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# IDEALS IN HAAGERUP TENSOR PRODUCT OF $C^*$–TERNARY RINGS AND TROS

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*Abstract.* We characterize the maximal, prime and primitive ideals of Haagerup tensor product $M\otimes^h B$ of a TRO $M$ and a $C^*$-algebra $B$.

## 1. Introduction

A $C^*$-ternary ring ($C^*$-tring) $(M,[.,.,.],\|.\|)$ consists of a complex Banach space $(M,\|.\|)$ and a ternary product $[.,.,.]: M^3\to M$ which is linear in the first and third variable, conjugate linear in the second variable and associative as:

$$[[x,y,z],u,v]=[x,y,[z,u,v]]=[x,[u,z,y],v].$$

Moreover, the norm satisfies $\|[x,x,x]\|=\|x\|^3$ and $\|[x,y,z]\|\leqslant\|x\|\|y\|\|z\|$. For instance, any ternary ring of operator (TRO) is a $C^*$-tring such as $B(\mathscr{H},\mathscr{K})$, the space of all bounded operators from a Hilbert space $\mathscr{H}$ to a Hilbert space $\mathscr{K}$, $M_{n,k}$ the $n\times k$ complex matrices or a $C^*$-algebra. It can be seen that every $C^*$-tring has an operator space structure [7, 21].

Pluta and Russo ([15], Proposition 2.7) assigned a $C^*$-algebra $\mathscr{A}(M)$ corresponding to a $C^*$-tring $M$. The referee and one of the coauthors of [15] have pointed out that Proposition 2.7 is not correct as stated (see [16]). In fact, if $M$ is a $C^*$-tring and there is a $C^*$-norm on $\mathscr{A}(M)$ then $M$ is isomorphic to a TRO. In this case $\mathscr{A}(M)$ is $C^*$-isomorphic to the linking $C^*$-algebra of $M$. In general $\mathscr{A}(M)$ is a Banach algebra having an approximate identity, which has been studied in [17].

Ideals of the Banach algebra arising from Haagerup tensor product $A\otimes^h B$ of $C^*$-algebras $A$ and $B$ were investigated in [1] and [3]. In [11], the Haagerup tensor product $M\otimes^h B$ of $C^*$-tring $M$ and $C^*$-algebra $B$ has been discussed in detail. One may note that the Haagerup tensor product is associative, injective but not necessarily symmetric.

In the present paper, we initiate a study of the ideal structure of the Banach space $M\otimes^h B$. After preliminaries about ideals of $C^*$-tring and $\varepsilon$-ideals of $M\otimes^h B$ in Section 2, we present prime ideals of a TRO $M$ in the next section. For a TRO $M$, we establish

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a homeomorphism between prime ideals of $M$ and $\mathscr{A}(M)$. We also show that if $M$ or $B$ is exact then there is a one-to-one correspondence between prime ideals of injective tensor product $M \otimes^{\text{tmin}} B$ and prime ideals of $M$ and $B$. In Section 4, it has been shown that if $M$ is a TRO then every maximal ideal of $M \otimes^h B$ has the form $I \otimes^h B + M \otimes^h J$ for some maximal ideals $I$ and $J$ of $M$ and $B$ respectively.

Subsequently, we introduce prime ideal, $i$-prime ideal and $\varepsilon$-prime ideal of $M \otimes^h B$ and study their relationship. Let $I$ and $J$ be ideals of $M$ and $B$ respectively and let $\pi : M \to M/I$ and $\rho : B \to B/J$ be the quotient maps. Then $\pi \otimes^{\text{tmin}} \rho : M \otimes^{\text{tmin}} B \to M/I \otimes^{\text{tmin}} B/J$ is a ternary homomorphism. We show that if $M$ or $B$ is exact, then $\ker(\pi \otimes^{\text{tmin}} \rho) = M \otimes^{\text{tmin}} J + I \otimes^{\text{tmin}} B$. This paves the way to establish that if $M$ or $B$ is exact then every prime ideal of $M \otimes^h B$ is of the form $I \otimes^h B + M \otimes^h J$ for some prime ideals $I$ and $J$ of $M$ and $B$. Finally, we describe primitive ideals of $M \otimes^h B$ in terms of primitive ideals of $M$ and $B$.

Throughout this paper, $M$ denotes a $C^*$-tring or a TRO whenever required and $B$ a $C^*$-algebra.

## 2. Preliminaries

A closed subspace $I$ of $M$ is called an ideal of $M$ provided $[I,M,M] + [M,M,I] \subseteq I$. By an ideal we shall always mean a closed ideal, unless otherwise stated. If $I$ is an ideal of $M$ then $[M,I,M] \subseteq I$ ([9], Remark 2.7). Let $\text{Id}(M)$ denotes the space of all ideals of $M$. We recall the $\tau_w$-topology defined on $\text{Id}(M)$. A subbasis for $\tau_w$-topology is given by the sets of the form $U(J) = \{I \in \text{Id}(M) : I \not\supseteq J\}$, where $J \in \text{Id}(M)$. If $M$ is a TRO then it is known that the map $\theta : \text{Id}(M) \to \text{Id}(\mathscr{A}(M))$ defined as $\theta(I) = \mathscr{A}(I)$ is a homeomorphism ([18], Proposition 2.7), ([10], Proposition 2.4).

DEFINITION 1. A linear mapping $\phi : M \to B(\mathscr{H}, \mathscr{K})$ is called a representation of $M$ if $\phi$ preserves the ternary structure i.e. $\phi([x,y,z]) = \phi(x)\phi(y)^*\phi(z)$.

In [10], it was shown that there is a one to one correspondence between (irreducible) representations of $M$ and $\mathscr{A}(M)$.

The Haagerup norm on the algebraic tensor products of $M$ and $B$ is defined, for $x \in M \otimes B$, by

$$||x||_h = \inf\left\{ ||a|| ||b|| : a = (a_{1j})_{1\times n}, b = (b_{j1})_{n\times 1} \text{ and } x = \sum_{j=1}^{n} a_{1j} \otimes b_{j1} \right\}.$$

The Haagerup tensor product $M \otimes^h B$ is then the completion of $M \otimes B$ in this norm. For more details, the reader is referred to [7]. It can be seen that $M \otimes^h B$ may neither be a $C^*$-tring nor a Banach algebra in general. Moreover, $M \otimes^{\text{tmin}} B$ is a $C^*$-tring and if $M$ happens to be a TRO then $\mathscr{A}(M \otimes^{\text{tmin}} B) = \mathscr{A}(M) \otimes^{\text{min}} B$. In [11], the concept of $\varepsilon$-ideals and $i$-ideals were introduced. We recall the definitions for convenience of the reader.

DEFINITION 2. A closed subspace $P$ of $M \otimes^h B$ is called an $\varepsilon$-ideal if $P = \varepsilon^{-1}(Q)$ for some closed ideal $Q$ of $M \otimes^{\text{tmin}} B$, where $\varepsilon : M \otimes^h B \to M \otimes^{\text{tmin}} B$ is the natural injective map. We shall regard $M \otimes^h B$ as a subspace of $M \otimes^{\text{tmin}} B$ with a different norm. It is easy to conclude that $P$ is an $\varepsilon$-ideal if and only if $P = Q \cap (M \otimes^h B)$, where $Q$ is a closed ideal in $M \otimes^{\text{tmin}} B$. If $M$ is a $C^*$-algebra then every $\varepsilon$-ideal of $M \otimes^h B$ is an ideal and conversely.

DEFINITION 3. If $M$ is a TRO, then a closed subspace $P$ of $M \otimes^h B$ is called an $i$-ideal if $P = i^{-1}(Q)$ for some closed ideal $Q$ of $\mathscr{A}(M) \otimes^h B$, where $i : M \otimes^h B \to \mathscr{A}(M) \otimes^h B$ is the isometry obtained by injectivity of the Haagerup tensor product. Of course, $P$ is an $i$-ideal if and only if $P = Q \cap (M \otimes^h B)$, where $Q$ is a closed ideal in $\mathscr{A}(M) \otimes^h B$.

It is known that a closed subspace $P$ of $M \otimes^h B$ is an $\varepsilon$-ideal if and only if $P$ is an $i$-ideal ([11], Proposition 4.12).

