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# ON $2\times 2$ POSITIVE MATRICES OF $\tau$–MEASURABLE OPERATORS

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*Abstract.* Let $\mathscr{M}$ be a semi-finite von Neumann algebra. We proved the following inequalities are hold and equivalent:

(i) If $x,y \in L_{\log_+}(\mathscr{M})$ are self-adjoint operators such that $\pm y \leqslant x$, then $y \preccurlyeq_{\log} x$.

(ii) If $a,b \in \mathscr{M}$, $x,y \in L_{\log_+}(\mathscr{M})$ and $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geqslant 0$, then

$$a^*zb + b^*z^*a \preccurlyeq_{\log} a^*xa + b^*yb.$$

(iii) If $x,y,z \in L_{\log_+}(\mathscr{M})$ and $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geqslant 0$, then $z^*+z \preccurlyeq_{\log} x+y$.

(iv) If $x,y \in L_{\log_+}(\mathscr{M})$ are positive operators, then $x-y \preccurlyeq_{\log} x+y$.

(v) If $x,y,z \in L_{\log_+}(\mathscr{M})$ and $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geqslant 0$, then $z^* \oplus z \preccurlyeq_{\log} x \oplus y$.

(vi) If $x,y \in L_{\log_+}(\mathscr{M})$ are normal operators and $z \in L_{\log_+}(\mathscr{M})$ is positive operator, then for any contraction $a \in \mathscr{M}$,

$$|za(x+y)a^*z| \preccurlyeq_{\log} za(|x|+|y|)a^*z.$$

## 1. Introduction

We denote the set of all $n\times n$ complex matrices by $\mathbb{M}_n$ and by $\mathbb{M}_2(\mathbb{M}_n)$ the set of all $2\times 2$ block matrices, i.e.,

$$\mathbb{M}_2(\mathbb{M}_n) = \left\{ \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix},\ x_{i,j} \in \mathbb{M}_n,\ i,j = 1,2 \right\}.$$

We use the direct sum notation $x \oplus y$ for the block-diagonal matrix $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$. Bourin proved that if $\begin{pmatrix} a & c \\ c^* & b \end{pmatrix}$ and $\begin{pmatrix} a & c^* \\ c & b \end{pmatrix}$ are positive block-matrix with entries in $\mathbb{M}_n$, then

$$\prod_{j=1}^{k} s_j(c) \leqslant \prod_{j=1}^{k} s_j(a^{\frac{1}{2}} b^{\frac{1}{2}}), \qquad k = 1,2,\cdots,n, \tag{1}$$

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where $s_j(x)$ $(j = 1, 2, \cdots, n)$ is singular value of $x \in \mathbb{M}_n$ (see [12, Theorem 4.1]).

Let $x, y \in \mathbb{M}_n$ be Hermitian matrices such that $\pm y \leqslant x$. In general,

$$s_j(y) \leqslant s_j(x), \qquad j = 1, 2, \cdots, n$$

not holds (see [3, p. 121]). But, Bourin, Hirzallah and Kittaneh [1] proved that the following relation holds.

$$\prod_{j=1}^{k} s_j(y) \leqslant \prod_{j=1}^{k} s_j(x), \qquad k = 1, 2, \cdots, n. \tag{2}$$

Notice that (2) can also be concluded from the inequality (2.4) in [10] (also see [3, Theorem 4.1]). On the other hand,

$$0 \leqslant \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x+y & 0 \\ 0 & x-y \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} x & y \\ y & x \end{pmatrix}$$

and $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ is a unitary operator in $\mathbb{M}_2(\mathcal{M})$, where $1$ is the identity matrix in $\mathbb{M}_n$. Hence, we have that for $x, y \in \mathbb{M}_n$ are Hermitian matrices $\pm y \leqslant x$ if and only if $\begin{pmatrix} x & y \\ y & x \end{pmatrix} \geqslant 0$ (see [7]). Therefore, (2) also follows from (1).

