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ON $M-class-c-wA_k^*(a,b)$ OPERATORS

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Abstract. Let L be a bounded linear operator on a complex Hilbert space H. If

$$|L^{*k}|^{2ac} \leqslant M^c(|L^{*k}|^a|L^k|^{2b}|L^{*k}|^a)^{\frac{ac}{k(a+b)}}$$

and

$$M^c|L^k|^{2bc} \geqslant (|L^k|^b|L^{*k}|^{2a}|L^k|^b)^{\frac{bc}{k(a+b)}},$$

then L is called M-class-c-$wA_k^*(a,b)$ for some positive integers c,k,M and $a,b \in (0,1]$. This study aims to derive the structural relationship of the operators using some of the well-known inequalities. Then the study focuses on $M-class-c-wA_k^*(a,b)$ operator's spectral and algebraic properties in $L^2(\lambda)$ space. Furthermore, the Kronecker product results are also explored.

1. Introduction

Let H denotes a non-zero complex Hilbert space and $B(H)$ denotes C^* algebra on H. Throughout this article $ker(L)$, $R(L)$ denotes null space and range of L on $B(H)$ respectively. If L is an operator then its adjoint is denoted by L^*. Furuta [10], defined class A operators as $|L^2| \geqslant |L|^2$. Panayappan [13], defined class A_k operators as $|L|^2 \leqslant (|L^{k+1}|^{\frac{2}{k+1}})$ for some positive integer k. Panayappan [14], studied spectral properties of class A_k^* operators as $|L^k|^{\frac{2}{k}} \geqslant |L^*|^2$, where k takes positive integer values. Uchiyamma [25], defined class $A(s,t)$ operators as $|L(s,t)|^{\frac{2t}{s+t}} \geqslant |L|^{2t}$ and derived some spectral properties. As an extension of w-hyponormal operators, class $wA(s,t)$ operators is defined as $|L^*|^{2t} \leqslant (|L^*|^t|L|^{2s}|L^*|^t)^{\frac{t}{s+t}}$ and $|L|^{2s} \geqslant (|L|^s|L^*|^{2t}|L|^s)^{\frac{s}{s+t}}$. Prasad [16], introduced class $p-wA(s,t)$ operators and Cho [4, 5, 6], worked on the spectrum, Putnam-Fuglede (P-F) theorem, and Quasi-similarity of $class-p-wA(s,t)$ operators. Shanmugapriya [21], defined $M-class-A_k^*$ operators as $|L^*|^2 \leqslant M(|L^k|^{\frac{2}{k}})$ where M and k are positive integers. Pradeep [15], defined $m-quasi-totally-(\alpha,\beta)$-normal operators and studied the structural, spectral properties of the operators. The above studies motivated us to work on $M-class-A_k^*(a,b)$ and $M-class-wA_k^*(a,b)$ operators.

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An operator L is called $M-class-A_k^*(a,b)$ if

$$|L^{*k}|^{2a} \leqslant M(|L^{*k}|^a|L^k|^{2b}|L^{*k}|^a)^{\frac{a}{k(a+b)}}$$

for each $(a,b) \in (0,1]$ and for some positive integer k and M.

For each $(a,b) \in (0,1]$ and for some positive integrs k and M, an operator L is called $M-class-wA_k^*(a,b)$ if

$$|L^{*k}|^{2a} \leqslant M(|L^{*k}|^a|L^k|^{2b}|L^{*k}|^a)^{\frac{a}{k(a+b)}}$$

and

$$M|L^k|^{2b} \geqslant (|L^k|^b|L^{*k}|^{2a}|L^k|^b)^{\frac{b}{k(a+b)}}.$$

If $L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $a=b=k=1$ and $M=2$ then using simple calculation, it is observed that L is $M-class-wA_k^*(a,b)$.

If $L = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, for $a=b=k=1$, L is $M-class-wA_k^*(a,b)$ for any $M>0$. If $a=b=1$ and $k=2$ then L is not $M-class-A_k^*(a,b)$ for any $M<7$.

REMARK 1.1. If $a=b=k=1$ then $M-class-A_k^*(a,b)$ coincides with $M-class-A_k^*$ and if $M=1$ then the operator coincides with class-A_k^* which obviously coincides with class-A_k and class-A.

Hence,

$$class-A \subset class-A_k \subset class-A_k^* \subset M-class-A_k^* \subset M-class-A_k^*(a,b).$$

REMARK 1.2. If $M=k=1$ then $M-class-A_k^*(a,b)$ coincides with class $A(s,t)$ and $M-class-wA_k^*(a,b)$ coincides with class-$wA(s,t)$.

Hence,

$$class-A(s,t) \subset M-class-A_k^*(a,b), \quad class-wA(s,t) \subset M-class-wA_k^*(a,b).$$

In this article, we focused on to define $M-class-c-wA_k^*(a,b)$ operators which coincides with $M-class-wA_k^*(a,b)$ for $c=1$ and derive its structural relationship. The next sections deals with $M-class-c-wA_k^*(a,b)$ operator's spectral and algebraic properties in $L^2(\lambda)$ space. Furthermore, the Kronecker product results are also explored.

For each $(a,b) \in (0,1]$ and for some positive integrs c,k and M, an operator L is called $M-class-c-wA_k^*(a,b)$ if

$$|L^{*k}|^{2ac} \leqslant M^c(|L^{*k}|^a|L^k|^{2b}|L^{*k}|^a)^{\frac{ac}{k(a+b)}} \tag{1.1}$$

and

$$M^c|L^k|^{2bc} \geqslant (|L^k|^b|L^{*k}|^{2a}|L^k|^b)^{\frac{bc}{k(a+b)}}. \tag{1.2}$$

Clearly, $M-class-wA_k^*(a,b) \subset M-class-c-wA_k^*(a,b)$.

If $L = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, for $a=b=k=1$, then L is $M-class-c-wA_k^*(a,b)$ for any $c,\, M > 0$.

2. Structural inequalities of $M-class-c-wA_k^*(a,b)$ operators

This section deals with some of the structural properties of the operators which are derived from the well-known inequalities. Let us start with the following Lemma.