Let $M \otimes^{\text{tmax}} B$ be the maximal $C^*$-tring tensor product of $M$ and $B$. We may note that $\|x\|_{\text{tmax}} \leqslant \|x\|_h$ for all $x \in M \otimes B$ [11]. For $C^*$-algebras $A$ and $B$ there is a one-to-one correspondence between representations of $A \otimes^{\max} B$ and $*$-representations of $A \otimes^h B$. Indeed, if $\rho$ is a representation of $A \otimes^{\max} B$ then $\rho\varepsilon'$ is a $*$-representation of $A \otimes^h B$ ($\varepsilon' : A \otimes^h B \to A \otimes^{\max} B$ is natural contractive homomorphism). If $\pi$ is a $*$-representation of $A \otimes^h B$ then by ([1], Lemma 5.12) there is a (unique) representation $\rho$ of $A \otimes^{\max} B$ such that $\pi = \rho\varepsilon'$. The proof of the following result is immediate.

PROPOSITION 1. *Let $M$ be a TRO and $B$ a $C^*$-algebra. Let $\pi$ be a (irreducible) $*$-representation of $\mathscr{A}(M) \otimes^h B$ then there exist (irreducible) representation $\rho$ of $M \otimes^{tmax} B$ such that $\pi = \mathscr{A}(\rho)\tilde{\varepsilon}'$, where $\tilde{\varepsilon}' : \mathscr{A}(M) \otimes^h B \to \mathscr{A}(M) \otimes^{max} B$ is the natural injective homomorphism.*

## 3. Prime ideals of min tensor product of $C^*$-trings

If $I$, $J$ and $K$ are ideals in $M$, then define

$$IJK = \overline{\text{span}}\{[a,b,c] : a \in I, b \in J, c \in K\}$$

It is easy to check that $IJK$ is an ideal of $M$.

LEMMA 1. *Let $I$, $J$ and $K$ be ideals of $C^*$-tring $M$. Then*

$$IJK = I \cap J \cap K.$$

*Proof.* Note that as $I$, $J$ and $K$ are ideals of $M$, therefore $IJK \subseteq I$, $IJK \subseteq J$ and $IJK \subseteq K$ which implies $IJK \subseteq I \cap J \cap K$. Conversely, let $x \in I \cap J \cap K$. Since $I \cap J \cap K$ is a $C^*$-tring and that every element of $C^*$-tring has a cube root ([17], Page 6 footnote), therefore there exists $y \in I \cap J \cap K$ such that $x = [y,y,y] \in IJK$. $\square$

PROPOSITION 2. *Let $M$ be a $C^*$-tring and $L$ an ideal in $M$. Then $L$ satisfies $(P1)$ if and only if it satisfies $(P2)$, where*

$(P1)$ *For any three ideals $I$, $J$ and $K$ of $M$ satisfying $IJK \subseteq L$, either $I \subseteq L$ or $J \subseteq L$ or $K \subseteq L$.*

$(P2)$ *For any pair of ideals $I$ and $J$ satisfying $I \cap J \subseteq L$, either $I \subseteq L$ or $J \subseteq L$.*

*Proof.* These statements are obviously equivalent for the ideal $M$ or $\{0\}$, so we assume that $L$ is a proper closed ideal in $M$. Suppose that $L$ satisfies $(P1)$, and let $I$ and $J$ be ideals such that $I \cap J \subseteq L$ then $IJI \subseteq I$ and $IJI \subseteq J$ so $IJI \subseteq I \cap J \subseteq L$. Thus either $I \subseteq L$ or $J \subseteq L$, proving that $L$ satisfies $(P2)$. Suppose now that $L$ is an ideal in $M$ satisfying $(P2)$, and let $I$, $J$ and $K$ be ideals such that $IJK \subseteq L$ then by Lemma 1, $I \cap J \cap K \subseteq L$. Thus either $I \subseteq L$ or $J \subseteq L$ or $K \subseteq L$. $\square$

We say that an ideal $L$ in $M$ is prime if it satisfies $(P1)$ or $(P2)$. One may easily note that $\{0\}$ and $K(\mathscr{H},\mathscr{K})$, the space of compact operators from a Hilbert space $\mathscr{H}$ to a Hilbert space $\mathscr{K}$ are prime ideals of the $C^*$-tring $B(\mathscr{H},\mathscr{K})$.

For a closed ideal $I$ of a $C^*$-tring $M$, let $\tilde{\mathscr{A}}(I)$ denotes the restriction of $\mathscr{A}(I)$ from $M \oplus M$ to $I \oplus I$.

LEMMA 2. *Let $I$ and $J$ be closed ideals of $C^*$-tring $M$, then $I+J$ is also closed.*

*Proof.* For closed ideals $I$ and $J$ of $C^*$-tring $M$, $\tilde{\mathscr{A}}(I)$ and $\tilde{\mathscr{A}}(J)$ are closed ideals of $\mathscr{A}(M)$ ([17], Propositions 4.1,4.2). Let $x \in \overline{I+J}$. Since $\tilde{\mathscr{A}}(I)$ and $\tilde{\mathscr{A}}(J)$ have bounded approximate identity so $\tilde{\mathscr{A}}(I)+\tilde{\mathscr{A}}(J)$ in $\mathscr{A}(M)$ is closed ([6], Proposition 2.4), thus

$$\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \in \overline{\tilde{\mathscr{A}}(I)+\tilde{\mathscr{A}}(J)} = \tilde{\mathscr{A}}(I)+\tilde{\mathscr{A}}(J) = \begin{bmatrix} \tilde{L}(I)+\tilde{L}(J) & I+J \\ \bar{I}+\bar{J} & \tilde{R}(I)+\tilde{R}(J) \end{bmatrix}$$

so $x \in I+J$. $\square$

EXAMPLE 1. Let $X$ and $Y$ be compact Hausdorff spaces. Furthermore, assume that $X'$ is a proper non void open and closed subset of $X$. Let $C(Y)$ be the algebra of complex-valued continuous functions on $Y$ with usual operations. Let $M = C_t(X, C(Y))$ be the set of continuous functions from $X$ into $C(Y)$. Define $\chi : X \to \{0_{C(Y)}, 1_{C(Y)}\}$ by

$$\chi(t) = \begin{cases} 1_{C(Y)}, & t \in X' \\ 0_{C(Y)}, & \text{otherwise.} \end{cases}$$

For $f,g,h \in C_t(X,C(Y))$ put

$$[f,g,h](x) = (2\chi(x) - 1_{C(Y)}) f(x) \overline{g(x)} h(x)$$

Then $(C_t(X, C_t(Y)), [.,.,.], \|.\|_{\sup})$ is a commutative $C^*$-tring ( i.e. $[a,b,c] = [c,b,a]$ for all $a,b,c \in M$) which is not a TRO.

Let $C_t(X\times Y)$ be the set of complex valued continuous functions on $X\times Y$. Define $\chi' : X\times Y \to \{0,1\}$ by

$$\chi'((x,y)) = \begin{cases} 1, & (x,y)\in X'\times Y \\ 0, & \text{otherwise.} \end{cases}$$

For $f,g,h \in C_t(X\times Y)$, put

$$[f,g,h](x,y) = (2\chi'(x,y)-1)f(x,y)\overline{g(x,y)}h(x,y)$$

Then $(C_t(X,C(Y)),[.,.,.],\|.\|_{\sup})$ is a commutative $C^*$-tring. Define

$$\psi : C_t(X,C(Y)) \to C_t(X\times Y)$$

by

$$\psi(f)(x,y) = f(x)(y)$$

It is not difficult to see that $\psi$ is an isomorphism of $C^*$-trings. Let $C(X\times Y)$ be the algebra of complex valued continuous functions on $X\times Y$ with usual operations. Let $V$ be a closed subset of $X\times Y$. Define, $I(V)=\{f\in C_t(X\times Y) : f(x,y)=0, \forall (x,y)\in V\}$. If $V=\{(a,b)\}$, we denote $I(V)$ by $I_{a,b}$. Note that $I(V)$ is a closed ideal of $C_t(X\times Y)$. It is easy to see that a closed subspace $I$ is an ideal of $C_t(X\times Y)$ if and only if $I$ is an ideal of $C(X\times Y)$. Thus, closed ideals of $C_t(X\times Y)$ are of the form $I(V)$ for some closed set $V$ of $X\times Y$. In particular, maximal ideals of $C_t(X\times Y)$ are of the form $I_{a,b}=\{f\in C_t(X\times Y)) : f(a,b)=0\}$ for some $(a,b)\in X\times Y$. Also ideals of the form $I_{a,b}$ are prime. In fact, there are no closed prime ideals other than the maximal ones.