If $x, y \in \mathbb{M}_n$ are positive matrices, then $x - y, x + y$ are Hermitian matrices such that $\pm(x - y) \leqslant x + y$. By (2),

$$x - y \preccurlyeq_{\log} x + y. \tag{3}$$

Conversely, if $x, y \in \mathbb{M}_n$ are Hermitian matrices $\pm y \leqslant x$, then $\frac{x-y}{2}, \frac{x+y}{2} \in \mathbb{M}_n$ are positive matrices. Using (3), one get (2). In [2, Proposition 1.1], Bourin and Lee proved that if $a \in \mathbb{M}_n$ is positive and $x, y \in \mathbb{M}_n$ are normal matrices, then for all $p \geqslant 1$,

$$\prod_{j=1}^{k} s_j(|a(x+y)a|^p) \leqslant \prod_{j=1}^{k} s_j(2^{p-1} a^p (|x|^p + |y|^p) a^p), \qquad k = 1, 2, \cdots, n. \tag{4}$$

It is clear that (4) implies (3).

Let $(\mathcal{M}, \tau)$ be a semi-finite von Neumann algebra. We denote by $L_0(\mathcal{M})$ the set of all $\tau$-measurable operators and by $\mu_t(x)$ the generalized singular number of $x \in L_0(\mathcal{M})$. In this paper, we generalize (4) for operators in $L_{\log_+}(\mathcal{M})$ (see next section for definition). We prove the following inequalities are equivalent:

(i) If $x, y \in L_{\log_+}(\mathcal{M})$ are self-adjoint operators such that $\pm y \leqslant x$, then $y \preccurlyeq_{\log} x$.

(ii) If $a, b \in \mathcal{M}$, $x, y \in L_{\log_+}(\mathcal{M})$ and $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geqslant 0$, then

$$a^* z b + b^* z^* a \preccurlyeq_{\log} a^* x a + b^* y b.$$

(iii) If $x, y \in L_{\log_+}(\mathcal{M})$ and $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geqslant 0$, then $z^* + z \preccurlyeq_{\log} x + y$.

(iv) If $x,y \in L_{\log_+}(\mathcal{M})$ are positive operators, then $x-y \preccurlyeq_{\log} x+y$.

(v) If $x,y \in L_{\log_+}(\mathcal{M})$ and $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geqslant 0$, then $z^* \oplus z \preccurlyeq_{\log} x \oplus y$.

(vi) If $x,y \in L_{\log_+}(\mathcal{M})$ are normal operators and $z \in L_{\log_+}(\mathcal{M})$ is positive operator, then for any contraction $a \in \mathcal{M}$,

$$|za(x+y)a^*z| \preccurlyeq_{\log} za(|x|+|y|)a^*z.$$

Using this result and an Araki-Lieb-Thirring type inequality in the $\tau$-measurable operator case ([8, Lemma 3.1]), we extend the (4) and [2, Corollary 2.10 and 2.13] to the $\tau$-measurable case.

## 2. Preliminaries

Let $\Omega = (0,\alpha)$ $(0 < \alpha \leqslant \infty)$ be equipped with the usual Lebesgue measure $\mu$. We denote by $L_0(\Omega)$ the space of $\mu$-measurable real-valued functions $f$ on $\Omega$ such that $\mu(\{\omega \in \Omega : |f(\omega)| > s\}) < \infty$ for some $s$. The decreasing rearrangement function $f^* : [0,\infty) \mapsto [0,\infty]$ for $f \in L_0(\Omega)$ is defined by

$$f^*(t) = \inf\{s > 0 : \mu(\{\omega \in \Omega : |f(\omega)| > s\}) \leqslant t\}$$

for $t \geqslant 0$. If $f,g \in L_0(\Omega)$ such that $\int_0^t f^*(s)ds \leqslant \int_0^t g^*(s)ds$ for all $t \geqslant 0$, $f$ is said to be *majorized* by $g$, denoted by $f \preccurlyeq g$. Let $E$ be a quasi-Banach subspace of $L_0(\Omega)$, simply called a quasi-Banach function space on $\Omega$ in the sequel. $E$ is said to be *symmetric* if, for $f \in E$ and $g \in L_0(\Omega)$ such that $g^*(t) \leqslant f^*(t)$ for all $t \geqslant 0$, one has $g \in E$ and $\|g\|_E \leqslant \|f\|_E$; $E$ is *fully symmetric* if, for $f \in L_0(\Omega)$ and $g \in E$ such that $f \preccurlyeq g$, we have $f \in E$ and $\|f\|_E \leqslant \|g\|_E$.