LEMMA 2.1. *If $L = U|L|$ then $|L^{-1}| = |L^*|^{-1}$ and $|(L^{-1})^*| = |L|^{-1}$.*

THEOREM 2.2. *If L is $M-class-c-wA_k^*(a,b)$ and it is invertible, then L^{-1} is $M-class-c-wA_k^*(b,a)$.*

Proof. Consider,

$$\begin{aligned} \left\{M^c(|(L^{-k})|^a|(L^{-k})^*|^{2b}|(L^{-k})|^a)^{\frac{ac}{k(a+b)}}\right\} &= \left\{M^c(|(L^{*k})|^{-a}|(L)^k|^{-2b}|(L^*)^k|^{-a})^{\frac{ac}{k(a+b)}}\right\} \\ &\leqslant \left\{|L^{*k}|^{-2ac}\right\} = |L^{-k}|^{2ac}. \\ \left\{(|(L^{-k})^*|^b|(L^{-k})|^{2a}|(L^{-k})^*|^b)^{\frac{bc}{k(a+b)}}\right\} &= \left\{(|L^k|^{-b}|L^{*k}|^{-2a}|L^k|^{-b})^{\frac{bc}{k(a+b)}}\right\} \\ &\geqslant M^c|L^k|^{-2bc} = M^c|(L^{-k})^*|^{2bc}. \quad \square \end{aligned}$$

THEOREM 2.3. *If L is $M-class-c-wA_k^*(a,b)$ then*

$$|L^{*k}|^{2ac} - M^c(|L^{*k}|^a|L^k|^{2b}|L^{*k}|^a)^{\frac{ac}{k(a+b)}} \leqslant M^c|L^k|^{2bc} - (|L^k|^b|L^{*k}|^{2a}|L^k|^b)^{\frac{bc}{k(a+b)}}.$$

Proof. Consider

$$\begin{aligned} A &= \left[|L^{*k}|^{2ac} \leqslant [M^c(|L^{*k}|^a|L^k|^{2b}|L^{*k}|^a)^{\frac{ac}{k(a+b)}}]\right] \\ B &= M^c\left[|L^k|^{2bc} \geqslant [(|L^k|^b|L^{*k}|^{2a}|L^k|^b)^{\frac{bc}{k(a+b)}}]\right]. \end{aligned}$$

By definition $A \leqslant B$, then

$$\left[|L^{*k}|^{2ac} \leqslant [M^c(|L^{*k}|^a|L^k|^{2b}|L^{*k}|^a)^{\frac{ac}{k(a+b)}}]\right] \leqslant \left[M^c|L^k|^{2bc} \geqslant [(|L^k|^b|L^{*k}|^{2a}|L^k|^b)^{\frac{bc}{k(a+b)}}]\right]$$

$$|L^{*k}|^{2ac} - M^c(|L^{*k}|^a|L^k|^{2b}|L^{*k}|^a)^{\frac{ac}{k(a+b)}} \leqslant M^c|L^k|^{2bc} - (|L^k|^b|L^{*k}|^{2a}|L^k|^b)^{\frac{bc}{k(a+b)}}. \quad \square$$

PROPOSITION 2.4. *The generalized Aluthge transformation is* $L(a,b) = |L|^a U|L|^b$. *For* $M-class-c-wA_k^*(a,b)$, L *enjoys the following property*

$$M^c|L^k(a,b)|^{\frac{2ac}{k(a+b)}} \geqslant |L^{*k}|^{2ac}$$

and

$$|L^k|^{2bc} \geqslant M^{-c}|L^k(a,b)^*|^{\frac{2bc}{k(a+b)}}.$$

PROPOSITION 2.5. *If* L *is* $M-class-c-wA_k^*(a,b)$ *then*

$$M^c|L^k(a,b)|^{\frac{2\rho}{k(a+b)}} \geqslant |L^k|^{2\rho} \geqslant M^{-c}|(L^k(a,b))^*|^{\frac{2\rho}{k(a+b)}}$$

for all $\rho \in (0, \min(ac, bc)]$.

THEOREM 2.6. *If* L *is* $M-class-c-wA_k^*(a,b)$ *then for* $M=1$
$(a1)$ *For* $\lambda \in [0,1]$

$$\lambda\left\{|L^k(a,b)|^{\frac{2\rho}{k(a+b)}} - |L^k(a,b)^*|^{\frac{2\rho}{k(a+b)}}\right\}$$
$$\geqslant (\lambda-1) + \left\{|L^k(a,b)|^{\frac{2\rho}{k(a+b)}} - |L^k(a,b)^*|^{\frac{2\rho}{k(a+b)}}\right\}^{\lambda}.$$

$(a2)$ *For* $\lambda > 1$

$$\left\{|L^k(a,b)|^{\frac{2\rho}{k(a+b)}} - |L^k(a,b)^*|^{\frac{2\rho}{k(a+b)}}\right\}^{\frac{1}{\lambda}}$$
$$\geqslant \lambda\left\{|L^k(a,b)|^{\frac{2\rho}{k(a+b)}} - |L(a,b)^*|^{\frac{2\rho}{k(a+b)}}\right\} + (1-\lambda).$$

$(a3)$ *For* $\lambda < 0$

$$\lambda\left\{|L^k(a,b)|^{\frac{2\rho}{k(a+b)}} - |L^k(a,b)^*|^{\frac{2\rho}{k(a+b)}}\right\} + (1-\lambda)$$
$$\leqslant \left\{|L^k(a,b)|^{\frac{2\rho}{k(a+b)}} - |L^k(a,b)^*|^{\frac{2\rho}{k(a+b)}}\right\}.$$

Further, $(a1)$, $(a2)$ *and* $(a3)$ *are mutually equivalent.*

Proof. Consider $f(x) = \lambda x + 1 - \lambda - x^{\lambda}$ for a positive x and $\lambda \in [0,1]$.
Since, $f(x)$ is a positive function, for positive operator L and $\lambda \in [0,1]$, we have the inequality, $\lambda L + (1-\lambda) \geqslant L^{\lambda}$,
Hence,

$$\lambda\left\{[|L^{*k}|^{2ac} - M^c(|L^{*k}|^a|L^k|^{2b}|L^{*k}|^a)^{\frac{ac}{k(a+b)}}] \leqslant [M^c|L^k|^{2bc} - (|L^k|^b|L^{*k}|^{2a}|L^k|^b|^{\frac{bc}{k(a+b)}}]\right\}$$
$$+(1-\lambda)$$
$$\geqslant \left\{[|L^{*k}|^{2ac} - M^c(|L^{*k}|^a|L^k|^{2b}|L^{*k}|^a)^{\frac{ac}{k(a+b)}}] \leqslant [M^c|L^k|^{2b} - (|L^k|^b|L^{*k}|^{2a}|L^k|^b)^{\frac{bc}{k(a+b)}}]\right\}^{\lambda}.$$

By Proposition 2.5,

$$\lambda\left\{|L^k(a,b)|^{\frac{2\rho}{k(a+b)}} - |L^k(a,b)^*|^{\frac{2\rho}{k(a+b)}}\right\} \geqslant (\lambda-1)+\left\{|L^k(a,b)|^{\frac{2\rho}{k(a+b)}} - |L^k(a,b)^*|^{\frac{2\rho}{k(a+b)}}\right\}^{\lambda}$$

for $\lambda \in [0,1]$.

