$ for all $n \geqslant 1$, and by assumption $\|x - \frac{1}{n}\| = r(x - \frac{1}{n}) \leqslant \frac{1}{n}$. Hence, $x = 0$.

Since $\|x\| = r(x)$ for every $x \in \mathscr{G}$, the spectral radius is a Banach algebra norm on A by Theorem 3. $\square$

It follows from this theorem that, in a non-commutative C^*-algebra, the relation $\|x\| = r(x)$, true for all normal elements ([1], Lemma 6.17, p. 270) is not satisfied for all the invertible elements of the algebra because otherwise the algebra would be commutative.

REMARK 2. Theorem 8 corresponds to Theorem 3 of [9].

The following result extends Theorem 2 of [9] to almost commutative Banach algebras.

THEOREM 9. *Let A be a unital Banach algebra and $x \in \mathscr{G}$. If x is an almost central element of A (i.e. $\overline{x} \in \mathscr{C}(A/\mathscr{R}(A))$), then x satisfies the condition (B) if and only if $\|x^{-1}\| = r(x^{-1})$.*

Proof. If $\|x^{-1}\| = r(x^{-1})$, let $s = x - \lambda_0$ where $\lambda_0 \in \sigma(x)$ and $|\lambda_0| = \min\{|\lambda|, \lambda \in \sigma(x)\} = \frac{1}{r(x^{-1})}$; s is a singular element and $s \in \overline{B}(x, \frac{1}{\|x^{-1}\|}) \cap (A \backslash \mathscr{G})$. Therefore, x satisfies condition (B).

Conversely, if $\overline{x}$ is central in $A/\mathscr{R}(A)$, then $r(x^{-1}y) = r(\overline{x}^{\,-1}\overline{y}) \leqslant r(\overline{x}^{\,-1})r(\overline{y}) \leqslant r(x^{-1})\|y\|$ for all $y \in A$, hence $B(x, \frac{1}{r(x^{-1})}) \subset \mathscr{G}$. Therefore, if $s \in \overline{B}(x, \frac{1}{\|x^{-1}\|}) \cap (A \backslash \mathscr{G})$ then $s \in \overline{B}(x, \frac{1}{r(x^{-1})}) \cap (A \backslash \mathscr{G})$ since $B(x, \frac{1}{\|x^{-1}\|}) \subset B(x, \frac{1}{r(x^{-1})}) \subset \mathscr{G}$. As a result, $\|x^{-1}\| = r(x^{-1})$. $\square$

EXAMPLE 2. Consider the algebra $A = \left\{x \in \mathscr{M}_2(\mathbb{C}) : x = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right\}$ endowed with the norm $\|x\| = \max\{|a| + |b|, |c|\}$. A is a noncommutative Banach algebra with radical $\mathscr{R}(A) = \{y \in \mathscr{M}_2(\mathbb{C}) : y = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\}$.

The algebra A does not satisfy the condition (B): consider the invertible element $m = \begin{bmatrix} 1 & i \\ 0 & 2 \end{bmatrix} \in A$, then $\|m^{-1}\| = \frac{3}{2}$. A matrix $s = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in A$ is singular if and only if $ac = 0$. Therefore, if $a = 0$, then $\|m - s\| \geqslant 1 > \frac{1}{\|m^{-1}\|} = \frac{2}{3}$, and if $c = 0$, then $\|m - s\| \geqslant 2 > \frac{1}{\|m^{-1}\|} = \frac{2}{3}$. Hence $\overline{B}(m, \frac{1}{\|m^{-1}\|}) \cap (A \backslash \mathscr{G}) = \phi$.

The algebra A is almost commutative: if $x = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ and $x_1 = \begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}$ then

$$xx_1 - x_1x = \begin{bmatrix} 0 & b_1(a-c) - b(a_1 - c_1) \\ 0 & 0 \end{bmatrix} \in \mathscr{R}(A).$$

By Theorem 9, the invertible matrix $x = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in A$ satisfies the condition (B) if and only if $\|x^{-1}\| = r(x^-1)$, that is, if and only if

$$\max\left\{\frac{1}{|a|} + \frac{|b|}{|ac|}, \frac{1}{|c|}\right\} = \max\left\{\frac{1}{|a|}, \frac{1}{|c|}\right\}.$$

This will be the case if and only if $b = 0$ or $|a| = |b| + |c|$. In both cases, $s = x - \lambda_0 I$ (where $\lambda_0 = a$ if $|a| \leqslant |c|$ and $\lambda_0 = c$ if $|a| > |c|$) is a singular element on the boundary of the ball centered at x with radius $\frac{1}{\|x^{-1}\|}$. $\square$

REMARK 3. An almost commutative Banach algebra which satisfies condition (B) is semi-simple, hence commutative.

REFERENCES

[1] G. R. ALLAN AND H. G. DALES, *Introduction to Banach Spaces and Algebras*, Oxford University Press, 2011.

[2] B. AUPETIT, *Propriétés spectrales des Algèbres de Banach*, Lecture Notes in Math., 735, Springer-Verlag, 1979.

[3] O. H. CHEIKH, *Algèbres de Banach presque commutatives*, C. R. Math. Acad. Sc. Canada, vol. XVII, no. 6, 1995, 243–247.

[4] O. H. CHEIKH, *Commutativity Criteria in Banach Algebras*, C. R. Math. Acad. Sc. Canada, vol. 37, no. 3, 2015, 89–93.

[5] C. LE PAGE, *Sur quelques conditions entraînant la commutativité dans les algèbres de Banach*, C. R. Acad. Sci. Paris, Sér. A 265, 1967, 235–237.

[6] V. PTÁK AND J. ZEMÁNEK, *On uniform continuity of spectral radius in Banach algebras*, manuscripta math. 20, 1977, 177–189.

[7] C. E. RICKART, *General theory of Banach algebras*, The University series in higher mathematics. R. E. Krieger Pub. Co., 1974.

[8] G. SEBASTIAN AND S. DANIEL, *On the open ball centered at an invertible element of a Banach Algebra*, Oper. Matrices. 12, 1, 2018, 19–25.

[9] G. SEBASTIAN AND S. DANIEL, *A characterizing property of commutative Banach algebras may not be sufficient only on the invertible element elements of a Banach Algebra*, Comptes Rendus Mathematique 356, 6, 2018, 594–596.

[10] J. ZEMÀNEK, *Spectral radius characterizations of commutativity in Banach algebras*, Studia Math. 61, 1977, 257–268.

(Received September 1, 2021)

Cheikh O. Hamoud
Department of Mathematics
Université de Nouakchott
Mauritania
e-mail: cheikhouldhamoud@yahoo.com
cheikh.hamoud@una.mr