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CONVERSE OF FUGLEDE THEOREM

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Abstract. In this paper, we investigate when subnormal operators T_1 and T_2 are quasinormal provided their product is quasinormal. Also, we obtain as a corollary that subnormal n-th roots of a quasinormal operator are quasinormal, and thus we answer the question asked by Curto et al. in [4]. Also, we give sufficient conditions for quasinormal (subnormal) operators T_1 and T_2 to be normal if their product is normal. In other words, we find sufficient conditions for the converse of the Fuglede Theorem and also make a connection with the theory of subnormal pairs.

1. Introduction

Let $\mathscr{H}$ be a complex Hilbert space and let $\mathfrak{B}(\mathscr{H})$ denote the algebra of bounded linear operators on $\mathscr{H}$. An operator T is said to be *normal* if $T^*T = TT^*$, *quasinormal* if T commutes with T^*T, i.e., $TT^*T = T^*T^2$, *subnormal* if $T = N|_{\mathscr{H}}$, where N is normal and $N(\mathscr{H}) \subseteq \mathscr{H}$, and *hyponormal* if $T^*T \geqslant TT^*$. It is well known that

$$\text{normal} \Rightarrow \text{quasinormal} \Rightarrow \text{subnormal} \Rightarrow \text{hyponormal}.$$

Obviously, if T is a subnormal operator, then its normal extension N is an upper-triangular operator matrix given by

$$N = \begin{bmatrix} T & A \\ 0 & B^* \end{bmatrix} : \begin{pmatrix} \mathscr{H} \\ \mathscr{H}^\perp \end{pmatrix} \mapsto \begin{pmatrix} \mathscr{H} \\ \mathscr{H}^\perp \end{pmatrix},$$

for some $A \in \mathfrak{B}(\mathscr{H}^\perp, \mathscr{H})$ and $B \in \mathfrak{B}(\mathscr{H}^\perp)$. For more information on theory of upper-triangular operator matrices we refer a reader to [10].

For $S, T \in \mathfrak{B}(\mathscr{H})$ let $[S,T] = ST - TS$. We say that an n-tuple $\mathbf{T} = (T_1, \ldots, T_n)$ of operators on $\mathscr{H}$ is (jointly) *hyponormal* if the operator matrix

$$[\mathbf{T}^*, \mathbf{T}] := \begin{bmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{bmatrix}$$

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is positive on the direct sum of n copies of $\mathscr{H}$ (cf. [1], [5], [6]). The n-tuple $\mathbf{T}$ is said to be *normal* if $\mathbf{T}$ is commuting and each T_i is normal, and *subnormal* if $\mathbf{T}$ is the restriction of a normal n-tuple to a common invariant subspace. For $i,j,k \in \{1,2,\dots,n\}$, $\mathbf{T}$ is called *matricially quasinormal* if each T_i commutes with each $T_j^*T_k$, $\mathbf{T}$ is (jointly) *quasinormal* if each T_i commutes with each $T_j^*T_j$, and *spherically quasinormal* if each T_i commutes with $\sum_{j=1}^n T_j^*T_j$. As shown in [2] and [13], we have

$$\text{normal} \Rightarrow \text{matricially quasinormal} \Rightarrow \text{(jointly) quasinormal} \\ \Rightarrow \text{spherically quasinormal} \Rightarrow \text{subnormal.}$$

On the other hand, the results in [7] and [13] show that the inverse implications do not hold.

In a recent paper [4], R. E. Curto, S. H. Lee and J. Yoon, partially motivated by the results of their previous articles [8, 9], asked the following question:

PROBLEM 1. *Let T be a subnormal operator, and assume that T^2 is quasinormal. Does it follow that T is quasinormal?*

With the additional assumption of left invertibility they showed that a left invertible subnormal operator T whose square T^2 is quasinormal must be quasinormal (see [4, Theorem 2.3]). It remained an open question whether this is true in general without any assumption about left invertibility until the paper [14] was published. Moreover, the authors proved even stronger result:

THEOREM 2. [14] *Let $T \in \mathfrak{B}(\mathscr{H})$ be a subnormal operator such that T^n is quasinormal for some $n \in \mathbb{N}$. Then T is quasinormal.*