Result (a1)$\Longleftrightarrow$ (a2).

To show the result (a2), assume $\lambda > 1$. Then (a1) is equal to

$$\left\{[|L^{*k}|^{2ac} - M^c(|L^{*k}|^a|L^k|^{2b}|L^{*k}|^a)^{\frac{ac}{k(a+b)}}] \leqslant [M^c|L^k|^{2bc} - (|L^k|^b|L^{*k}|^{2a}|L^k|^b)^{\frac{bc}{k(a+b}}]\right\}^{1/\lambda}$$
$$\leqslant \frac{1}{\lambda}\left\{[|L^{*k}|^{2ac} - M^c(|L^{*k}|^a|L^k|^{2b}|L^{*k}|)^a)^{\frac{ac}{k(a+b)}}] \leqslant [M^c|L^k|^{2bc} - (|L^k|^b|L^{*k}|^{2a}|L^k|^b)^{\frac{bc}{k(a+b)}}]\right\} + \left(1-\frac{1}{\lambda}\right)$$
$$\cong \lambda\left\{[|L^{*k}|^{2ac} - M^c(|L^{*k}|^a|L^k|^{2b}|L^{*k}|)^a)^{\frac{ac}{k(a+b)}}] \leqslant [M^c|L^k|^{2bc} - (|L^k|^b|L^{*k}|^{2a}|L^k|^b)^{\frac{bc}{k(a+b)}}]\right\}^{\frac{1}{\lambda}}$$
$$\leqslant \left\{[|L^{*k}|^{2ac} - M^c(|L^{*k}|^a|L^k|^{2b}|L^{*k}|)^a)^{\frac{ac}{k(a+b)}}] \leqslant [M^c|L^k|^{2bc} - (|L^k|^b|L^{*k}|^{2a}|L^k|^b)^{\frac{bc}{k(a+b)}}]\right\} + (\lambda-1).$$

By Proposition 2.5,

$$\lambda\left\{|L^k(a,b)|^{\frac{2\rho}{k(a+b)}} - |L^k(a,b)^*|^{\frac{2\rho}{k(a+b)}}\right\}^{\frac{1}{\lambda}} \leqslant \left\{|L^k(a,b)|^{\frac{2\rho}{k(a+b)}} - |L^k(a,b)^*|^{\frac{2\rho}{k(a+b)}}\right\} + (\lambda-1).$$

Now, substitute $L^{\frac{1}{\lambda}} = S$

$$\left\{|L^k(a,b)|^{\frac{2\rho}{k(a+b)}} - |L^k(a,b)^*|^{\frac{2\rho}{k(a+b)}}\right\}^{\frac{1}{\lambda}} \geqslant \lambda\left\{|L^k(a,b)|^{\frac{2\rho}{k(a+b)}} - |L^k(a,b)^*|^{\frac{2\rho}{k(a+b)}}\right\} + (1-\lambda).$$

Therefore, (a1)$\Longleftrightarrow$(a 2).

Similarly, by using the inequality $\lambda + (1-\lambda)L^{-1} \leqslant L^{\lambda-1}$ for any $\lambda > 1$ and taking $\mu = 1-\lambda < 0$ and $S = L^{-1}$, (a3) will be attained.

Thus (a2)$\Longleftrightarrow$(a3).

So, (a1), (a2) and (a3) are mutually equivalent. □

THEOREM 2.7. *If L is M-class c-$wA_k^*(a,b)$ then*

$$\begin{aligned}&\left[M^c(|L^{*k}|^a|L^k|^{2b}|L^{*k}|^a)^{\frac{ac}{k(a+b)}} - |L^{*k}|^{2a}\right]\\ \geqslant\ &[M^c|L^k|^{2bc} - (|L^k|^b|L^{*k}|^{2a}|L^k|^b)^{\frac{bc}{k(a+b)}}]\\ &+\lambda\Big\{[M^c(|L^{*k}|^a|L^k|^{2bc}|L^{*k}|^a)^{\frac{ac}{k(a+b)}} - |L^{*k}|^{2ac}]\\ &-[M^c|L^k|^{2bc} - (|L^k|^b|L^{*k}|^{2a}|L^k|^b)^{\frac{bc}{k(a+b)}}]\Big\}.\end{aligned}$$

Proof. Let

$$A = M^c(|L^{*k}|^a|L^k|^{2b}|L^{*k}|^a)^{\frac{ac}{k(a+b)}} - |L^{*k}|^{2ac}$$

and

$$B = M^c|L^k|^{2bc} - [(|L^k|^b|L^{*k}|^{2a}|L^k|^b)^{\frac{bc}{k(a+b)}}].$$

Consider the well known inequality, $(1-\lambda)A + \lambda B \geqslant [(1-\lambda)A^{-1} + \lambda B^{-1}]^{-1}$

$$\begin{aligned}&(1-\lambda)\left[M^c(|L^{*k}|^a|L^k|^{2b}|L^{*k}|^a)^{\frac{ac}{k(a+b)}} - |L^{*k}|^{2ac}\right]\\ &+\lambda\left[M^c|L^k|^{2bc} - (|L^k|^b|L^{*k}|^{2a}|L^k|^b)^{\frac{bc}{k(a+b)}}\right]\\ \geqslant\ &\Big[(1-\lambda)[M^c(|L^{*k}|^a|L^k|^{2b}|L^{*k}|^a)^{\frac{ac}{k(a+b)}} - |L^{*k}|^{2ac}]^{-1}\\ &+\lambda[M^c|L^k|^{2bc} - (|L^k|^b|L^{*k}|^{2a}|L^k|^b)^{\frac{bc}{k(a+b)}}]^{-1}\Big]^{-1}\\ \Rightarrow\ &\left[M^c(|L^{*k}|^a|L^k|^{2b}|L^{*k}|^a|^{\frac{ac}{k(a+b)}} - |L^{*k}|^{2ac}\right]\\ &-\lambda\Big[\{M^c(|L^{*k}|^a|L^k|^{2b}|L^{*k}|^a)^{\frac{ac}{k(a+b)}} - |L^{*k}|^{2ac}\}\\ &-\{M^c|L^k|^{2bc} - (|L^k|^b|L^{*k}|^{2a}|L^k|^b)^{\frac{bc}{k(a+b)}}\}\Big]\\ \geqslant\ &\Big[(1-\lambda)[M^c(|L^{*k}|^a|L^k|^{2b}|L^{*k}|^a)^{\frac{ac}{k(a+b)}} - |L^{*k}|^{2ac}]^{-1}\\ &+\lambda[M^c|L^k|^{2bc} - (|L^k|^b|L^{*k}|^{2a}|L^k|^b)^{\frac{bc}{k(a+b)}}]^{-1}\Big]^{-1}\end{aligned}$$