We always denote by $\mathcal{M}$ a semi-finite von Neumann algebra with a faithful normal finite trace $\tau$ and by $L_0(\mathcal{M})$ the set of all $\tau$-measurable operators associated with $(\mathcal{M},\tau)$. For $x \in L_0(\mathcal{M})$, the distribution function $\lambda_{\cdot}(x)$ of $x$ is defined by $\lambda_t(x) = \tau(e_{(t,\infty)}(|x|))$ for $t > 0$, where $e_{(t,\infty)}(|x|)$ is the spectral projection of $|x|$ in the interval $(t,\infty)$, and the generalized singular numbers $\mu_{\cdot}(x)$ of $x$ by

$$\mu_t(x) = \inf\{s > 0 : \lambda_s(x) \leqslant t\} \quad \text{for} \quad t > 0.$$

Let $E$ be a symmetric quasi-Banach function space on $(0,\alpha)$ $(\tau(1) = \alpha)$. Define

$$E(\mathcal{M},\tau) = \{x \in L_0(\mathcal{M}) : \|\mu(x)\|_E < \infty\}, \qquad \|x\|_E = \|\mu(x)\|_E.$$

Then $(E(\mathcal{M},\tau), \|\cdot\|_E)$ is a quasi-Banach space. We call it noncommutative symmetric space and denote by $E(\mathcal{M})$ (see [15, 17]).

If $x,y \in L_0(\mathcal{M})$, then we shall say that $x$ is submajorized by $y$, written $x \preccurlyeq y$, if and only if $\mu(x) \preccurlyeq \mu(y)$.

Let

$$L_{\log_+}(\mathcal{M}) = \{x \in L_0(\mathcal{M}) : \log_+ |x| \in L_1(\mathcal{M}) + \mathcal{M}\},$$

where $\log_+ t = \{\log t, 0\}$, $t > 0$. We recall that $L_{\log_+}(\mathscr{M})$ is a $*$-algebra and

$$L_1(\mathscr{M}) + \mathscr{M} \subset L_{\log_+}(\mathscr{M}) \subset L_0(\mathscr{M}).$$

For $x \in L_{\log_+}(\mathscr{M})$ and $t \in (0, \tau(1))$, the determinant function associated with $x$ is defined by

$$\Delta_t(x) = e^{\int_0^t \log \mu_s(x) ds}.$$

From the definition and [6, Lemma 2.5], we easy deduce that if $x \in L_{\log_+}(\mathscr{M})$ and $t > 0$, then

$$\Delta_t(x) = \Delta_t(x^*) = \Delta_t(|x|) \tag{5}$$

and

$$\Delta_t(x^r) = \Delta_t(x)^r, \quad \text{if } r > 0 \text{ and } x \text{ is positive.} \tag{6}$$

For the determinant function, we have the following Weyl inequality:

$$\Delta_t(xy) \leqslant \Delta_t(x)\Delta_t(y), \qquad \forall x, y \in L_{\log_+}(\mathscr{M}), \qquad \forall t > 0 \tag{7}$$

(see [4, Theorem 4.2]). Recall that if $x, y \in L_{\log_+}(\mathscr{M})$ and the product $xy$ is self adjoint, then

$$\Delta_t(xy) \leqslant \Delta_t(yx), \qquad t > 0. \tag{8}$$

If $x, y \in L_{\log_+}(\mathscr{M})$ such that

$$\int_0^t \log \mu_s(x) ds \leqslant \int_0^t \log \mu_s(y) ds, \qquad t > 0,$$

$x$ is said to be logarithmically submajorized by $y$, denoted by $x \preccurlyeq_{\log} y$. It is clear that $x \preccurlyeq_{\log} y$ if and only if $\Delta_t(x) \leqslant \Delta_t(y)$ for all $t > 0$. For $f(t) = e^t$ using [5, Lemma 4.1], we get that $x \preccurlyeq_{\log} y$ implies $x \preccurlyeq y$.