In Section 2, we give a generalization of Theorem 2. The crucial step is the following observation:

We can reformulate Problem 1 as follows: *Let $\mathbf{T} = (T,T)$ be a subnormal pair and assume that $T \cdot T$ is quasinormal. Does it follow that T is quasinormal?*

This also gives us the motivation for the following problems:

PROBLEM 3. *Let $\mathbf{T} = (T_1,T_2)$ be a subnormal pair such that T_1T_2 is quasinormal. Find sufficient conditions for T_1 and T_2 to be quasinormal.*

PROBLEM 4. *Let $\mathbf{T} = (T_1,T_2)$ be a (jointly) quasinormal pair such that T_1T_2 is normal. Find sufficient conditions for T_1 and T_2 to be normal.*

PROBLEM 5. *Let $\mathbf{T} = (T_1,T_2)$ be a subnormal pair such that T_1T_2 is normal. Find sufficient conditions for T_1 and T_2 to be normal.*

Problem 3–5 are closely related to celebrated Fuglede Theorem, and especially with its most famous corollary:

THEOREM 6. [12] *Let T and N be bounded operators on a complex Hilbert space with N being normal. If $TN = NT$, then $TN^* = N^*T$.*

THEOREM 7. [12] *If M and N are commuting normal operators, then MN is also normal.*

Thus, Problem 5 can be treated as a converse of Fuglede Theorem.

2. Results

The starting point in our discussion will be the following lemma:

LEMMA 1. [4] *Let $T \in \mathfrak{B}(\mathscr{H})$ be a subnormal operator with normal extension*

$$N = \begin{bmatrix} T & A \\ 0 & B^* \end{bmatrix} : \begin{pmatrix} \mathscr{H} \\ \mathscr{H}^\perp \end{pmatrix} \mapsto \begin{pmatrix} \mathscr{H} \\ \mathscr{H}^\perp \end{pmatrix}.$$

Then T is quasinormal if and only if $A^*T = 0$ and normal if and only if $A = 0$.

LEMMA 2. *Let $\mathbf{T} = (T_1, T_2)$ be a subnormal pair with the normal extension $\mathbf{N} = (N_1, N_2)$ such that T_2 is quasinormal and T_1T_2 is normal. If T_1 is left invertible, then T_2 is normal.*

Proof. Let

$$N_1 = \begin{bmatrix} T_1 & A_1 \\ 0 & B_1^* \end{bmatrix}, \quad N_2 = \begin{bmatrix} T_2 & A_2 \\ 0 & B_2^* \end{bmatrix}$$

be the normal extensions for T_1 and T_2, respectively, defined on $\mathscr{K} = \mathscr{H} \oplus \mathscr{H}^\perp$. Since $N_1N_2 = N_2N_1$, by Fuglede Theorem, N_1N_2 is normal. Thus,

$$N_1N_2 = \begin{bmatrix} T_1T_2 & T_1A_2 + A_1B_2^* \\ 0 & (B_2B_1)^* \end{bmatrix}$$

is a normal extension for T_1T_2. Operator T_1T_2 is normal, so, by Lemma 1, we have that $T_1A_2 + A_1B_2^* = 0$, i.e., $T_1A_2 = -A_1B_2^*$. Since T_1 is left invertible, there exists $C_1 \in \mathfrak{B}(\mathscr{H})$ such that $A_2 = -C_1A_1B_2^*$. From here, $\mathscr{N}(B_2^*) \subseteq \mathscr{N}(A_2)$ so $A_2|_{\mathscr{N}(B_2^*)} = 0$.

From $N_2^*N_2 = N_2N_2^*$ it follows that $A_2^*T_2 = B_2^*A_2^*$. Since T_2 is quasinormal, by Lemma 1 we have that $A_2^*T_2 = 0$ and so $A_2B_2 = 0$. Thus, $A_2|_{\mathscr{R}(B_2)} = 0$, and by continuity, $A_2|_{\overline{\mathscr{R}(B_2)}} = 0$. Since $\mathscr{L} = \mathscr{N}(B_2^*) \oplus \overline{\mathscr{R}(B_2)}$, it follows that $A_2 = 0$ so T_2 is normal, by Lemma 1. □