on simplification we get the desired result. □

THEOREM 2.8. *If L is $M-class-c-wA_k^*(a,b)$ operator then*

$$\left[M^c(|L^{*k}|^a|L^k|^{2b}|L^{*k}|^a)^{\frac{\rho\alpha}{k(a+b)}}\right] \geqslant \left[M^{-c}[|L^k|^b|L^{*k}|^{2a}|L^k|^b]^{\frac{\rho\alpha}{k(a+b)}}\right]$$

for any $\alpha \in (0,1]$.

Proof. Let

$$L_1 = M^c|L^k|^{2bc} - \left[(|L^k|^b|L^{*k}|^{2a}|L^k|^b)^{\frac{bc}{k(a+b)}}\right]$$

and

$$L_2 = \left[M^c(|L^{*k}|^a|L^k|^{2b}|L^{*k}|^a)^{\frac{ac}{k(a+b)}}\right] - |L^{*k}|^{2ac}.$$

By Proposition 2.5,

$$\left[M^c(|L^{*k}|^a|L^k|^{2b}|L^{*k}|^a)^{\frac{\rho}{k(a+b)}}\right] \geqslant \left[M^{-c}(|L^k|^b|L^{*k}|^{2a}|L^k|^b)^{\frac{\rho}{k(a+b)}}\right].$$

By Lowner-Heinz inequality, if $L_1 \geqslant L_2 \geqslant 0$ ensures $(L_1)^\alpha \geqslant (L_2)^\alpha$ for any $\alpha \in (0,1]$. □

THEOREM 2.9. *If L is $M-class-c-wA_k^*(a,b)$ operator then*
(i) $|L^k|^{\frac{\omega^2}{\beta}} \geqslant |L^k|^{2\omega}$,
(ii) $|L^k|^{2\omega} \geqslant |L^k|^{\frac{\omega^2}{2\beta}}$.

Proof. Given L is $M-class-c-wA_k^*(a,b)$ operator then

$$|L^{*k}|^{2ac} \geqslant \left[M^c(|L^{*k}|^a|L^k|^{2b}|L^{*k}|^a)^{\frac{ac}{k(a+b)}}\right]$$

and

$$M^c|L^k|^{2bc} \geqslant \left[(|L^k|^b|L^{*k}|^{2a}|L^k|^b)^{\frac{bc}{k(a+b)}}\right].$$

By Proposition 2.5,

$$M^c\left(|L^{*k}|^a|L^k|^{2b}|L^{*k}|^a\right)^{\frac{\rho}{k(a+b)}} \geqslant |L^k|^{2\rho} \geqslant M^{-c}\left(|L^k|^b|L^{*k}|^{2a}|L^k|^b\right)^{\frac{\rho}{k(a+b)}}$$

for all $\rho \in (0, \min(ac,bc)]$.

Let $A = M^c(|L^{*k}|^a|L^k|^{2b}|L^{*k}|^a)^{\frac{\rho}{k(a+b)}}$ and $B = |L^k|^{2\rho}$.
Let $s = r = \frac{k(a+b)}{\rho}$ for $\rho \in (0, \min(ac,bc)]$ and take $M = 1$.

(i) By Furuta inequality,

$$\left\{[|L^k|^{2\rho}]^{\frac{r}{2}}[(|L^{*k}|^a|L^k|^{2b}|L^{*k}|^a)^{\frac{\rho}{k(a+b)}}]^s[|L^k|^{2\rho}]^{\frac{r}{2}}\right\}^{\frac{1+r}{s+r}} \geqslant [|L^k|^{2\rho}]^{1+r}.$$

Take $\beta = k(a+b)$ and $\omega = \rho + \beta$.

Then simple calculation yields $|L^k|^{\frac{\omega^2}{\beta}} \geqslant [|L^k|^{2\omega}]$.

(ii) Let $A = |L^k|^{2\rho}$ and $B = M^{-c}(|L^k|^b|L^{*k}|^{2a}|L^k|^b)^{\frac{\rho}{k(a+b)}}$.
Let $M = 1$ and $s = r = \frac{k(a+b)}{\rho}$ for $\rho \in (0, \min(ac,bc)]$.
By Furuta's inequality,

$$\{|L^k|^{2\rho}\}^{1+\frac{k(a+b)}{\rho}} \geqslant \left\{[|L^k|^{2\rho}]^{\frac{k(a+b)}{2\rho}}[(|L^k|^b|L^{*k}|^{2a}|L^k|^b)^{\frac{\rho}{k(a+b)}}]^{\frac{k(a+b)}{\rho}}[|L^k|^{2\rho}]^{\frac{k(a+b)}{2\rho}}\right\}^{\frac{\omega}{2\beta}}$$

$$\Rightarrow |L^k|^{2\omega} \geqslant |L^k|^{\frac{\omega^2}{2\beta}}. \quad \square$$

3. Spectral and algebraic properties of $M-class-c-wA_k^*(a,b)$ operators

In this section, the spectral and algebraic properties of $M-class-c-wA_k^*(a,b)$ operators are studied using the Althuge transformation.The following references are used to derive the results [6], [7], [8], [10], [11], [14], [16], [17], [18], [19], [20], [21], [22], [23], [24].