We recall the well-known equality:

$$e^{\frac{1}{t}\int_0^t log|f(s)|ds} = \lim_{p \to 0} \left( \frac{1}{t} \int_0^t |f(s|^p ds \right)^{\frac{1}{p}}, \tag{9}$$

$$\text{if } \int_0^t |f(s|^p ds < +\infty \text{ for some } p > 0$$

(see page 71 of [13]).

We remark that if $\mathscr{M} = \mathbb{M}_m$ and $\tau$ is the standard trace, then

$$\mu_t(x) = s_j(x), \quad t \in [j-1, j), \qquad j = 1, 2, \cdots, m.$$

Hence, if $x, y \in \mathbb{M}_m$, then $x \preccurlyeq y$ is equivalent to

$$\sum_{j=1}^k s_j(x) \leqslant \sum_{j=1}^k s_j(y), \qquad 1 \leqslant k \leqslant m,$$

$x \preccurlyeq_{\log} y$ is equivalent to

$$\prod_{j=1}^{k} s_j(x) \leqslant \prod_{j=1}^{k} s_j(y), \qquad 1 \leqslant k \leqslant m.$$

We will denote the semi-finite von Neumann algebra

$$\mathbb{M}_2(\mathscr{M}) = \left\{ \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}, \, x_{i,j} \in \mathscr{M}, \, i,j = 1,2 \right\}$$

on Hilbert space $\mathscr{H} \oplus \mathscr{H}$ by $\mathbb{M}_2(\mathscr{M})$, which is associated with the semi-finite trace $Tr \otimes \tau$.

We will use the following result (see [14, Proposition 3]), if $x \in L_0(\mathscr{M})$, then

$$\mu_t(x \oplus x^*) = \mu_{\frac{t}{2}}(x), \qquad t > 0. \tag{10}$$

## 3. Main results

First, we extend (2) to the semi-finite von Neumann algebra case.

LEMMA 1. *Let $x,y \in \mathscr{M}$ be self-adjoint operators such that $\pm y \leqslant x$. Then*

$$y \preccurlyeq_{\log} x.$$

*Proof.* We use the method in the proof of [1, inequality (1.6)]. It is clear that

$$0 \leqslant \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x+y & 0 \\ 0 & x-y \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} x & y \\ y & x \end{pmatrix},$$

$\begin{pmatrix} x+y & 0 \\ 0 & x-y \end{pmatrix} \geqslant 0$ and $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ is a unitary operator in $\mathbb{M}_2(\mathscr{M})$. It follows that $\begin{pmatrix} x & y \\ y & x \end{pmatrix} \geqslant 0$. Using [9, Lemma 2.2], we obtain that there exists a contraction $a$ such that $y = x^{\frac{1}{2}} a x^{\frac{1}{2}}$. By (7) and (8), we get that

$$\begin{aligned} \Delta_t(y) &= \Delta_t(x^{\frac{1}{2}} a x^{\frac{1}{2}}) \leqslant \Delta_t(xa) \\ &= e^{\int_0^t \log \mu_s(xa)ds} \leqslant e^{\int_0^t \log \|a\| \mu_s(a)ds} \\ &\leqslant e^{\int_0^t \log \mu_s(x)ds} = \Delta_t(x), \qquad t > 0. \quad \square \end{aligned}$$

LEMMA 2. *Let $x,y \in L_{\log_+}(\mathscr{M})$ be positive operators. Then $x - y \preccurlyeq_{\log} x + y$.*

*Proof.* First assume that $x,y$ are self-adjoint operators in $\mathscr{M}$. Since $\pm(x-y) \leqslant x+y$, by Lemma 1, the result holds.

If $x, y \in L_{\log_+}(\mathscr{M})$. Set $x_n = xe_{[0,n]}(x)$, $y_n = ye_{[0,n]}(y)$ for $n \in \mathbb{N}$. Then $x_n, y_n \in \mathscr{M}$ are positive operators, $x_n \leqslant x$, $y_n \leqslant y$, $x_n \to x$ and $y_n \to y$ in measure. Using [6, Lemma 3.4 and 2.5 (iii)], Fatou's lemma and the first case, we get

$$\begin{aligned}\int_0^t \log \mu_s(x-y)ds &\leqslant \int_0^t \liminf_{n\to\infty} \log \mu_s(x_n - y_n)ds \leqslant \liminf_{n\to\infty} \int_0^t \log \mu_s(x_n - y_n)ds \\ &\leqslant \liminf_{n\to\infty} \int_0^t \log \mu_s(x_n + y_n)ds \leqslant \int_0^t \log \mu_s(x+y)ds. \quad \square\end{aligned}$$