Let Prime$(M)$ denotes the space of Prime ideals of $M$, then Prime$(M)$ inherits subspace topology from Id$(M)$. In the next proposition, we establish that the map $\theta$ defined in Section 2 is a homeomorphism between prime ideals of $M$ and $\mathscr{A}(M)$.

PROPOSITION 3. *Let $M$ be a TRO then* Prime$(M)$ *is homeomorphic to* Prime$(\mathscr{A}(M))$.

*Proof.* Suppose $L$ is a prime ideal of $M$ and let $I'\cap J' \subseteq \mathscr{A}(L)$ for some ideals $I'$ and $J'$ of $\mathscr{A}(M)$. We may assume that $I'=\mathscr{A}(I)$ and $J'=\mathscr{A}(J)$ for some ideals $I$ and $J$ of $M$.Then $\mathscr{A}(I)\cap\mathscr{A}(J)\subseteq\mathscr{A}(L)$ which implies $\mathscr{A}(I\cap J)\subseteq\mathscr{A}(L)$, thus $I\cap J\subseteq L$ so either $I\subseteq L$ or $J\subseteq L$ as $L$ is a prime ideal. The proof of the converse is along similar lines, so we omit it. □

Recall that an ideal $I$ of $M$ is called modular if there exists $e$ and $f$ in $M$ such that $a-[a,e,f]\in I$ for every $a\in M$. It is easy to see that for separable Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, $K(\mathscr{H},\mathscr{K})$ is the only non-trivial modular ideal of $B(\mathscr{H},\mathscr{K})$. An ideal $I$ of $M$ is called primitive if it is quotient of a maximal modular ideal i.e. $I=(J:M)=\{a\in M : [a,M,M]\subseteq J\}$ for some maximal modular ideal $J$ of $M$. Moreover, a closed ideal $I$ of $M$ is primitive if and only if $I$ is kernel of some nonzero irreducible representation ([10], Theorem 2.8).

COROLLARY 1.

*(a) Every primitive ideal is prime and every maximal modular ideal is prime.*

*(b) If $M$ is separable, then every prime ideal is primitive.*

*Proof.*

(a) Let $I$ be a primitive ideal of $M$, then by ([10], Theorem 2.6(4)), $\mathscr{A}(I)$ is a primitive ideal of $\mathscr{A}(M)$. As primitive ideals of $C^*$-algebras are prime ([13], Theorem 5.4.5), so $\mathscr{A}(I)$ is prime and therefore $I$ is prime by above proposition. The other part follows immediately from ([10], Proposition 2.5).

(b) Let $I$ be a prime ideal of $M$, then $\mathscr{A}(I)$ is a prime ideal of $\mathscr{A}(M)$. Since $M$ is separable, so $\mathscr{A}(M)$ is separable. Thus, by ([14], Theorem 4.3.6), $\mathscr{A}(I)$ is primitive and hence $I$ is primitive ([10], Theorem 2.6). $\square$

We now turn our attention to describe prime ideals of operator space injective tensor product. For $C^*$-trings $M$ and $N$, let $M\otimes^{\text{tmin}}N$ denotes the operator space injective tensor product of $M$ and $N$. Note that $M\otimes^{\text{tmin}}N$ is a $C^*$-tring. By taking $M=C_t(X)$ and $N$ as any $C^*$-algebra, we can obtain other $C^*$-trings which are not TROs.

PROPOSITION 4. *Let $M_i$ and $N_i$ $(i=1,2)$ be $C^*$-trings. Let $f_i : M_i \to N_i$ be ternary homorphisms for $i=1,2$. Then $f_1\otimes f_2$ continuously extends to a ternary homomorphism $f_1\otimes^{tmin} f_2 : M_1\otimes^{tmin} M_2 \to N_1\otimes^{tmin} N_2$. Moreover, $f_1\otimes^{tmin} f_2$ is injective if $f_1$ and $f_2$ are so.*

*Proof.* By ([8], Proposition 3.11) each $f_i$ is contraction. Also, for each $n\in\mathbb{N}$,

$$(f_i)_n : M_n(M_i) \to M_n(N_i) : [v_{i,j}] \to [f_i(v_{i,j})]$$

is also a ternary homomorphism, and thus a contraction. Hence $f_i$ is a complete contraction. Since injective tensor product of operator spaces is injective therefore $f_1\otimes f_2$ continuously extends by density to a completely bounded map $f_1\otimes^{\text{tmin}} f_2 :$ $M_1\otimes^{\text{tmin}} M_2 \to N_1\otimes^{\text{tmin}} N_2$. The extended map $f_1\otimes^{\text{tmin}} f_2$ is also a ternary homomorphism. Moreover, if each $f_i$ is injective then $f_i$ is complete isometry, and therefore $f_1\otimes^{\text{tmin}} f_2$ is also complete isometry. $\square$

COROLLARY 2. *Let $I$ and $J$ be closed ideals of $C^*$-trings $M$ and $N$ respectively then $I\otimes^{tmin} J$ is a closed ideal of $M\otimes^{tmin} N$.*

EXAMPLE 2. Let $I$ be an ideal of $M=C_t(X,C(Y))$ in Example 1. Define $e$ and $f$ in $C_t(X,C(Y))$ as $e(x)=1_{C(Y)}$ and $f(x)=2\chi(x)-1_{C(Y)}$ for all $x\in X$. Then, we have $h-[h,e,f]=0\in I$ for every $h\in C_t(X\times Y)$, so $I$ is modular. One can verify that $I\otimes^{\text{tmin}}B+M\otimes^{\text{tmin}}J$ is a closed modular ideal of $M\otimes^{\text{tmin}}B$, where $J$ is a modular ideal (Lemma 2). In particular, $I\otimes^{\text{tmin}}B$ is modular.

If $M$ is a TRO, using ([11], Proposition 4.6) and ([4], Lemma 2.12), it is not difficult to see that every nonzero ideal of $M\otimes^{\text{tmin}} B$ has a nonzero elementary tensor. We may combine Corollary 2, Lemma 2, ([11], Proposition 4.6) and Proposition 4 to obtain the following.

COROLLARY 3. *If $I$ and $J$ are prime ideals of $M$ and $B$ respectively, then $I\otimes^{tmin} B+M\otimes^{tmin} J$ is also a prime ideal of $M\otimes^{tmin} B$.*

DEFINITION 4. A $C^*$-tring $M$ is said to be exact if the functor $M\otimes^{\text{tmin}} -$ is exact; i.e., for each $C^*$-tring $N$ and ideal $J$ of $N$ the sequence

$$0\to M\otimes^{\text{tmin}} J\to M\otimes^{\text{tmin}} N\to M\otimes^{\text{tmin}} N/J\to 0$$

is exact.

EXAMPLE 3. It is easy to see that every finite dimensional $C^*$-tring is exact. If $M$ is commutative $C^*$-tring, then using ([15], Lemma 1.1), $R(M)$ is commutative, so $R(M)$ is exact. From ([8], Corollary 5.17), it is known that $M$ is exact if and only if $R(M)$ is exact, so $M$ is an exact $C^*$-tring. In particular, $C_t(X,C(Y))$ in Example 1 is exact. Also, it can be seen that $K(\mathscr{H},\mathscr{K})$ and $M_{n,k}$ are exact.

LEMMA 3. *Let $M$ be an exact TRO, then $\mathscr{A}(M)$ is an exact $C^*$-algebra.*

*Proof.* Let $J$ be an ideal of $B$ and $0\to J\to B\to B/J\to 0$ be an exact sequence. Since $M$ is exact, so the sequence

$$0\to M\otimes^{\text{tmin}} J\to M\otimes^{\text{tmin}} B\to M\otimes^{\text{tmin}} B/J\to 0$$

is exact. So the sequence

$$0\to \mathscr{A}(M\otimes^{\text{tmin}} J)\to \mathscr{A}(M\otimes^{\text{tmin}} B)\to \mathscr{A}(M\otimes^{\text{tmin}} B/J)\to 0$$

is exact by ([9], Proposition 2.9). But then the sequence

$$0\to \mathscr{A}(M)\otimes^{\min} J\to \mathscr{A}(M)\otimes^{\min} B\to \mathscr{A}(M)\otimes^{\min} B/J\to 0$$

is exact by ([11], Proposition 4.6). Thus, $\mathscr{A}(M)$ is exact $C^*$-algebra. $\square$

In view of Corollary 3, we obtain a canonical map

$$\text{Prime}(M)\times\text{Prime}(B)\to\text{Prime}(M\otimes^{\text{tmin}} B)$$

given by

$$(I,J)\to I\otimes^{\text{tmin}} B+M\otimes^{\text{tmin}} J.$$

THEOREM 1. *If $M$ is an exact TRO or $B$ is exact then* $\text{Prime}(M\otimes^{tmin} B)$ *is homeomorphic to* $\text{Prime}(M)\times\text{Prime}(B)$.