DEFINITION 3.1. Take $L=U|L|$ be the decomposition of L and let $a,b>0$. Then, the generalized Aluthge transformation is defined as follows:

$$L(a,b)=\widetilde{L}_{a,b}=|L|^aU|L|^b$$

and

$$L^*(a,b)=\widetilde{L}_{a,b}^*=(\widetilde{L}_{(a,b)})^*=|L|^bU^*|L|^a.$$

THEOREM 3.2. *Assume the polar decomposition of L. Then, L is $M-class-c-wA_k^*(a,b)$, if and only if*

$$\left[M^c|\widetilde{L}_{a,b}^k|^{\frac{2ac}{k(a+b)}}-|L^k|^{2ac}\right]\geqslant 0$$

and

$$0\geqslant\left[M^{-c}|\widetilde{L}_{a,b}^{k*}|^{\frac{2bc}{k(a+b)}}-|L^k|^{2bc}\right].$$

COROLLARY 3.3. *If L is $M-class-c-wA_k^*(a,b)$ then $\widetilde{L}_{a,b}$ is $\frac{\min(ac,bc)}{k(a+b)}-M-hyponormal$. For $M=k=1$ it is $\frac{\min(ac,bc)}{a+b}-hyponormal$.*

LEMMA 3.4. *Let $A,B\geqslant 0$. For each $p,r\geqslant 0$, the Furuta inequality implies that*
(i) $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}}\geqslant B^r$ and
(ii) $(A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}\leqslant A^p$.

LEMMA 3.5. *(i) For any $p_1\geqslant 1$ and $r_1\geqslant 0$, if $A_1\geqslant B_1$, then Furuta inequality implies that $(B_1^{\frac{r_1}{2}}A_1^{p_1}B_1^{\frac{r_1}{2}})^{\frac{1+r_1}{p_1+r_1}}\geqslant B_1^{1+r_1}$.*

(ii) For any $p_2\geqslant 1$ and $r_2\geqslant 0$, if $A_2\geqslant B_2$ then Furuta inequality implies that $(A_2)^{1+r_2}\geqslant(A_2^{\frac{r_2}{2}}B_2{}^{p_2}A_2^{\frac{r_2}{2}})^{\frac{1+r_2}{p_2+r_2}}$.

THEOREM 3.6. *If L is $M-class-c-wA_k^*(a,b)$ and $0<a\leqslant a_1$, $0<b\leqslant b_1$, then L is $M_1-class-1-wA_1^*(a_1,b_1)$.*

Proof. Let L be $M-class-c-wA_k^*(a,b)$. Then

$$|L^{*k}|^{2ac}\leqslant\left[M^c(|L^{*k}|^a|L^k|^{2b}|L^{*k}|^a)^{\frac{ac}{k(a+b)}}\right]$$

and

$$M|L^k|^{2bc} \geqslant \left[(|L^k|^b|L^{*k}|^{2a}|L^k|^b)^{\frac{bc}{k(a+b)}}\right].$$

Let $A_1=[M^c(|L^{*k}|^a|L^k|^{2b}|L^{*k}|^a)^{\frac{ac}{k(a+b)}}]$, $B_1=|L^*|^{2ac}$.

Since $A_1 \geqslant B_1$ by Lemma 3.5 $(B_1^{\frac{r_1}{2}}A_1^{p_1}B_1^{\frac{r_1}{2}})^{\frac{1+r_1}{p_1+r_1}} \geqslant B_1^{1+r_1}$ for any $p_1 \geqslant 1$ and $r_1 \geqslant 0$.

Let $\beta \geqslant a$, $p_1=\frac{a+b}{a} \geqslant 1$, $r_1=\frac{\beta-a}{a} \geqslant 0$ and $c=k=1$.

Then $[M_1|L^*|^\beta|L|^{2b}|L^*|^\beta]^{\frac{\beta c}{b+\beta}} \geqslant |L^*|^{2\beta c}$.

Let

$$\begin{aligned} f_s(\beta) &= \left[|L|^b|L^*|^{2\beta}|L|^b\right]^{\frac{b}{b+\beta}} \qquad (\text{for } \beta \geqslant a) \\ &= \left\{[|L|^b|L^*|^{2\beta}|L|^b]^{\frac{b+2\beta}{b+\beta}}\right\}^{\frac{b}{b+2\beta}} \\ &\geqslant \left\{|L|^b|L^*|^\beta|L^*|^{2\beta}|L^*|^\beta|L|^b\right\}^{\frac{b}{b+2\beta}} = f_s(\beta+\delta). \end{aligned}$$

Therefore, $f_s(\beta)$ is decreasing for $\beta \geqslant a$.

Then $M|L^k|^{2b} \geqslant [(|L^k|^b|L^{*k}|^{2a}|L^k|^b)^{\frac{b}{k(a+b)}}] = f_s(a) \geqslant f_s(a_1) = [|L|^b|L^*|^{2a_1}|L|^b]^{\frac{b}{b+a_1}}$.

Let $A_2=M^c|L^k|^{2bc}$ and $B_2=[|L^k|^b|L^{*k}|^{2a_1}|L^k|^b]^{\frac{bc}{b+a_1}}$.

Since $A_2 \geqslant B_2$, by Lemma 3.5 for any $c=k=1$, $p_2 \geqslant 1$ and $r_2 \geqslant 0$, $(A_2)^{1+r_2} \geqslant (A_2^{\frac{r_2}{2}}B_2{}^{p_2}A_2^{\frac{r_2}{2}})^{\frac{1+r_2}{p_2+r_2}}$.

Let $p_2=\frac{b+a_1}{b} \geqslant 1$, $r_2=\frac{b_1-b}{b} \geqslant 0$.

Then $M_1|L|^{2b_1} \geqslant [|L|^{b_1}|L^*|^{2a_1}|L|^{b_1}]^{\frac{b_1}{a_1+b_1}}$.

Similarly, $[M_1|L^*|^{a_1}|L|^{2b_1}|L^*|^{a_1}]^{\frac{a_1}{a_1+b_1}} \geqslant |L^*|^{2a_1}$. □

Let $\sigma(L), \sigma_p(L)$ and $\sigma_a(L)$ be the spectrum and approximate point spectrum of $L \in B(H)$ respectively. Let λ be a approximate eigen value of L if there exist unit vectors x_n such that $(L-\lambda)x_n \to 0$, $(L-\lambda)^*x_n \to 0$. Let $\sigma_{na}(L)$ denotes the set of all approximate eigenvalues of L.

LEMMA 3.7. *Let $L \in B(H)$ be a $M-class-c-wA_k^*(a,b)$ operator. If $(L-\lambda)x_n \to 0$, then*

$$(L-\lambda)^*x_n, (|L|-r)x_n, (U-e^{i\theta})x_n, (U-e^{i\theta})^*x_n \to 0.$$

Proof. Assume $a+b=1$, $0 \neq \lambda = re^{i\theta} \in C$.