THEOREM 1. *The following statements are equivalent:*

(i) *If $x, y \in L_{\log_+}(\mathscr{M})$ are self-adjoint operators such that $\pm y \leqslant x$, then $y \preccurlyeq_{\log} x$.*

(ii) *If $a, b \in \mathscr{M}$, $x, y \in L_{\log_+}(\mathscr{M})$ and $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geqslant 0$, then*

$$a^* zb + b^* z^* a \preccurlyeq_{\log} a^* xa + b^* yb.$$

(iii) *If $x, y, z \in L_{\log_+}(\mathscr{M})$ and $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geqslant 0$, then $z^* + z \preccurlyeq_{\log} x + y$.*

(iv) *If $x, y \in L_{\log_+}(\mathscr{M})$ are positive operators, then $x - y \preccurlyeq_{\log} x + y$.*

(v) *If $x, y, z \in L_{\log_+}(\mathscr{M})$ and $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geqslant 0$, then $z^* \oplus z \preccurlyeq_{\log} x \oplus y$.*

(vi) *If $x, y \in L_{\log_+}(\mathscr{M})$ are normal operators and $z \in L_{\log_+}(\mathscr{M})$ is positive operator, then for any contraction $a \in \mathscr{M}$,*

$$|za(x+y)a^* z| \preccurlyeq_{\log} za(|x| + |y|)a^* z.$$

*Proof.* (i) $\Rightarrow$ (ii) If $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geqslant 0$, we use the method in the proof of [3, Theorem 3.2 and 4.4] to obtain that

$$\begin{pmatrix} a^* xa + b^* yb + a^* zb + b^* z^* a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a^* & b^* \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \geqslant 0$$

and

$$\begin{pmatrix} a^* xa + b^* yb - a^* zb - b^* z^* a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a^* & -b^* \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \begin{pmatrix} a & 0 \\ -b & 0 \end{pmatrix} \geqslant 0.$$

Hence, $a^* xa + b^* yb \geqslant \pm(a^* zb + b^* z^* a)$, so by (i), we obtain (ii).

(ii) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (i) If $x, y \in L_{\log_+}(\mathscr{M})$ are self-adjoint operators such that $\pm y \leqslant x$, then $\begin{pmatrix} x & y \\ y & x \end{pmatrix} \geqslant 0$. By (iii), we deduce that (i) holds.

(i) $\Rightarrow$ (iv) If $x,y\in L_{\log_+}(\mathscr{M})$ are positive operators, then $\pm(x-y)\leqslant x+y$. By (i), we obtain (iv).

(iv) $\Rightarrow$ (i) If $x,y\in L_{\log_+}(\mathscr{M})$ are self-adjoint operators such that $\pm y\leqslant x$, then $x-y\geqslant 0$, $x+y\geqslant 0$. By (iv), we obtain (i).

(i) $\Rightarrow$ (v) Since $\begin{pmatrix} x & z\\ z^* & y\end{pmatrix}\geqslant 0$ and $\begin{pmatrix} x & -z\\ -z^* & y\end{pmatrix}\geqslant 0$, we get that

$$\begin{pmatrix} x & 0\\ 0 & y\end{pmatrix}\geqslant \pm\begin{pmatrix} 0 & z\\ z^* & 0\end{pmatrix}.$$

Hence, by (i), $z^*\oplus z\preccurlyeq_{\log} x\oplus y$.