LEMMA 3. *Let $\mathbf{T} = (T_1, T_2)$ be a subnormal pair with the normal extension $\mathbf{N} = (N_1, N_2)$ such that T_2 is quasinormal and T_1T_2 is normal. If $\mathscr{R}(T_1) = \overline{\mathscr{R}(T_1)} \subseteq \overline{\mathscr{R}(T_2^*)}$ and $\mathscr{N}(T_1) = \mathscr{N}(T_2)$, then T_2 is normal.*

Proof. Since $\mathscr{R}(T_1) = \overline{\mathscr{R}(T_1)} \subseteq \overline{\mathscr{R}(T_2^*)}$ and $\mathscr{N}(T_1) = \mathscr{N}(T_2)$ we have that operators T_1 and T_2 have representations

$$T_1 = \begin{bmatrix} T_1^1 & 0 \\ 0 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} T_2^1 & 0 \\ 0 & 0 \end{bmatrix},$$

respectively, with respect to $\mathscr{H} = \mathscr{N}(T_2)^{\perp} \oplus \mathscr{N}(T_2)$ decomposition. It follows that

$$N_1 = \begin{bmatrix} T_1^1 & 0 & A_1^1 \\ 0 & 0 & A_1^2 \\ 0 & 0 & B_1^* \end{bmatrix} : \begin{pmatrix} \mathscr{N}(T_2)^{\perp} \\ \mathscr{N}(T_2) \\ \mathscr{H}^{\perp} \end{pmatrix} \mapsto \begin{pmatrix} \mathscr{N}(T_2)^{\perp} \\ \mathscr{N}(T_2) \\ \mathscr{H}^{\perp} \end{pmatrix}$$

is a normal extension for T_1^1 and

$$N_2 = \begin{bmatrix} T_2^1 & 0 & A_2^1 \\ 0 & 0 & A_2^2 \\ 0 & 0 & B_2^* \end{bmatrix} : \begin{pmatrix} \mathscr{N}(T_2)^{\perp} \\ \mathscr{N}(T_2) \\ \mathscr{H}^{\perp} \end{pmatrix} \mapsto \begin{pmatrix} \mathscr{N}(T_2)^{\perp} \\ \mathscr{N}(T_2) \\ \mathscr{H}^{\perp} \end{pmatrix}$$

is a normal extension for T_2^1. Since $N_1N_2 = N_2N_1$, operator pair $\mathbf{T}^1 = (T_1^1, T_2^1)$ is subnormal. From quasinormality of T_2 we have that T_2^1 is quasinormal, and since T_1T_2 is normal, it follows that $T_1^1T_2^1$ is normal.

Obviously, $\mathscr{R}(T_1^1) = \mathscr{R}(T_1)$, so $\mathscr{R}(T_1^1)$ is closed. Now let $x \in \mathscr{N}(T_1^1) \subseteq \mathscr{N}(T_2)^{\perp}$. Then $P_{\mathscr{N}(T_2)^{\perp}} T_1 x = 0$ and from $\mathscr{R}(T_1) \subseteq \mathscr{N}(T_2)^{\perp}$ we have that $T_1 x = 0$, i.e., $x \in \mathscr{N}(T_1) = \mathscr{N}(T_2)$. It must be $x = 0$ so $\mathscr{N}(T_1^1) = \{0\}$. Therefore, T_1^1 is left invertible.

If we apply Lemma 2 to operator pair $\mathbf{T}^1 = (T_1^1, T_2^1) \in \mathfrak{B}(\mathscr{N}(T_2)^{\perp})^2$, we conclude that T_2^1 is normal. Now it directly follows that T_2 is also normal. □

COROLLARY 1. *Let $\mathbf{T} = (T_1, T_2)$ be a (jointly) quasinormal pair such that T_1T_2 is normal. If $\mathscr{R}(T_1) = \mathscr{R}(T_2)$ is closed and $\mathscr{N}(T_1) = \mathscr{N}(T_2)$, then $\mathbf{T}$ is normal.*