By Corollary 3.3

$$(L(a,b)-re^{i\theta})x_n = (L-re^{i\theta})x_n \to 0 \Rightarrow (L(a,b)-re^{i\theta})^*x_n \to 0.$$

Hence,

$$(L(a,b)-re^{i\theta})^*(L(a,b)-re^{i\theta})x_n \to 0 \Rightarrow (|L(a,b)|^2-r^2)x_n \to 0$$

and $(|L(a,b)|^{\min(ac,bc)} - r^{\min(ac,bc)})x_n \to 0$ similarly, $(|L(a,b)^*|^{\min(ac,bc)} - r^{\min(ac,bc)})x_n \to 0$. Hence, $\left\langle (|L|^{\min(ac,bc)} - r^{\min(ac,bc)})x_n, x_n \right\rangle \to 0$.

If $r \in (0, \min(ac,bc)]$ then $\frac{r}{2} \in (0, \min(ac,bc)]$. So, $\left\langle (|L|^{\frac{\min(ac,bc)}{2}} - r^{\frac{\min(ac,bc)}{2}})x_n, x_n \right\rangle \to 0$. Then $\left\| (|L|^{\frac{\min(ac,bc)}{2}} - r^{\frac{\min(ac,bc)}{2}})x_n \right\|^2 \to 0$. So,

$$(|L| - r)x_n \to 0 \Rightarrow \langle |L|x_n, x_n \rangle - r$$

and

$$\|(|L| - r)x_n\|^2 = r^2 - 2r^2 + r^2 = 0 \Rightarrow (|L| - r)x_n \to 0.$$

Since,

$$(L - re^{i\theta})x_n \to 0 = U(|L| - r)x_n + r(U - e^{i\theta})x_n \to 0.$$

We know that $r > 0$, therefore $(U - e^{i\theta})x_n \to 0 \Rightarrow (U - e^{i\theta})^* x_n \to 0$.

Hence, $(L - re^{i\theta})^* x_n \to 0$. □

THEOREM 3.8. *Let $L = U|L| \in B(H)$ be $M-class-c-wA_k^*(a,b)$ operator for $a,b \in (0,1]$ and M,k,c be any positive integer. Let $0 \neq \lambda = re^{i\theta}$ then $ker(L-\lambda) = ker(L(a,b) - \lambda_{a+b})$ where $\lambda_{a+b} = r^{a+b}e^{i\theta}$.*

Proof. Let $x \in ker(L-\lambda)$. Then $|L|x = rx$ and $Ux = e^{i\theta}x$. So, $L(a,b)x = r^{a+b}e^{i\theta}x$. Hence, $ker(L-\lambda) \subset ker(L(a,b) - \lambda_{a+b})$. Conversely, let $x \in ker(L(a,b) - \lambda_{a+b})$.

Since, L is $\frac{\min(ac,bc)}{(a+b)}$-hyponormal for $M = k = 1$ (Corollary 3.3). So, $(L(a,b) - \lambda_{a+b})^* x = 0$ and $|L(a,b)| = r^{a+b}x, |L(a,b)^*|x = r^{a+b}x$.

Since, $|L(a,b)|^{\frac{2\min(ac,bc)}{a+b}} \geqslant |L|^{2\min(ac,bc)} \geqslant |L(a,b)^*|^{\frac{2\min(ac,bc)}{a+b}}$. We have,

$$|L(a,b)|^{\frac{2\min(ac,bc)}{a+b}} - |L|^{2\min(ac,bc)} \leqslant |L(a,b)|^{\frac{2\min(ac,bc)}{a+b}} - |L(a,b)^*|^{\frac{2\min(ac,bc)}{a+b}} \geqslant 0.$$

Hence, $|L|^{2\min(ac,bc)}x = r^{2\min(ac,bc)}x$.

Since, $r^{a+b}e^{-i\theta}x = L(a,b)^*x = |L|^b U^* |L|^a x = r^b |L|^b U^* |L|^a x$. So, $L^*x = \overline{\lambda}x$ and $\|(L-\lambda)x\|^2 = 0$.

This implies that $ker(L(a,b) - \lambda_{a+b}) \subset ker(L-\lambda)$. □

THEOREM 3.9. *Let L is $M-class-c-wA_k^*(a,b)$. Let $\lambda = re^{i\theta}$ be an isolated point of $\sigma(L)$ and $r > 0$. Then the Riesz idempotent E for L with respect to $re^{i\theta}$ is self adjoint with $ran(E) = ker(L - re^{i\theta}) = ker(L - re^{i\theta})^*$ and coincides with Riesz idempotent $E(a,b)$ for $L(a,b)$ with respect to $re^{i\theta}$.*

Proof. The proof of this theorem is immediate by using the above theorem. □

THEOREM 3.10. *If L is $M-class-c-wA_k^*(a,b)$, then L^* is $M-class-c-wA_k^*(a,b)$.*

Proof. From the definition of $M-class-c-wA_k^*(a,b)$, for every $x\in H$.

$$\left[M^c(|L^{*k}|^a|L^k|^{2b}|L^{*k}|^a)^{\frac{ac}{k(a+b)}}-|L^{*k}|^{2ac}\right]\geqslant 0$$
$$M^c\left[(L^kL^{*k})^{\frac{a}{2}}(L^{*k}L^k)^b(L^kL^{*k})^{\frac{a}{2}})^{\frac{ac}{k(a+b)}}\right]\geqslant (L^kL^{*k})^{ac}$$
$$M^c\left((L^kL^{*k})^{*\frac{a}{2}}(L^{*k}L^k)^{*b}(L^kL^{*k})^{*\frac{a}{2}}\right)^{\frac{ac}{k(a+b)}}\geqslant (L^kL^{*k})^{*ac}$$
$$M^c\left(|L^{*k}|^{*a}|L^k|^{*2b}|L^{*k}|^{*a}\right)^{\frac{ac}{k(a+b)}}\geqslant |L^{*k}|^{*2ac}.$$

Likely,

$$M^c|L^k|^{2bc}\geqslant\left[(|L^k|^b|L^{*k}|^{2a}|L^k|^b)^{\frac{bc}{(a+b)}}\right]$$
$$M^c(L^{*k}L^k)^b\geqslant\left[(L^{*k}L^k)^{\frac{b}{2}}(L^kL^{*k})^a(L^{*k}L^k)^{\frac{b}{2}}\right]^{\frac{bc}{k(a+b)}}$$
$$M^c||L^k|^{*2bc}\geqslant\left[(|L^k|^{*b}|L^{*k}|^{*2a}|L^k|^{*b})^{\frac{bc}{k(a+b)}}\right].$$