(v) $\Rightarrow$ (vi) Let $x=u|x|$ be the polar decomposition of $x$. Then $x=|x|^{\frac{1}{2}}u|x|^{\frac{1}{2}}$ and

$$\begin{pmatrix} |x| & x\\ x^* & |x|\end{pmatrix}=\begin{pmatrix} |x|^{\frac{1}{2}} & 0\\ 0 & |x|^{\frac{1}{2}}\end{pmatrix}\begin{pmatrix} 1 & u\\ u^* & 1\end{pmatrix}\begin{pmatrix} |x|^{\frac{1}{2}} & 0\\ 0 & |x|^{\frac{1}{2}}\end{pmatrix}.$$

It is clear that $\begin{pmatrix} 1 & u\\ u^* & 1\end{pmatrix}\geqslant 0$, and so $\begin{pmatrix} |x| & x\\ x^* & |x|\end{pmatrix}\geqslant 0$. Similarly, $\begin{pmatrix} |y| & y\\ y^* & |y|\end{pmatrix}\geqslant 0$. Hence, $\begin{pmatrix} |x|+|y| & x+y\\ x^*+y^* & |x|+|y|\end{pmatrix}\geqslant 0$. Therefore,

$$\begin{pmatrix} za(|x|+|y|)a^*z & za(x+y)a^*z\\ za(x^*+y^*)a^*z & za(|x|+|y|)a^*z\end{pmatrix}=\begin{pmatrix} za & 0\\ 0 & za\end{pmatrix}\begin{pmatrix} |x|+|y| & x+y\\ x^*+y^* & |x|+|y|\end{pmatrix}\begin{pmatrix} a^*z & 0\\ 0 & a^*z\end{pmatrix}\geqslant 0.$$

Using (v) and (10), we obtain that

$$\begin{aligned}\Delta_{\frac{t}{2}}(za(x+y)a^*z)^2 &= \Delta_t(za(x+y)a^*z\oplus za(x^*+y^*)a^*z)\\ &\leqslant \Delta_t(za(|x|+|y|)a^*z\oplus za(|x|+|y|)a^*z)\\ &= \Delta_{\frac{t}{2}}(za(|x|+|y|)a^*z)^2, \qquad t>0.\end{aligned}$$

It follows that

$$za(x+y)a^*z\preccurlyeq_{\log} za(|x|+|y|)a^*z.$$

(vi) $\Rightarrow$ (iv) is trivial. □

REMARK 1. From (v) of Theorem 1 it follows that [11, Theorem 3.1] holds for operators in $L_{\log_+}(\mathscr{M})$.

Now, we extend [2, Corollary 2.10 and 2.13] to the $\tau$-measurable case.

PROPOSITION 1. *Let $p \geqslant 1$, $z \in L_{\log_+}(\mathscr{M})$ be positive operator and $a \in \mathscr{M}$ be a contraction.*

(i) *If $x_i \in L_{\log_+}(\mathscr{M})$, $i = 1,2,\cdots,m$ are normal operators, then*

$$\left|za\Big(\frac{\sum_{i=1}^m x_i}{m}\Big)a^*z\right|^p \preccurlyeq_{\log} \left(za\Big(\frac{\sum_{i=1}^m |x_i|}{m}\Big)a^*z\right)^p \preccurlyeq_{\log} z^p a\Big(\frac{\sum_{i=1}^m |x_i|^p}{m}\Big)a^*z^p.$$

(ii) *If $x \in L_{\log_+}(\mathscr{M})$, then*

$$\left|za\Big(\frac{x+x^*}{2}\Big)a^*z\right|^p \preccurlyeq_{\log} \left(za\Big(\frac{|x|+|x^*|}{2}\Big)a^*z\right)^p \preccurlyeq_{\log} z^p a\Big(\frac{|x|^p+|x^*|^p}{2}\Big)a^*z^p.$$

*Proof.* (i) Let $Z = \begin{pmatrix} z & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$, $X = \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & x_m \end{pmatrix}$, $A = \frac{1}{\sqrt{m}}\begin{pmatrix} a & a & \cdots & a \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$.
By (vi) of Theorem 1,

$$|ZAXA^*Z| \preccurlyeq_{\log} ZA|X|A^*Z,$$

hence,

$$\left|za\Big(\frac{\sum_{i=1}^m x_i}{m}\Big)a^*z\right|^p \preccurlyeq_{\log} \left(za\Big(\frac{\sum_{i=1}^m |x_i|}{m}\Big)a^*z\right)^p.$$