Proof. Since T_1 and T_2 are hyponormal, we have $\mathscr{R}(T_i) \subseteq \mathscr{R}(T_i^*)$, $i = 1,2$. Thus, if $\mathscr{R}(T_1) = \mathscr{R}(T_2)$ is closed, then $\mathscr{R}(T_1) = \overline{\mathscr{R}(T_1)} \subseteq \overline{\mathscr{R}(T_2^*)}$ and $\mathscr{R}(T_2) = \overline{\mathscr{R}(T_2)} \subseteq \overline{\mathscr{R}(T_1^*)}$. The conclusion now follows directly from Lemma 3. □

Combining Lemma 2, Lemma 3 and Corollary 1 we obtain the following theorem:

THEOREM 8. *Let $\mathbf{T} = (T_1, T_2)$ be a (jointly) quasinormal pair such that T_1T_2 is normal. Then $\mathbf{T}$ is normal if one of the following conditions holds:*

- (i) *T_1 or T_2 is right invertible;*
- (ii) *T_1 and T_2 are left invertible;*
- (iii) *$\mathscr{R}(T_i) = \overline{\mathscr{R}(T_i)} \subseteq \overline{\mathscr{R}(T_j^*)}$ for $i \neq j$ and $\mathscr{N}(T_1) = \mathscr{N}(T_2)$;*
- (iv) *$\mathscr{R}(T_1) = \mathscr{R}(T_2)$ is closed and $\mathscr{N}(T_1) = \mathscr{N}(T_2)$.*

Proof. (i) Without loss of generality, assume that T_1 is right invertible. Then T_1^* is left invertible and $\mathscr{N}(T_1) \subseteq \mathscr{N}(T_1^*) = \{0\}$, as T_1 is hyponormal. Thus T_1 is invertible. From quasinormality of T_1 now follows that T_1 is normal. Operator T_2 is normal by Lemma 2. Thus, $\mathbf{T}$ is normal.

The rest of the proof follows directly from Lemma 2, Lemma 3 and Corollary 1. □

REMARK 1. In Corollary 1 and Theorem 8 it is enough to assume that T_1 and T_2 are quasinormal instead of (joint) quasinormality of $\mathbf{T} = (T_1, T_2)$. We will show in the sequel that we can actually remove quasinormality condition on one (or both) of the coordinate operators.

REMARK 2. Although condition (iv) of Theorem 8 actually implies condition (iii) of the same theorem (as shown in the proof of Corollary 1), we listed it due to its elegant form.

This concludes our consideration of Problem 4. We now shift our focus to "implied quasinormality problem" and the converse of Fuglede Theorem.

The following lemma, similar in spirit to Lemma 1, will be a major tool for giving an answer to Problems 3 and 5. We present it here in a slightly different form:

LEMMA 4. [3, Lemma 3.1] *Let $T \in \mathfrak{B}(\mathcal{H})$ be a subnormal operator. If N is a normal extension for T, then T is quasinormal if and only if $\mathcal{H}$ is invariant for N^*N.*

Proof. Let N be the normal extension of T on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^{\perp}$ given by

$$N = \begin{bmatrix} T & A \\ 0 & B^* \end{bmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{H}^{\perp} \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{H} \\ \mathcal{H}^{\perp} \end{pmatrix},$$

and let $P \in \mathfrak{B}(\mathcal{K})$ be the orthogonal projection onto $\mathcal{H}$. Note that $\mathcal{H}$ is invariant for N^*N if and only if $PN^*NP = N^*NP$. A direct computation shows that

$$N^*NP = \begin{bmatrix} T^*T & 0 \\ A^*T & 0 \end{bmatrix} \quad \text{and} \quad PN^*NP = \begin{bmatrix} T^*T & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, $PN^*NP = N^*NP$ if and only if $A^*T = 0$. The conclusion now follows from Lemma 1. $\square$

For any operator $A \in \mathfrak{B}(\mathcal{H})$ let $Comm(A)$ denote the commutant of A, i.e., $Comm(A) = \{B \in \mathfrak{B}(\mathcal{H}) : AB = BA\}$.