Therefore, L^* is $M-class-c-wA_k^*(a,b)$. □

Now, let us discuss some properties in $L^2(\lambda)$. Let C be the non-empty set of complex numbers and let f be a complex-valued measurable function on X. The essential range of f is denoted by $eR(f)$ and $eR(f)=\{\lambda\in C,\mu(f^{-1}(G))\neq 0\}$. A point $z\in C$ is in the joint spectrum $\sigma_{jp}(L)$ if there exits a vector x such that $Lx=zx$ and $L^*x=\bar{z}x$. For a composition operator L, $L=U|L|$ where $|L|f=\sqrt{h}f$ and $Uf=\frac{1}{\sqrt{h\circ L}}(f\circ L)$. Althuge transformation for composition operator is $L(r,1-r)=|L|^rU|L|^{1-r}$ and $L(r,1-r)f=(\frac{h}{(h\circ L)})^{\frac{r}{2}}(f\circ L)$. $L(r,1-r)$ is weighted composition operator with weight $\pi=(\frac{h}{h\circ L})^{\frac{r}{2}}$ where $0<r<1$. For $a,b>0$, $L(a,b)=|L|^aU|L|^b=\frac{h^{\frac{a}{2}}(h^{\frac{b}{2}}\circ L)}{\sqrt{(h\circ L)}}f\circ L$ and so $L(a,b)$ is weighted composition opertator with weight $w=\frac{h^{\frac{a}{2}}(h^{\frac{b}{2}}\circ L)}{\sqrt{h\circ L}}$.

Take $J_1=h[E(w^2)]\circ L^{-1}$, $J_2=h_2[E_2(w_2^2)]\circ L^{-2}$ and $K=w(h\circ L)E(w)$.

THEOREM 3.11. *L is $M-class-c-A_1^*(a,b)$ operator if and only if $M^cJ_1^{\frac{ac}{a+b}}\geqslant h^{ac}$.*

Proof. $L(a,b)$ is weighted composition operator with weight $w=\frac{h^{\frac{a}{2}}(h^{\frac{b}{2}}\circ L)}{\sqrt{h\circ L}}$, where $h=\frac{d\lambda L^{-1}}{d\lambda}$ be the Radon-Nikodym derivative. Then $|L(a,b)|f=\sqrt{J_1}f$ holds by Proposition 2.5. It is known that $|L|=h^{\frac{1}{2}}$. Hence, $M^c|L(a,b)|^{\frac{2ac}{(a+b)}}\geqslant|L|^{2ac}$ because L is $M-class-c-A_1^*(a,b)$ and $M^cJ_1^{\frac{ac}{(a+b)}}\geqslant h^{ac}$. □

COROLLARY 3.12. *L is $M-class-c-wA_1^*(a,b)$ operator if and only if $M^c J_1^{\frac{ac}{(a+b)}} \geqslant h^{ac}$.*

THEOREM 3.13. *L is $M-class-c-wA_1^*(a,b)$ operator if and only if $M^c J_1^{\frac{ac}{(a+b)}} \geqslant h^{ac}$ and $K^{\frac{bc}{(a+b)}} \leqslant M^c h^{bc}$.*

Proof. The proof is immediate from Proposition 2.5 and Theorem 3.11 □

THEOREM 3.14. *If L is $M-class-c-wA_k^*(a,b)$ operator with $0<a,b,a+b \leqslant 1$, then $\sigma(L) \subset z \in C : |z|^2 \in eR(f)$.*

Proof. Given L is $M-class-c-wA_k^*(a,b)$ with $0<a,b,a+b \leqslant 1$. Let $L(a,b)$ is weighted composition operator with weight $w = \frac{h^{\frac{b}{2}}(h^{\frac{a}{2}}\circ L)}{\sqrt{(h\circ L)}}$, $L(a,b)^* L(a,b) f = J_1 f$.

Hence, $\sigma(L(a,b)^* L(a,b)) = eR(J_1)$. If $x \in eR(J_1)$ then $z \in \sigma(L(a,b))$ such that $|Z|^2 = x$ and $eR(J_1) = |z|^2 : z \in \sigma(L(a,b))$.

So, $\sigma(L(a,b)) \subset z \in C : |z|^2 \in eR(J_1) \Rightarrow \sigma(L) \subset z \in C : |Z|^2 \in eR(J_1)$. □

THEOREM 3.15. *Let L be $M-class-c-wA_k^*(a,b)$ operator with $0<a,b,a+b \leqslant 1$. Then $\sigma_p(L) \subset z \in C : |z|^2 = J_1$.*

Proof. Let $z \in \sigma_{jp}(L(a,b))$. There exists a non-zero f such that

$$L(a,b)^* L(a,b) f = |z|^2 f.$$

Since $L(a,b)^* L(a,b) f = J_1 f$ for $f \in L^2(\mu)$ and since $\sigma_p(L) = \sigma(L(a,b))$, it easily follows that $\sigma_p(L) \subset z \in C : |z|^2 = J_1$. □

PROPOSITION 3.16. *If L is compact, then $\sigma(L) = \sigma_p(L)$.*

THEOREM 3.17. *Let $L^2(\lambda)$ be infinite dimensional, then no $M-class-c-wA_k^*(a,b)$ operator in $L^2(\lambda)$ is compact.*

Proof. Assume that L is compact $M-class-c-wA_k^*(a,b)$ operator and $a,b>0$ in $L^2(\lambda)$. Since $\sigma_p(L(a,b))$ is contained in the unit circle, $\sigma(L(a,b))$ is in the unit circle. Then $\sigma(L)$ is in the unit circle, so L is unitary, which contradicts the assumption.

Hence the proof. □

THEOREM 3.18. *Let L be $M-class-c-wA_k^*(a,b)$ operator with $0<a,b,a+b \leqslant 1$ and $1 \notin eR(J_1)$. Then $\sigma(L) = \sigma_w(L)$.*

Proof. Given L is $M-class-c-wA_k^*(a,b)$ operator with $0<a,b,a+b\leqslant 1$ and $1\notin eR(J_1)$. Then Weyl's theorem holds for L and so $\frac{\sigma(L)}{\sigma_w(L)}=\pi_{00}(L)$ holds where $\pi_{00}(L)$ is the set of all isolated eigenvalues of the infinite multiplicity of $\sigma(L)$. So, it is enough to show that $\pi_{00}(L)$ is empty. If $0\in\pi_{00}(L)$, then L has a closed range and so $L$is invertible, which contradicts that L is not invertible. Let $z\neq 0$ is an eigen value of $\sigma(L)$ with $|z|\neq 1$. Since $\sigma_p(L)$ has symmetry about zero except on $|z|=1$. Hence, $z\neq\pi_{00}(L)$. □

4. Kronecker product of $M-class-c-wA_k^*(a,b)$ operators

In this section, the tensor product results of $M-class-c-wA_k^*(a,b)$ are derived using the references from [1], [2], [3], [4], [5], [9], [12], [13], [15]. The results follow from the following Lemma:

LEMMA 4.1. *Let $L_1,L_2\in B(H)$, $S_1,S_2\in B(K)$ be non-negative operators. If $L_1\neq 0$ and $S_1\neq 0$ then the below conditions are alike*

(1) $L_1\otimes S_1\geqslant L_2\otimes S_2$.