*Proof.* If $M$ or $B$ is exact then $\mathscr{A}(M)$ or $B$ is exact by above lemma. Thus using ([4], Proposition 2.16 and 2.17), $\text{Prime}(\mathscr{A}(M)\otimes^{\text{tmin}} B)$ is homeomorphic to $\text{Prime}(\mathscr{A}(M))\times\text{Prime}(B)$, which is homeomorphic to $\text{Prime}(M)\times\text{Prime}(B)$ by Proposition 3. $\square$

## 4. Maximal ideals of $M \otimes^h B$

In the remaining sections of the paper, we assume $M$ to be a TRO.

We classify all $\varepsilon$-ideals of $M \otimes^h B$ which are maximal. As noted in ([11], Remark 4.22), if $U_1$ and $U_2$ are maximal ideals of $M$ and $B$ respectively then $U_1 \otimes^h B + M \otimes^h U_2$ is maximal $\varepsilon$-ideal. We first note that the following diagram

$$\begin{array}{ccc} M \otimes^h B & \xrightarrow{\varepsilon} & M \otimes^{\mathrm{tmin}} B \\ {\scriptstyle i}\downarrow & & {\scriptstyle j}\downarrow \\ \mathscr{A}(M) \otimes^h B & \xrightarrow[\tilde{\varepsilon}]{} & \mathscr{A}(M) \otimes^{\min} B \end{array}$$

is commutative i.e. $j\varepsilon = \tilde{\varepsilon} i$. The maps $i = i_M \otimes \mathrm{id}_B$ and $j$ are isometry. Moreover the maps $\varepsilon$ and $\tilde{\varepsilon}$ are injective and contractive ([5], Proposition 2) and ([11], Proposition 4.9).

LEMMA 4. *Let $I$ and $J$ be ideals of $M$ and $B$ respectively, then*

*(a)* $j^{-1}(\mathscr{A}(I) \otimes^{min} J) = I \otimes^{tmin} J$.

*(b)* $\tilde{\varepsilon}(\mathscr{A}(I) \otimes^h J) \subseteq \mathscr{A}(I) \otimes^{min} J$.

*(c) For* $\{\mathscr{A}(I_i) \otimes^h J_i\}_{i=1}^n$ *a finite collection of product ideals in* $\mathscr{A}(M) \otimes^h B$, *we have,*

$$i^{-1}\Big(\sum_{i=1}^n \mathscr{A}(I_i) \otimes^h J_i\Big) = \sum_{i=1}^n (I_i \otimes^h J_i).$$

*Proof.*

(a) Since $I \otimes J \subseteq j^{-1}(\mathscr{A}(I) \otimes^{\min} J)$ and $j^{-1}(\mathscr{A}(I) \otimes^{\min} J)$ is closed so $I \otimes^{\mathrm{tmin}} J \subseteq j^{-1}(\mathscr{A}(I) \otimes^{\min} J)$. Conversely, let $x \in j^{-1}(\mathscr{A}(I) \otimes^{\min} J)$ i.e.

$$\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} = j(x) \in \mathscr{A}(I) \otimes^{\min} J = \overline{\mathscr{A}(I) \otimes J}^{\min}$$

So there is a sequence $(x_n) \in \mathscr{A}(I) \otimes J$ such that $\|x_n - j(x)\|_{\min} \to 0$ as $n \to \infty$. Suppose $x_n = \sum_{i=1}^n \begin{bmatrix} A_i & f_i \\ \overline{g_i} & B_i \end{bmatrix} \otimes J_i$, where $f_i \in I$, $A_i \in L(I)$, $B_i \in R(I)$, $\overline{g_i} \in \overline{I}$ and $J_i \in J$. Let $N = (I \otimes^{\mathrm{tmin}} J) \oplus R(I \otimes^{\mathrm{tmin}} J)$. Since we have the $C^*$-isomorphism between $\mathscr{A}(I) \otimes^{\min} J$ and $\mathscr{A}(I \otimes^{\mathrm{tmin}} J)$ ([12], Proposition 3.1), so using $\left\| \begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix} \right\| \geqslant \|f\|$ ([17], Proof of Theorem 2.7) we have

$$\begin{aligned} \|x_n - j(x)\|_{\min} &= \left\| \begin{bmatrix} \sum_{i=1}^n A_i \otimes J_i & \sum_{i=1}^n f_i \otimes J_i - x \\ \sum_{i=1}^n \overline{g_i} \otimes J_i & \sum_{i=1}^n B_i \otimes J_i \end{bmatrix} \right\|_{B(N)} \\ &\geqslant \left\| \sum_{i=1}^n f_i \otimes J_i - x \right\|_{\mathrm{tmin}}. \end{aligned}$$

Thus $\sum_{i=1}^{n} f_i \otimes J_i \xrightarrow{\text{tmin}} x$ as $n \to \infty$. Hence $x \in I \otimes^{\text{tmin}} J$.

(b) Follows immediately using continuity of $\tilde{\varepsilon}$.

(c) It is sufficient to prove the result for $n = 2$. Let $K_1 = \mathscr{A}(I_1) \otimes^h J_1$ and $K_2 = \mathscr{A}(I_2) \otimes^h J_2$. Note that $i^{-1}(K_1 + K_2)$ is closed and contains $I_1 \otimes J_1 + I_2 \otimes J_2$ therefore $i^{-1}(K_1 + K_2)$ also contains $I_1 \otimes^h J_1 + I_2 \otimes^h J_2$. Conversely, let $z \in i^{-1}(K_1 + K_2)$ i.e. $i(z) = x + y$ for some $x \in K_1$ and $y \in K_2$ so $j\varepsilon(z) = \tilde{\varepsilon}(i(z)) = \tilde{\varepsilon}(x) + \tilde{\varepsilon}(y) \in \tilde{\varepsilon}(K_1) + \tilde{\varepsilon}(K_2) \subseteq \mathscr{A}(I_1) \otimes^{\min} J_1 + \mathscr{A}(I_2) \otimes^{\min} J_2 = \mathscr{A}(I_1 \otimes^{\text{tmin}} J_1 + I_2 \otimes^{\text{tmin}} J_2)$. Therefore $\varepsilon(z) \in I_1 \otimes^{\text{tmin}} J_1 + I_2 \otimes^{\text{tmin}} J_2$ using $(a)$, which gives $z \in I_1 \otimes^h J_1 + I_2 \otimes^h J_2$ by ([11], Proposition 4.17). □

For a $C^*$-tring $M$, let $\nu(M)$ denotes the number of closed ideals in $M$ where we count both 0 and $M$. From ([9], Proposition 2.21), it is clear that $\nu(M) = \nu(\mathscr{A}(M))$. The next result characterizes all the $\varepsilon$-ideals of $M \otimes^h B$ in the case where $M$ or $B$ has finitely many $\varepsilon$-ideals.

COROLLARY 4. *If $\nu(M)$ is finite then every $i$-ideal ($\varepsilon$-ideal) of $M \otimes^h B$ is a finite sum of product ideals.*

*Proof.* Let $T_1$ be an $i$-ideal of $M \otimes^h B$ i.e. $T_1 = i^{-1}(T_2)$, for some ideal $T_2$ of $\mathscr{A}(M) \otimes^h B$. Since $\nu(M) = \nu(\mathscr{A}(M))$ so $\nu(\mathscr{A}(M))$ is also finite and therefore by ([1], Theorem 5.3), $T_2 = \sum_{i=1}^{n} \mathscr{A}(I_i) \otimes^h J_i$ for some ideals $I_i$ and $J_i$ of $M$ and $B$ respectively. Thus, by Lemma 4,

$$T_1 = i^{-1}\Big(\sum_{i=1}^{n} \mathscr{A}(I_i) \otimes^h J_i\Big) = \sum_{i=1}^{n} I_i \otimes^h J_i. qed$$

REMARK 1. In ([11], Example 4.22), all $\varepsilon$-ideals of $B(\mathscr{H}, \mathscr{K}) \otimes^h B(\mathscr{L})$, where $\mathscr{H}$, $\mathscr{K}$ and $\mathscr{L}$ are infinite dimensional separable Hilbert spaces were classified. The previous corollary gives an elementary proof of the same classification.