LEMMA 5. *Let $\mathbf{T} = (T_1, T_2)$ be a subnormal pair with the normal extension $\mathbf{N} = (N_1, N_2)$ such that T_1T_2 is quasinormal. Then T_2 is quasinormal if one of the following conditions holds:*

(i) *$Comm(N_1^*N_1N_2^*N_2) \subseteq Comm(N_2^*N_2)$;*

(ii) *T_1 is quasinormal and right invertible;*

(iii) *T_1 is quasinormal and N_1 is left invertible.*

Proof. (i) Let $\mathbf{N} = (N_1, N_2) \in \mathfrak{B}(\mathcal{K})^2$ be the normal extension for $\mathbf{T} = (T_1, T_2)$ where $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^{\perp}$, and let $P \in \mathfrak{B}(\mathcal{K})$ be the orthogonal projection onto $\mathcal{H}$.

Since $N_1N_2 = N_2N_1$, by Fuglede Theorem, N_1N_2 is a normal extension for T_1T_2. Also, T_1T_2 is quasinormal, so we have that

$$P(N_1N_2)^*(N_1N_2)P = (N_1N_2)^*(N_1N_2)P,$$

by Lemma 4. By taking adjoints,

$$(N_1N_2)^*(N_1N_2)P = P(N_1N_2)^*(N_1N_2),$$

and thus P commutes with $(N_1N_2)^*(N_1N_2) = N_1^*N_1N_2^*N_2$. The last equality follows from Fuglede Theorem. Since P commutes with $N_1^*N_1N_2^*N_2$ we have that P commutes with $N_2^*N_2$, and so $\mathscr{H}$ is invariant for $N_2^*N_2$. Therefore, T_2 is quasinormal, by Lemma 4.

(ii) As in the proof of Theorem 8, we have that T_1 is invertible normal operator. Using the fact that T_1T_2 is quasinormal and T_1 and T_2 commute, Fuglede Theorem implies

$$T_1T_1^*T_1T_2T_2^*T_2 = T_1^*T_1T_1T_2^*T_2T_2.$$

Multiplying from the left side with $(T_1T_1^*T_1)^{-1}$ it follows that T_2 is quasinormal.

(iii) As shown in part (i), we have that P commutes with $N_1^*N_1N_2^*N_2$, i.e., $PN_1^*N_1N_2^*N_2 = N_1^*N_1N_2^*N_2P$. By assumption, T_1 is quasinormal, and so P commutes with $N_1^*N_1$ (Lemma 4). Hence, $N_1^*N_1PN_2^*N_2 = N_1^*N_1N_2^*N_2P$, The left invertibility of N_1 now implies that $N_1^*N_1$ is invertible and thus $PN_2^*N_2 = N_2^*N_2P$. The quasinormality of T_2 now follows from Lemma 4. □

Theorem 2 now follows as a simple corollary of Lemma 5 and the following theorem:

THEOREM 9. (see [15, Theorem 12.12]) *If $n \in \mathbb{N}$, then the commutants of a positive operator and it's n-th root coincide.*

COROLLARY 2. [14] *Let $T \in \mathfrak{B}(\mathscr{H})$ be a subnormal operator such that T^n is quasinormal for some $n \in \mathbb{N}$. Then T is quasinormal.*

Proof. Let N be a normal extension for T and let $T_1 = T^{n-1}$ and $T_2 = T$. Then $\mathbf{T} = (T_1, T_2)$ is a subnormal pair with the normal extension $\mathbf{N} = (N_1, N_2) = (N^{n-1}, N)$. Note that $N_1^*N_1N_2^*N_2 = (N^*N)^n$ and so the first condition of Lemma 5 is satisfied, by Theorem 9. Thus, $T_2 = T$ is quasinormal. □

Using Lemma 5 and the same technique as in the proof of Lemma 3, we can prove the next lemma:

LEMMA 6. *Let $\mathbf{T} = (T_1, T_2)$ be a subnormal pair with the normal extension $\mathbf{N} = (N_1, N_2)$ such that T_1 and T_1T_2 are quasinormal. If $\mathscr{R}(T_1) = \overline{\mathscr{R}(T_2^*)}$ and $\mathscr{N}(T_2) \subseteq \mathscr{N}(T_1)$, then T_2 is quasinormal.*