(2) $L_1\leqslant cL_1$ *and* $S_1\leqslant c^{-1}S_2$ *for* $c>0$.

LEMMA 4.2. *Let $L=U_L|L|$ and $S=U_S|S|$ be the polar decomposition of $L\in S(H)$ and $S\in S(K)$ respectively. Then the following assertions hold:*

(1) $|L\otimes S|=|L|\otimes|S|$.

(2) $L\otimes S=(U_L\otimes U_S)(|L|\otimes|S|)$.

(3) $(\widetilde{L\otimes S})_{a,b}=\widetilde{L_{a,b}}\otimes\widetilde{S_{a,b}}$ *for* $a,b>0$.

THEOREM 4.3. *Let $L\in B(H)$ and $S\in B(K)$. $L\otimes S$ is $M-class-c-wA_k^*(a,b)$ if and only if L and S are $M-class-c-wA_k^*(a,b)$.*

Proof. Let $L\in B(H)$ and $S\in B(K)$ be $M-class-c-wA_k^*(a,b)$ operator. For convenience take $M=1$.
Then

$$|\widetilde{L}^k_{a,b}|^{\frac{2ac}{k(a+b)}}\geqslant|L^k|^{2ac}$$

and

$$|\widetilde{S}^k_{a,b}|^{\frac{2bc}{k(a+b)}}\geqslant|L^k|^{2bc}.$$

$$\begin{aligned}|(\widetilde{L^k\otimes S^k})_{a,b}|^{\frac{2ac}{k(a+b)}}&=|\widetilde{L}^k_{a,b}\otimes\widetilde{S}^k_{a,b}|^{\frac{2ac}{k(a+b)}}=|\widetilde{L}^k_{a,b}|^{\frac{2ac}{k(a+b)}}\otimes|\widetilde{S}^k_{a,b}|^{\frac{2ac}{k(a+b)}}\\&=|L^k\otimes S^k|^{2ac}.\end{aligned}$$

In the same way, $|L^k\otimes S^k|^{2bc}\geqslant|(\widetilde{L^k\otimes S^k})^*_{a,b}|^{\frac{2bc}{k(a+b)}}$.

Now, suppose that $L\otimes S$ is $M-class-c-wA_k^*(a,b)$. Then

$$|\widetilde{(L^k\otimes S^k)}_{a,b}|^{\frac{2ac}{k(a+b)}}=|\widetilde{L^k_{a,b}}|^{\frac{2ac}{k(a+b)}}\otimes|\widetilde{S^k_{a,b}}|^{\frac{2ac}{k(a+b)}}=|L^k\otimes S^k|^{2ac}$$

and

$$|L^k\otimes S^k|^{2bc}\geqslant|\widetilde{(L^k\otimes S^k)}^{*}_{a,b}|^{\frac{2ac}{k(a+b)}}$$

so for $d>0$, $d|(\widetilde{L}^k_{a,b}|^{\frac{2ac}{k(a+b)}}\geqslant|L^k|^{2ac}$ and $d^{-1}|(\widetilde{S}^k_{a,b}|^{\frac{2ac}{k(a+b)}}\geqslant|S^k|^{2ac}$.

Let x be a unit vector. Then

$$\begin{aligned}\||L|^a x\|^{2c}&=\left\langle|L|^{2ac}x,x\right\rangle\leqslant\left\langle d|\widetilde{L}^k_{a,b}|^{\frac{2ac}{k(a+b)}}x,x\right\rangle\\&=d\,\||L|^a\|^{2c}\qquad\text{for } d\geqslant 1.\end{aligned}$$

Also,

$$\||S|^a x\|^{2c}=\left\langle|S|^{2ac}x,x\right\rangle\geqslant\left\langle d^{-1}|\widetilde{S}^k_{a,b}|^{\frac{2ac}{k(a+b)}}x,x\right\rangle=d^{-1}\,\||S|^a\|^{2c}$$

for $d^{-1}\geqslant 1$.

Hence, $d=1$ and $|\widetilde{L}^k_{a,b}|^{\frac{2ac}{k(a+b)}}\geqslant|L^k|^{2ac}$ and $|\widetilde{S}^k_{a,b}|^{\frac{2ac}{k(a+b)}}\geqslant|S^k|^{2ac}$.

Similarly, $|\widetilde{L}^{*k}_{a,b}|^{\frac{2bc}{k(a+b)}}\geqslant|L^k|^{2bc}$ and $|\widetilde{S}^{*k}_{a,b}|^{\frac{2bc}{k(a+b)}}\geqslant|S^k|^{2bc}$.

Thus L and S are $M-class-c-wA_k^*(a,b)$. □

5. Concluding remark

In this article, the $M-class-c-wA_k^*(a,b)$ operator is defined then few examples are discussed to show the inclusion relation, later the spectral and algebraic properties, the kronecker product results are determined. Now, consider $M=k=1$ and $|L|^2=P$, $|L^*|^2=R$ then the equations (1.1) and (1.2) becomes

$$R^{ac}\leqslant[R^{\frac{a}{2}}P^bR^{\frac{a}{2}}]^{\frac{ac}{a+b}} \tag{5.1}$$

and

$$P^{bc}\geqslant[P^{\frac{b}{2}}R^aP^{\frac{b}{2}}]^{\frac{bc}{a+b}} \tag{5.2}$$

It is obvious to think that (5.1) is equivalent to (5.2) for any $a,b,c>0$, but it is not true for the following example:

Let $P=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$ and $R=\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$.

Then $P^{bc}-[P^{\frac{b}{2}}R^aP^{\frac{b}{2}}]^{\frac{bc}{a+b}}\geqslant 0$ but $[R^{\frac{a}{2}}P^bR^{\frac{a}{2}}]^{\frac{ac}{a+b}}-R^{ac}\leqslant 0$ for any $a,b,c>0$.

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