Since $\|x\|_{\text{tmax}} \leqslant \|x\|_h$ for all $x \in M \otimes B$, so there is a contractive map $\varepsilon' : M \otimes^h B \to M \otimes^{\text{tmax}} B$ such that $\varepsilon'(a \otimes b) = a \otimes b$ for all $a \in M$ and $b \in B$. The map $\varepsilon$ has a natural factorization through $M \otimes^{\text{tmax}} B$ so using ([11], Proposition 4.9), we have

LEMMA 5. *The contractive map $\varepsilon' : M \otimes^h B \to M \otimes^{tmax} B$ is injective.*

Let $A$ and $B$ be $C^*$-algebras then it is known that there is a one-to-one correspondence between representations of $A \otimes^{\max} B$ and $*$-representations of $A \otimes^h B$. Motivated by this, we define the following.

DEFINITION 5. A linear map $\pi : M \otimes^h B \to B(\mathscr{H}, \mathscr{K})$ is called a representation of $M \otimes^h B$ if there exists a representation $\rho$ of $M \otimes^{\text{tmax}} B$ such that $\pi = \rho\varepsilon'$. $\pi$ is called irreducible if $\rho$ is irreducible.

LEMMA 6. *Let $\pi$ be a nonzero representation of $M \otimes^h B$ then* $\ker(\pi)$ *contains a nonzero elementary tensor.*

*Proof.* We have $\pi = \rho\varepsilon'$, where $\rho : M \otimes^{\text{tmax}} B \to B(\mathscr{H}, \mathscr{K})$ is a representation of $M \otimes^{\text{tmax}} B$. Since $\rho$ is a representation of $M \otimes^{\text{tmax}} B$ so $\mathscr{A}(\rho)$ is a representation of $\mathscr{A}(M) \otimes^{\max} B$ ([10], Proposition 2.1). Consider the commutative diagram,

$$\begin{array}{ccc} M \otimes^h B & \xrightarrow{\varepsilon'} & M \otimes^{\text{tmax}} B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j'} \\ \mathscr{A}(M) \otimes^h B & \xrightarrow[\tilde{\varepsilon'}]{} & \mathscr{A}(M) \otimes^{\max} B \end{array}$$

$\tilde{\pi} = \mathscr{A}(\rho)\tilde{\varepsilon'}$ is a representation of $\mathscr{A}(M) \otimes^h B$. First note that $\ker(\tilde{\pi}) \neq (0)$. For this let $x \in \ker(\pi)$, $x \neq 0$ so $\pi(x) = 0$. Since all maps in the diagram are injective so $\tilde{\varepsilon'}i(x) \neq 0$ and $\tilde{\pi}i(x) = \mathscr{A}(\rho)\tilde{\varepsilon'}i(x) = \mathscr{A}(\rho)j'\varepsilon'(x) = 0$. So $i(x) \in \ker(\tilde{\pi})$ and $i(x) \neq 0$. Thus $\ker(\tilde{\pi})$, is a nonzero closed ideal of $\mathscr{A}(M) \otimes^h B$. By ([1], Proposition 4.5), $\ker(\tilde{\pi})$ must contain a nonzero elementary tensor say $\begin{bmatrix} p & q \\ r & s \end{bmatrix} \otimes b \in \ker(\tilde{\pi})$. As

$$\mathscr{A}(\rho)\left(\begin{bmatrix} p & q \\ r & s \end{bmatrix} \otimes b\right) = \mathscr{A}(\rho)\tilde{\varepsilon'}\left(\begin{bmatrix} p & q \\ r & s \end{bmatrix} \otimes b\right) = 0.$$

So $\ker(\mathscr{A}(\rho))$ contains a nonzero elementary tensor. Thus $b \neq 0$. Now we claim that we can assume $q \neq 0$. If $q = 0$ and $r \neq 0$ then as $\ker(\mathscr{A}(\rho))$ is an ideal of the $C^*$-algebra $\mathscr{A}(M) \otimes^{\max} B$ so $\ker(\mathscr{A}(\rho))$ is a $*$-ideal, hence $\begin{bmatrix} p \otimes b & r \otimes b \\ 0 & s \otimes b \end{bmatrix} \in \ker(\mathscr{A}(\rho))$. Now if $q = 0$, $r = 0$ and $p \neq 0$ so there is $m \in M$ such that $pm \neq 0$. Consider

$$\left(\begin{bmatrix} p & 0 \\ 0 & s \end{bmatrix} \otimes b\right)\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \otimes b\right) \in \ker(\mathscr{A}(\rho))$$

which gives $\begin{bmatrix} 0 & pm \otimes b^2 \\ 0 & 0 \end{bmatrix} \in \ker(\mathscr{A}(\rho))$.

Thus we may assume $q \otimes b \neq 0$ and $\begin{bmatrix} p \otimes b & q \otimes b \\ r \otimes b & s \otimes b \end{bmatrix} \in \ker(\mathscr{A}(\rho)) = \mathscr{A}(\ker(\rho))$ ([10], Lemma 2.7). So $0 \neq q \otimes b \in \ker(\rho)$. Thus $\rho(q \otimes b) = 0$. Note that $\pi(q \otimes b) = \rho\varepsilon'(q \otimes b) = \rho(q \otimes b) = 0$ and $q \otimes b \neq 0$. Thus $\ker(\pi)$ contains a nonzero elementary tensor. □

LEMMA 7. *If $\pi : M \otimes^{tmin} B \to B(\mathscr{H}, \mathscr{K})$ is a representation, then there exist commuting representations $\pi_1 : M \to B(\mathscr{H} \oplus \mathscr{K})$ and $\pi_2 : B \to B(\mathscr{H} \oplus \mathscr{K})$ such that for all $a \in M$ and $b \in B$ we have*

$$\pi(a \otimes b) = \pi_1(a)\pi_2(b) = \pi_2(b)\pi_1(a).$$

*In particular, if $\pi$ is irreducible then $\pi_1$ and $\pi_2$ are factor representations in the sense that if $\mathscr{M}$ is a von Neumann algebra generated by $\{\pi_1(a) : a \in M\}$ then $\mathscr{M}$ is a factor i.e. center of $\mathscr{M}$ is $\mathbb{C}(I_{\mathscr{H}} \oplus I_{\mathscr{K}})$.*

*Proof.* Existence of $\pi_1$ and $\pi_2$ follows from ([20], Lemma IV.4.1), ([10], Proposition 2.1) and ([12], Proposition 3.1). If $\pi$ is irreducible, let $\pi_1(y_0)$ be in the center of $\mathscr{M}$. It can be shown that $\pi_1(y_0)\pi(x\otimes y) = \pi(x\otimes y)\pi_1(y_0)$ for all $x \in M$ and $y \in B$. So $\pi_1(y_0)$ is in the commutant of von Neumann generated by $\pi(x\otimes y)$ for $x \in M$, $y \in B$ which is same as the commutant of von Neumann algebra generated by $\mathscr{A}(\pi)(\mathscr{A}(M)\otimes^{\min} B)$ ([2], Lemma 4.4$(b)$). Since $\mathscr{A}(\pi)$ is irreducible, so the last commutant is equal to $\mathbb{C}(I_{\mathscr{H}} \oplus I_{\mathscr{K}})$. □

The next result gives the complete description of maximal $\varepsilon$-ideals of $M\otimes^h B$ in terms of maximal ideals of $M$ and $B$. The result generalizes ([1], Theorem 5.6).