Proof. Since $\mathscr{R}(T_1) = \overline{\mathscr{R}(T_2^*)}$ and $\mathscr{N}(T_2) \subseteq \mathscr{N}(T_1)$ we have that operators T_1 and T_2 have representations

$$T_1 = \begin{bmatrix} T_1^1 & 0 \\ 0 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} T_2^1 & 0 \\ 0 & 0 \end{bmatrix},$$

respectively, with respect to $\mathscr{H} = \mathscr{N}(T_2)^\perp \oplus \mathscr{N}(T_2)$ decomposition. It follows that

$$N_1 = \begin{bmatrix} T_1^1 & 0 & A_1^1 \\ 0 & 0 & A_1^2 \\ 0 & 0 & B_1^* \end{bmatrix} : \begin{pmatrix} \mathscr{N}(T_2)^\perp \\ \mathscr{N}(T_2) \\ \mathscr{H}^\perp \end{pmatrix} \mapsto \begin{pmatrix} \mathscr{N}(T_2)^\perp \\ \mathscr{N}(T_2) \\ \mathscr{H}^\perp \end{pmatrix}$$

is a normal extension for T_1^1 and

$$N_2 = \begin{bmatrix} T_2^1 & 0 & A_2^1 \\ 0 & 0 & A_2^2 \\ 0 & 0 & B_2^* \end{bmatrix} : \begin{pmatrix} \mathscr{N}(T_2)^\perp \\ \mathscr{N}(T_2) \\ \mathscr{H}^\perp \end{pmatrix} \mapsto \begin{pmatrix} \mathscr{N}(T_2)^\perp \\ \mathscr{N}(T_2) \\ \mathscr{H}^\perp \end{pmatrix}$$

is a normal extension for T_2^1. Since $N_1N_2 = N_2N_1$, operator pair $\mathbf{T}^1 = (T_1^1, T_2^1)$ is subnormal. From quasinormality of T_1 we have that T_1^1 is quasinormal, and since T_1T_2 is quasinormal, it follows that $T_1^1T_2^1$ is quasinormal.

Obviously, $\mathscr{R}(T_1^1) = \mathscr{R}(T_1) = \overline{\mathscr{R}(T_2^*)} = \mathscr{N}(T_2)^\perp$, so T_1^1 is onto. In other words, T_1^1 is right invertible.

We conclude that operator pair $\mathbf{T}^1 = (T_1^1, T_2^1) \in \mathfrak{B}(\mathscr{N}(T_2)^\perp)^2$ satisfies condition (ii) of Lemma 5, and so T_2^1 is quasinormal. Now it directly follows that T_2 is also quasinormal. □

In order to prove our next result, similar in spirit to Lemma 5, but also of independent interest, we need the following theorem:

THEOREM 10. [11] *Let A and B be operators with $\sigma(A) \cap \sigma(B) = \emptyset$. Then every operator that commutes with $A+B$ and with AB also commutes with A and B.*

THEOREM 11. *Let $\mathbf{T} = (T_1, T_2)$ be a spherically quasinormal pair with a normal extension $\mathbf{N} = (N_1, N_2)$ such that $\sigma(N_1^*N_1) \cap \sigma(N_2^*N_2) = \emptyset$. If T_1T_2 is quasinormal, then $\mathbf{T}$ is (jointly) quasinormal.*

Proof. Let N_i, $i = 1,2$, be the normal extensions of T_i on $\mathscr{K} = \mathscr{H} \oplus \mathscr{H}^\perp$ given by

$$N_i = \begin{bmatrix} T_i & A_i \\ 0 & B_i^* \end{bmatrix} : \begin{pmatrix} \mathscr{H} \\ \mathscr{H}^\perp \end{pmatrix} \mapsto \begin{pmatrix} \mathscr{H} \\ \mathscr{H}^\perp \end{pmatrix},$$

and let $P \in \mathfrak{B}(\mathscr{K})$ be the orthogonal projection onto $\mathscr{H}$. As in the proof of Lemma 5, we can show that quasinormality of T_1T_2 implies that P commutes with $N_1^*N_1N_2^*N_2$. We will also show that P commutes with $N_1^*N_1 + N_2^*N_2$.