Using [8, Lemma 3.1], we deduce that for any $r > 0$,

$$[(ZA|X|A^*Z)^p]^r \preccurlyeq (Z^pA|X|^pA^*Z^p)^r.$$

Applying (9), we obtain that $|ZAXA^*Z|^p \preccurlyeq_{\log} (ZA|X|A^*Z)^p$, i.e.,

$$\left(za\Big(\frac{\sum_{i=1}^m |x_i|}{m}\Big)a^*z\right)^p \preccurlyeq_{\log} z^p a\Big(\frac{\sum_{i=1}^m |x_i|^p}{m}\Big)a^*z^p.$$

(ii) From the proof of (v) $\Rightarrow$ (vi) of Theorem 1, we know that $\begin{pmatrix} |x^*| & x \\ x^* & |x| \end{pmatrix} \geqslant 0$. Hence,

$$\begin{pmatrix} |x| & x^* \\ x & |^*x| \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} |x^*| & x \\ x^* & |x| \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \geqslant 0,$$

and so $\begin{pmatrix} za(\frac{|x|+|x^*|}{2})za^* & za(\frac{x^*+x}{2})za^* \\ za(\frac{x+x^*}{2})za^* & za(\frac{|x^*|+|x|}{2})za^* \end{pmatrix} \geqslant 0$. Using (iii) of Theorem 1, we deduce that

$$za\Big(\frac{x^*+x}{2}\Big)za^* \preccurlyeq_{\log} za\Big(\frac{|x|+|x^*|}{2}\Big)za^*.$$

So, it follows that $\left|za(\frac{x+x^*}{2})a^*z\right|^p \preccurlyeq_{\log} \left(za(\frac{|x|+|x^*|}{2})a^*z\right)^p$.

Let $Z = \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix}$, $X = \begin{pmatrix} |x| & 0 \\ 0 & |x^*| \end{pmatrix}$ and $A = \frac{1}{\sqrt{2}}\begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix}$. The remainder of the proof follows exactly the same way as in the proof of (i). $\square$

COROLLARY 1. *The following statements are equivalent:*

(i) *If $x,y\in\mathbb{M}_n$ are Hermitian matrices and $\pm y\leqslant x$, then*

$$\prod_{j=1}^{k}s_j(y)\leqslant\prod_{j=1}^{k}s_j(x),\qquad k=1,2,\cdots,n.$$

(ii) *If $x,y,z,a,b\in\mathbb{M}_n$ and $\begin{pmatrix} x & z\\ z^* & y\end{pmatrix}\geqslant 0$, then*

$$\prod_{j=1}^{k}s_j(a^*zb+b^*z^*a)\leqslant\prod_{j=1}^{k}s_j(a^*xa+b^*yb),\qquad k=1,2,\cdots,n.$$

(iii) *If $x,y,\in\mathbb{M}_n$ and $\begin{pmatrix} x & z\\ z^* & y\end{pmatrix}\geqslant 0$, then*

$$\prod_{j=1}^{k}s_j(z+z^*)\leqslant\prod_{j=1}^{k}s_j(x+y),\qquad k=1,2,\cdots,n.$$

(iv) *If $x,y\in\mathbb{M}_n$ are positive semi-definite matrices, then*

$$\prod_{j=1}^{k}s_j(x-y)\leqslant\prod_{j=1}^{k}s_j(x+y),\qquad k=1,2,\cdots,n.$$

(v) *If $x,y,z\in\mathbb{M}_n$ and $\begin{pmatrix} x & z\\ z^* & y\end{pmatrix}\geqslant 0$, then*

$$\prod_{j=1}^{k}s_j(z\oplus z^*)\leqslant\prod_{j=1}^{k}s_j(x\oplus y),\qquad k=1,2,\cdots,n.$$

(vi) *If $x,y\in\mathbb{M}_n$ are normal matrix and $z\in\mathbb{M}_n$ is positive matrix, then for any contraction matrix $a\in\mathbb{M}_n$,*

$$\prod_{j=1}^{k}s_j(za(x+y)a^*z)\leqslant\prod_{j=1}^{k}s_j(za(|x|+|y|)a^*z),\qquad k=1,2,\cdots,n.$$

THEOREM 2. *Let $E$ be a fully symmetric Banach function space on $(0,\alpha)$ and $f$ be a continuous increasing function on $(0,\alpha)$ such that $f(0)=0$ and $t\to f(e^t)$ is convex. The following holds:*