THEOREM 2. *Let $P$ be a maximal $\varepsilon$-ideal of $M\otimes^h B$ then there exist maximal ideals $U_1$ and $U_2$ of $M$ and $B$ respectively such that*

$$P = U_1\otimes^h B + M\otimes^h U_2$$

*Proof.* Let $P$ be a maximal $\varepsilon$-ideal of $M\otimes^h B$, so there exists a proper ideal $Q$ of $M\otimes^{\mathrm{tmin}} B$ such that $P = \varepsilon^{-1}(Q)$. Let $\pi_0 : M\otimes^{\mathrm{tmin}} B \to \frac{M\otimes^{\mathrm{tmin}} B}{Q} = M_0$ be the quotient map. $M_0$ is a TRO so it admits an irreducible representation $\tilde{\pi} : M_0 \to B(\mathscr{H},\mathscr{K})$ corresponding to an irreducible representation of the $C^*$-algebra $\mathscr{A}(M_0)$ ([10], Proposition 2.1,2.2). Let $\pi = \tilde{\pi}\pi_0$. Then $\pi$ is an irreducible representation of $M\otimes^{\mathrm{tmin}} B$ annihilating $Q$. By above lemma, there exist representations $\pi_1$ of $M$ and $\pi_2$ of $B$ on $B(\mathscr{H}\oplus\mathscr{K})$ such that for all $a \in M$ and $b \in B$ we have

$$\pi(a\otimes b) = \pi_1(a)\pi_2(b) = \pi_2(b)\pi_1(a).$$

Define

$$U = U_1\otimes^h B + M\otimes^h U_2,$$

and

$$\tilde{U} = U_1\otimes^{\mathrm{tmin}} B + M\otimes^{\mathrm{tmin}} U_2,$$

where $U_1 = \ker(\pi_1)$ and $U_2 = \ker(\pi_2)$. First we claim that $P = U$. Note that if $a\otimes b \in U_1\otimes B$ then $\pi(a\otimes b) = \pi_1(a)\pi_2(b) = 0$ which implies $U_1\otimes^{\mathrm{tmin}} B \subseteq \ker(\pi)$, as $\ker(\pi)$ is closed. Similarly, $M\otimes^{\mathrm{tmin}} U_2 \subseteq \ker(\pi)$ which gives $\pi(\tilde{U}) = 0$ so $\pi(Q+\tilde{U}) = 0$. Thus

$$\pi\varepsilon(\varepsilon^{-1}(Q+\tilde{U})) \subseteq \pi(Q+\tilde{U}) = 0$$

so $\varepsilon^{-1}(Q+\tilde{U})$ is proper and $P \subseteq \varepsilon^{-1}(Q+\tilde{U})$, hence $P = \varepsilon^{-1}(Q+\tilde{U})$. Since $U = \varepsilon^{-1}(\tilde{U})$, so $U \subseteq P$. Let $q : M\otimes^h B \to \frac{M}{U_1}\otimes^h \frac{B}{U_2}$ be the quotient map with kernel $U$ ([1], Corollary 2.6). Note that the representations $\pi_1$ and $\pi_2$ induce faithful factor representations $\tilde{\pi}_1$ of $\frac{M}{U_1}$ and $\tilde{\pi}_2$ of $\frac{B}{U_2}$ on $\mathscr{H}\oplus\mathscr{K}$. Moreover, as $\pi_1$ and $\pi_2$ are commuting, so $\tilde{\pi}_1$ and $\tilde{\pi}_2$ are also commuting. The linear map $\tilde{\pi}_1.\tilde{\pi}_2 : \frac{M}{U_1} \otimes \frac{B}{U_2} \to B(\mathscr{H} \oplus \mathscr{K})$ preserves the ternary product. Also, for $x \in \frac{M}{U_1} \otimes \frac{B}{U_2}$ note that

$$\|\tilde{\pi}_1.\tilde{\pi}_2(x)\| \leqslant \|x\|_{\mathrm{tmax}}.$$

Thus, $\tilde{\pi}_1.\tilde{\pi}_2$ is a contractive map, so extends to a contraction from $\frac{M}{U_1} \otimes^{\mathrm{tmax}} \frac{B}{U_2} \to B(\mathscr{H} \oplus \mathscr{K})$. Since the Haagerup norm dominates the tmax norm ([11], Proposition 3.2), so there is an induced representation $\tilde{\pi}_1.\tilde{\pi}_2$ of $\frac{M}{U_1} \otimes^h \frac{B}{U_2}$ into $B(\mathscr{H} \oplus \mathscr{K})$. Consider the following commutative diagram

$$\begin{array}{ccc} M \otimes^h B & \xrightarrow{\quad q \quad} & \frac{M}{U_1} \otimes^h \frac{B}{U_2} \\ \Big\downarrow \varepsilon & & \Big\downarrow \tilde{\pi}_1.\tilde{\pi}_2 \\ M \otimes^{\mathrm{tmin}} B & \xrightarrow[\pi]{} & B(\mathscr{H} \oplus \mathscr{K}) \end{array}$$

so $\tilde{\pi}_1.\tilde{\pi}_2(q(P)) = 0$. Now we claim that $\tilde{\pi}_1.\tilde{\pi}_2$ is a faithful representation. Since $\tilde{\pi}_1$ and $\tilde{\pi}_2$ are faithful factor representations so by using ([20], Proposition IV.4.20), $\tilde{\pi}_1.\tilde{\pi}_2$ is faithful on the algebraic tensor product $\frac{M}{U_1} \otimes \frac{B}{U_2}$. If $\ker(\tilde{\pi}_1.\tilde{\pi}_2)$ were nonzero, then by Lemma 6, $\ker(\tilde{\pi}_1.\tilde{\pi}_2)$ would contain a nonzero elementary tensor, say $\overline{a} \otimes \overline{b}$. Thus $\tilde{\pi}_1.\tilde{\pi}_2(\overline{a} \otimes \overline{b}) = 0$, so $\tilde{\pi}_1.\tilde{\pi}_2$ would not be faithful on $\frac{M}{U_1} \otimes \frac{B}{U_2}$. Therefore, $\tilde{\pi}_1.\tilde{\pi}_2$ is a faithful representation i.e. $q(P) = 0$. Thus, $P \subseteq \ker(q) = U$, which establishes the equality. To show $U_1$ and $U_2$ are maximal, observe that

$$\frac{M \otimes^h B}{U} = \frac{M}{U_1} \otimes^h \frac{B}{U_2}$$

Since $U$ is maximal ideal, therefore $\frac{M \otimes^h B}{U}$ is simple, so by ([11], Proposition 4.16), $\frac{M}{U_1}$ and $\frac{B}{U_2}$ are simple which implies $U_1$ and $U_2$ are maximal ideals of $M$ and $B$ respectively. □

REMARK 2. For separable Hilbert spaces $\mathscr{H}$, $\mathscr{K}$ and $\mathscr{L}$, the only maximal ideal of $B(\mathscr{H},\mathscr{K}) \otimes^h B(\mathscr{L})$ is $B(\mathscr{H},\mathscr{K}) \otimes^h K(\mathscr{L}) + K(\mathscr{H},\mathscr{K}) \otimes^h B(\mathscr{L})$.

## 5. Prime ideals of $M \otimes^h B$

In this section our aim is to give a complete classification of prime ideals of $M \otimes^h B$. We first define prime ideals of $M \otimes^h B$.

DEFINITION 6. An $\varepsilon$-ideal $P$ of $M \otimes^h B$ is called a prime ideal if for any pair $I$ and $J$ of $\varepsilon$-ideals satisfying $I \cap J \subseteq P$, either $I \subseteq P$ or $J \subseteq P$.

DEFINITION 7. A closed subspace $P$ of $M \otimes^h B$ is called an $i$-prime ideal ($\varepsilon$-prime ideal) if $P = i^{-1}(Q)$ ($P = \varepsilon^{-1}(Q)$), for some prime ideal $Q$ of $\mathscr{A}(M) \otimes^h B$ ($M \otimes^{\mathrm{tmin}} B$).

REMARK 3. Let $P$ be an $i$-prime ideal of $M \otimes^h B$ ($P = i^{-1}(Q)$). Suppose $P_1$ and $P_2$ be $\varepsilon$-ideals of $M \otimes^h B$ satisfying $P_1 \cap P_2 \subseteq P$, then as $i$ is injective so

$$i(P_1) \cap i(P_2) \subseteq i(P_1 \cap P_2) \subseteq i(P) \subseteq Q$$

which gives $i(P_1) \subseteq Q$ or $i(P_2) \subseteq Q$ so $P_1 = i^{-1}(i(P_1)) \subseteq P$ or $P_2 = i^{-1}(i(P_2)) \subseteq P$. Thus every $i$-prime ideal of $M \otimes^h B$ is a prime ideal.