Since $\mathbf{T}$ is spherically quasinormal, we have that $A_1^*T_1 + A_2^*T_2 = 0$ [4, Theorem 2.8]. By direct computation,

$$\begin{aligned}(N_1^*N_1 + N_2^*N_2)P &= \begin{bmatrix} T_1^*T_1 + T_2^*T_2 & 0 \\ A_1^*T_1 + A_2^*T_2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} T_1^*T_1 + T_2^*T_2 & 0 \\ 0 & 0 \end{bmatrix} \\ &= P(N_1^*N_1 + N_2^*N_2)P.\end{aligned}$$

Thus,

$$(N_1^*N_1 + N_2^*N_2)P = P(N_1^*N_1 + N_2^*N_2)P,$$

and by taking adjoints, we have that

$$P(N_1^*N_1 + N_2^*N_2) = (N_1^*N_1 + N_2^*N_2)P.$$

Therefore, P commutes with $N_1^*N_1 + N_2^*N_2$.

By assumption, $\sigma(N_1^*N_1) \cap \sigma(N_2^*N_2) = \emptyset$, and so P commutes with $N_1^*N_1$ and $N_2^*N_2$ (Theorem 10). Hence, $\mathscr{H}$ is invariant for $N_1^*N_1$ and $N_2^*N_2$. By Lemma 4, T_1 and T_2 are quasinormal. Since T_1 commutes with $T_1^*T_1$ and $T_1^*T_1 + T_2^*T_2$, it also commutes with $T_2^*T_2$. Similarly, T_2 commutes with $T_1^*T_1$. Therefore, $\mathbf{T}$ is (jointly) quasinormal. □

Finally, we arrive at the main result of this section:

THEOREM 12. (Converse of Fuglede Theorem) *Let $\mathbf{T} = (T_1, T_2)$ be a subnormal pair with the normal extension $\mathbf{N} = (N_1, N_2)$ such that T_1T_2 is normal. Then $\mathbf{T}$ is normal if one of the following conditions holds:*

(i) T_1 or T_2 is right invertible quasinormal operator;

(ii) T_1 is quasinormal and N_1 and T_2 are left invertible, or T_2 is quasinormal and T_1 and N_2 are left invertible;

(iii) T_1 or T_2 is quasinormal, $\mathscr{R}(T_i) = \overline{\mathscr{R}(T_j^*)}$ for $i \neq j$ and $\mathscr{N}(T_1) = \mathscr{N}(T_2)$.

(iv) $Comm(N_1^*N_1N_2^*N_2) \subseteq Comm(N_1^*N_1) \cap Comm(N_2^*N_2)$ and any of the conditions $(i)-(iv)$ of Theorem 8 holds;

(v) $\mathbf{T}$ is spherically quasinormal, $\sigma(N_1^*N_1) \cap \sigma(N_2^*N_2) = \emptyset$ and any of the conditions $(i)-(iv)$ of Theorem 8 holds.

Proof. (i) Without loss of generality, assume that T_1 is right invertible quasinormal operator. By Lemma 5, it follows that T_2 is quasinormal. Thus, condition (i) of Theorem 8 is satisfied, so $\mathbf{T}$ is normal.

(ii) Without loss of generality, assume that T_1 is quasinormal and N_1 and T_2 are left invertible. By Lemma 5, it follows that T_2 is quasinormal. Also, left invertibility of

N_1 implies left invertibility of T_1. This means that condition (ii) of Theorem 8 holds, so $\mathbf{T}$ is normal.

(iii) Again, we may assume that T_1 is quasinormal. By Lemma 6, we have that T_2 is quasinormal. The condition (iii) of Theorem 8 is obviously satisfied in this case, and hence, $\mathbf{T}$ is normal.

(iv) Condition

$$Comm(N_1^*N_1N_2^*N_2) \subseteq Comm(N_1^*N_1) \cap Comm(N_2^*N_2)$$

implies that both T_1 and T_2 are quasinormal. Any condition of Theorem 8 is now sufficient for normality of $\mathbf{T}$.

(v) Conditions $\mathbf{T}$ is spherically quasinormal and $\sigma(N_1^*N_1) \cap \sigma(N_2^*N_2) = \emptyset$ implies that $\mathbf{T}$ is (jointly) quasinormal. As in the previous case, any condition of Theorem 8 is now sufficient for normality of $\mathbf{T}$. $\square$

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