(i) *If $x,y\in L_{\log_+}(\mathscr{M})$ are self-adjoint operators such that $\pm y\leqslant x$, then*

$$\|f(|y|)\|_E\leqslant\|f(|x|)\|_E.$$

(ii) *If* $a,b\in\mathscr{M}$, $x,y\in L_{\log_+}(\mathscr{M})$ *and* $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix}\geqslant 0$, *then*

$$\|f(|a^*zb+b^*z^*a|)\|_E\leqslant\|f(|a^*xa+b^*yb|)\|_E.$$

(iii) *If* $x,y\in L_{\log_+}(\mathscr{M})$ *and* $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix}\geqslant 0$, *then*

$$\|f(|z^*\oplus z|)\|_E\leqslant\|f(|x\oplus y|)\|_E.$$

COROLLARY 2. *Let* $E$ *be a fully symmetric Banach function space on* $(0,\alpha)$ *and* $f$ *be a continuous increasing function on* $(0,\alpha)$ *such that* $f(0)=0$ *and* $t\to f(e^t)$ *is convex. Then for any* $x,y\in E(\mathscr{M})$,

$$\|f(|x+x^*|)\|_E\leqslant\|f(|x|+|x^*|)\|_E$$

*and*

$$\|f(|x+y+x^*+y^*|)\|_E\leqslant\min\{\|f(|x+y|+|x^*+y^*|)\|_E,\|f(|x|+|x^*|+|y|+|y^*|)\|_E\}.$$

In the matrix case, the first inequality of Corollary 2 follows from [2, Corollary 2.13 ].

*Proof.* Since $\begin{pmatrix} |x|+|x^*| & x^*+x \\ x+x^* & |x^*|+|x| \end{pmatrix}\geqslant 0$, by (iii) of Theorem 1 and [5, Lemma 4.1], we obtain that $\|f(|x+x^*|)\|_E\leqslant\|f(|x^*|+|x|)\|_E$. Similarly,

$$\begin{pmatrix} |x|+|x^*|+|y|+|y^*| & x^*+x+y^*+y \\ x+x^*+y+y^* & |x^*|+|x|+|y^*|+|y| \end{pmatrix}\geqslant 0$$

and

$$\begin{pmatrix} |x+y|+|x^*+y^*| & x^*+y^*+x+y \\ x+y+x^*+y^* & |x^*+y^*|+|x+y| \end{pmatrix}\geqslant 0.$$

From these we get the second inequality. □

COROLLARY 3. *Let* $E$ *be a fully symmetric Banach function space on* $(0,\alpha)$ *and* $f$ *be a continuous increasing function on* $(0,\alpha)$ *such that* $f(0)=0$ *and* $t\to f(e^t)$ *is convex. Then for any* $a,b,c,d\in E(\mathscr{M})$,

$$\max\left\{\begin{array}{l}\|f(|ab^*+ba^*+cd^*+dc^*|)\|_E, \\ \|f(|ac^*+ca^*+bd^*+db^*|)\|_E, \\ \|f(|ad^*+da^*+bc^*+cb^*|)\|_E\end{array}\right\}\leqslant\|f(|a|^2+|b|^2+|c|^2+|d|^2|)\|_E.$$

*Proof.* Since $\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} |a|^2+|c|^2 & a^*b+c^*d \\ b^*a+d^*c & |b|^2+|d|^2 \end{pmatrix} \geqslant 0$, using (iii) of Theorem 1 and [5, Lemma 4.1], we obtain that

$$\|f(|ab^*+ba^*+cd^*+dc^*|)\|_E \leqslant \|f(|a|^2+|b|^2+|c|^2+|d|^2|)\|_E.$$

Similarly,

$$\|f(|ac^*+ca^*+bd^*+db^*|)\|_E \leqslant \|f(|a|^2+|b|^2+|c|^2+|d|^2|)\|_E$$

and

$$\|f(|ad^*+da^*+bc^*+cb^*|)\|_E \leqslant \|f(|a|^2+|b|^2+|c|^2+|d|^2|)\|_E. \quad \square$$

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