REMARK 4. Let $P$ be an $\varepsilon$-prime ideal of $M \otimes^h B$ i.e. $P = \varepsilon^{-1}(Q)$ for some prime ideal $Q$ of $M \otimes^{\text{tmin}} B$. Let $\tilde{Q} = \tilde{\varepsilon}^{-1}(j(Q))$, then $j(Q)$ is a prime ideal of $\mathscr{A}(M) \otimes^{\min} B$. Since $\tilde{\varepsilon}$ is an injective homomorphism and the range of $\tilde{\varepsilon}$ is dense in $\mathscr{A}(M) \otimes^{\min} B$ so $\tilde{Q}$ is a prime ideal of $\mathscr{A}(M) \otimes^h B$. We will show that $P = i^{-1}(\tilde{Q})$. Suppose $x \in P$ then $\varepsilon(x) \in Q$ so $\tilde{\varepsilon} i(x) = j(\varepsilon(x)) \in j(Q)$. Thus, $x \in i^{-1}\tilde{\varepsilon}^{-1}(j(Q)) = i^{-1}(\tilde{Q})$. Conversely, let $x \in i^{-1}(\tilde{Q})$, then $i(x) \in \tilde{Q} = \tilde{\varepsilon}^{-1}(j(Q))$ so by commutativity of the first diagram in Section 4, $i(x) \in \tilde{\varepsilon}^{-1}(j(Q)) = i\varepsilon^{-1}(Q)$ so $x \in P$. This shows that any $\varepsilon$-prime ideal of $M \otimes^h B$ is an $i$-prime ideal.

PROPOSITION 5. *If $P$ is $i$-prime ideal of $M \otimes^h B$ then $P = M \otimes^h J + I \otimes^h B$ for some prime ideals $I$ and $J$ of $M$ and $B$.*

*Proof.* Let $P$ be an $i$-prime ideal of $M \otimes^h B$ so $P = i^{-1}(Q)$, for some prime ideal $Q$ of $\mathscr{A}(M) \otimes^h B$. By ([1], Theorem 5.9) and Proposition 4, $Q = \mathscr{A}(I) \otimes^h B + \mathscr{A}(M) \otimes^h J$, for some prime ideals $I$ and $J$ of $M$ and $B$ respectively. Thus, by Lemma 4, $P = i^{-1}(\mathscr{A}(I) \otimes^h B + \mathscr{A}(M) \otimes^h J) = M \otimes^h J + I \otimes^h B$. □

Our next goal is to provide a complete characterization of the prime ideals of $M \otimes^h B$. Let $I$ and $J$ be ideals of $M$ and $B$ respectively. Furthermore, let $\pi : M \to M/I$ and $\rho : B \to B/J$ be the quotient maps. Then $\pi \otimes^{\text{tmin}} \rho : M \otimes^{\text{tmin}} B \to M/I \otimes^{\text{tmin}} B/J$ is a ternary homomorphism. To achieve our goal, we need to know the $\text{Ker}(\pi \otimes^{\text{tmin}} \rho)$. We begin with the case where $M$ and $B$ both are $C^*$-algebras. Let $A$ and $B$ be $C^*$-algebras with $I$ and $J$ ideals of $A$ and $B$ respectively. Then $\pi \otimes \rho : A \otimes^{\min} B \to A/I \otimes^{\min} B/J$ is a $C^*$-homomorphism. The following result establishes a useful formula for the $\ker(\pi \otimes \rho)$. This might be known, but we are unable to find a reference so including a proof for the convenience of the reader.

LEMMA 8. *If $A$ or $B$ is an exact $C^*$-algebra, then*

$$\ker(\pi \otimes \rho) = I \otimes^{min} B + A \otimes^{min} J.$$

*Proof.* Let $K = \ker(\pi \otimes \rho)$ and $K_0 = I \otimes^{\min} B + A \otimes^{\min} J$. Since $K$ is closed, so it is obvious that $K_0 \subseteq K$. Let $K_1 \otimes^{\min} K_2 \subseteq K$, where $K_1$ and $K_2$ are closed ideals of $A$ and $B$ respectively and let $a \otimes b \in K_1 \otimes K_2$. Since $K_0$ is closure of the sum of ideals generated by elementary tensors $a \otimes b \in K$ ([4], Lemma 2.12), so $\langle a \otimes b \rangle \subseteq K_0$. This in turn implies $K_1 \otimes K_2 \subseteq K_0$ and therefore $K_1 \otimes^{\min} K_2 \subseteq K_0$, as $K_0$ is closed. By ([4], Proposition 2.16, Proposition 2.17), $K$ is the closure of the sum of all elementary ideals $K_1 \otimes^{\min} K_2 \subseteq K$, where $K_1 \subseteq A$ and $K_2 \subseteq B$ are closed ideals and since $K_1 \otimes^{\min} K_2 \subseteq K_0$ for every elementary ideal $K_1 \otimes^{\min} K_2$ of $K$, so $K \subseteq K_0$. □

LEMMA 9. *If $M$ or $B$ is exact, then*

$$\ker(\pi \otimes^{tmin} \rho) = M \otimes^{tmin} J + I \otimes^{tmin} B$$

*Proof.* As $\pi \otimes^{\text{tmin}} \rho : M \otimes^{\text{tmin}} B \to M/I \otimes^{\text{tmin}} B/J$ is a ternary homomorphism, so applying the functor $\mathscr{A}$ and using ([11], Proposition 4.6) $\mathscr{A}(\pi \otimes^{\text{tmin}} \rho) : \mathscr{A}(M) \otimes^{\text{tmin}} B \to \mathscr{A}(M)/\mathscr{A}(I) \otimes^{\text{tmin}} B/J$ is a $C^*$-homomorphism. Thus, using above lemma, we have

$$\ker(\mathscr{A}(\pi \otimes^{\text{tmin}} \rho)) = \mathscr{A}(M) \otimes^{\text{min}} J + \mathscr{A}(I) \otimes^{\text{min}} B = \mathscr{A}(M \otimes^{\text{tmin}} J + I \otimes^{\text{tmin}} B)$$

Since $\ker(\mathscr{A}(\pi \otimes^{\text{tmin}} \rho)) = \mathscr{A}(\ker(\pi \otimes^{\text{tmin}} \rho))$ ([10], Lemma 2.7), so $\mathscr{A}(\ker(\pi \otimes^{\text{tmin}} \rho)) = \mathscr{A}(M \otimes^{\text{tmin}} J + I \otimes^{\text{tmin}} B)$. Thus, $\ker(\pi \otimes^{\text{tmin}} \rho) = M \otimes^{\text{tmin}} J + I \otimes^{\text{tmin}} B$. □

THEOREM 3. *(a) Let $I$ and $J$ be prime ideals of $M$ and $B$ respectively, then $M \otimes^h J + I \otimes^h B$ is a prime ideal ($\varepsilon$-prime ideal) of $M \otimes^h B$.*

*(b) Conversely, if $P$ is prime ideal of $M \otimes^h B$ and $M$ or $B$ is exact, then $P = M \otimes^h J + I \otimes^h B$ for some prime ideals $I$ and $J$ of $M$ and $B$ respectively.*

*Proof.*

(a) Note that $I \otimes^{\text{tmin}} B + M \otimes^{\text{tmin}} J$ is a prime ideal of $M \otimes^{\text{tmin}} B$ by Corollary 3. But $\varepsilon^{-1}(I \otimes^{\text{tmin}} B + M \otimes^{\text{tmin}} J) = I \otimes^h B + M \otimes^h J$. Thus $I \otimes^h B + M \otimes^h J$ is an $\varepsilon$-prime ideal of $M \otimes^h B$ so is a prime ideal by Remarks 3 and 4.

(b) Suppose that $P$ is a prime ideal in $M \otimes^h B$, say $P = \varepsilon^{-1}(Q)$, for some ideal $Q$ of $M \otimes^{\text{tmin}} B$. Without loss of generality, we may assume that $P$ is proper. By Zorn's lemma, choose an ideal $I$ not equal to $M$ of $M$ which is maximal with respect to the properties that $I \otimes^h B \subseteq P$ and $I \otimes^{\text{tmin}} B \subseteq Q$. Again choose an ideal $J$ not equal to $B$ of $B$ which is maximal with respect to the properties that $M \otimes^h J \subseteq P$ and $M \otimes^{\text{tmin}} J \subseteq Q$. Then $\tilde{Q} = (\pi \otimes^{\text{tmin}} \rho)(Q)$ is a closed ideal in $(M/I) \otimes^{\text{tmin}} (B/J)$ (Proposition 5 and ([8], Corollary 4.8)). The map $\pi \otimes \rho : M \otimes^h B \to (M/I) \otimes^h (B/J)$ is a quotient map ([1], Theorem 2.5). Define $\tilde{P} = \pi \otimes \rho(P)$, then by ([1], Corollary 2.7), $\tilde{P}$ is a closed subspace of $M/I \otimes^h B/J$.

If $\tilde{Q}$ is nonzero then, it contains a nonzero elementary tensor, say $c \otimes d$. By definition of $\tilde{Q}$, there exists $z \in Q$, such that $\pi \otimes \rho(z) = c \otimes d$. Choose, $a \in M$ and $b \in B$ such that $\pi(a) = c$ and $\rho(b) = d$, then $\pi \otimes \rho(a \otimes b) = c \otimes d$. So, by Lemma 6, $z - (a \otimes b) \in M \otimes^{\text{tmin}} J + I \otimes^{\text{tmin}} B \subseteq Q$ and therefore $a \otimes b \in Q$. Thus, there exists $a \otimes b \in Q$ such that $\pi(a) \otimes \rho(b) \neq 0$. The element $a \otimes b \in Q$ generates a product ideal $I_1 \otimes^{\text{tmin}} J_1$ in $Q$ and therefore $I_1 \otimes^h J_1$ is a product ideal in $P$ using ([11], Proposition 4.17). Define two product ideal in $M \otimes^h B$ by $K = M \otimes^h (J + J_1)$ and $L = (I + I_1) \otimes^h B$. Then,

$$K \cap L = (I + I_1) \otimes^h (J + J_1) = I \otimes^h J + I \otimes^h J_1 + I_1 \otimes^h J + I_1 \otimes^h J_1 \subseteq P.$$

But then as $P$ is prime, so $K \subseteq P$ or $L \subseteq P$. The choice of ideals $I$ and $J$ then implies $I_1 \subseteq I$ or $J_1 \subseteq J$. Thus, either $\pi(a) = 0$ or $\rho(b) = 0$, which is a contradiction, as $\pi(a) \otimes \rho(b) \neq 0$. This shows that $\tilde{Q} = 0$.

Next, we show that $\tilde{P} = 0$. Observe that the following diagram

$$\begin{array}{ccc} M \otimes^h B & \xrightarrow{\ \varepsilon\ } & M \otimes^{\text{tmin}} B \\ {\scriptstyle \pi\otimes\rho}\downarrow & & \downarrow{\scriptstyle \pi\otimes^{\text{tmin}}\rho} \\ M/I \otimes^h B/J & \xrightarrow[\varepsilon_1]{} & M/I \otimes^{\text{tmin}} B/J \end{array}$$

is commutative i.e. $\varepsilon_1(\pi \otimes \rho) = (\pi \otimes^{\text{tmin}} \rho)\varepsilon$. Moreover,

$$\varepsilon_1(\tilde{P}) = \varepsilon_1(\pi \otimes \rho(P)) = (\pi \otimes^{\text{tmin}} \rho)\varepsilon(P) \subseteq (\pi \otimes^{\text{tmin}} B)(Q) = 0,$$

which implies $\varepsilon_1(\tilde{P}) = 0$ and therefore $\tilde{P} = 0$ as $\varepsilon_1$ is injective. Thus, $P \subseteq \text{Ker}(\pi \otimes \rho) = M \otimes^h J + I \otimes^h B$, using ([1], Corollary 2.6). But $M \otimes^h J + I \otimes^h B \subseteq P$ (by choice of $I$ and $J$).

We now show that the ideals $I$ and $J$ must be prime. If $I_1$ and $I_2$ are closed ideals in $M$ such that $I_1 \cap I_2 \subseteq I$, then by ([19], Corollary 4.6),

$$(I_1 \otimes^h B) \cap (I_2 \otimes^h B) = (I_1 \cap I_2) \otimes^h B \subseteq P.$$

By hypothesis, $P$ contains either $I_1 \otimes^h B$ or $I_2 \otimes^h B$, and assume without loss of generality that $I_1 \otimes^h B \subseteq P$. Let $\phi$ be an arbitrary element of the annihilator $I^\perp \subseteq M^*$ of $I$, and choose a non-zero element $\psi \in J^\perp \subseteq B^*$. Then by ([7], Proposition 9.2.5), $\phi \otimes \psi \in (M \otimes^h B)^*$ and annihilates $P$ and so must annihilate $I_1 \otimes^h B$. Since $\phi \in I^\perp$ was arbitrary this forces $I_1 \subseteq I$, proving that $I$ is prime. A similar argument shows that $J$ is also prime. □

The next corollary which is a simple consequence of our results establishes a relationship of prime ideals, $i$-prime ideals, and $\varepsilon$-prime ideals of $M \otimes^h B$.

COROLLARY 5. *Let $P$ be a closed subspace of $M \otimes^h B$. If $M$ or $B$ is exact, then $P$ is a prime ideal if and only if $P$ is $\varepsilon$-prime ideal ($i$-prime ideal).*

*Proof.* In view of Remarks 3 and 4, we only need to show that every prime ideal is a $\varepsilon$-prime ideal. Suppose $P$ is prime ideal of $M \otimes^h B$, then by Theorem 3, $P = M \otimes^h J + I \otimes^h B$ for some prime ideals $I$ and $J$ of $M$ and $B$. By ([1], Theorem 5.9), $\mathscr{A}(I) \otimes^h B + \mathscr{A}(M) \otimes^h J$ is prime ideal of $\mathscr{A}(M) \otimes^h B$ and $P = \varepsilon^{-1}(\mathscr{A}(I) \otimes^h B + \mathscr{A}(M) \otimes^h J)$ ([11], Proposition 4.17). Thus, $P$ is a $\varepsilon$-prime ideal. □

## 6. Primitive ideals of $M \otimes^h B$

In this section we are going to study primitive ideals in the Haagerup tensor product of $M$ and $B$. As in ([3], Page 4), we have

LEMMA 10. *If $I_1$ and $I_2$ are proper closed ideals in $M$ and $J_1$ and $J_2$ proper closed ideals of $B$ then $M \otimes^h J_1 + I_1 \otimes^h B = M \otimes^h J_2 + I_2 \otimes^h B$ if and only if $I_1 = I_2$ and $J_1 = J_2$.*

DEFINITION 8. A closed subspace $P$ of $M \otimes^h B$ will be called a primitive ideal of $M \otimes^h B$ if $P = \ker(\pi)$, for some irreducible representation $\pi$ of $M \otimes^h B$.

THEOREM 4. *(a) If $I$ and $J$ are primitive ideals of $M$ and $B$ respectively then $I \otimes^h B + M \otimes^h J$ is a primitive ideal of $M \otimes^h B$.*

*(b) If $P$ is a primitive ideal of $M \otimes^h B$ then there exists prime ideals $I$ and $J$ of $M$ and $B$ such that $P = I \otimes^h B + M \otimes^h J$.*

*(c) If $P$ is primitive ideal of $M \otimes^h B$ and $M$ and $B$ are separable then there exists primitive ideals $I$ and $J$ of $M$ and $B$ such that $P = I \otimes^h B + M \otimes^h J$.*

*(d) Let $I$ be a closed ideal of $M$ then $I \otimes^h B$ is a primitive ideal of $M \otimes^h B$ if and only if $\mathscr{A}(I) \otimes^h B$ is a primitive ideal of $\mathscr{A}(M) \otimes^h B$.*

*Proof.*

(a) Let $I$ and $J$ be primitive ideals of $M$ and $B$ then by ([10], Theorem 2.6), $\mathscr{A}(I)$ is primitive ideal of $\mathscr{A}(M)$. Thus $\tilde{P} = \mathscr{A}(I) \otimes^h B + \mathscr{A}(M) \otimes^h J$ is primitive ideal of $\mathscr{A}(M) \otimes^h B$ ([1], Theorem 5.13) and therefore $\tilde{P} = \ker(\psi)$ for some irreducible $*$-representation $\psi$ of $\mathscr{A}(M) \otimes^h B$. Using Proposition 1, let $\psi = \mathscr{A}(\rho)\tilde{\varepsilon}'$, where $\rho$ is an irreducible representation of $M \otimes^{\mathrm{tmax}} B$. Define, $\pi = \rho\varepsilon'$, then $\pi$ is an irreducible representation of $M \otimes^h B$. We claim that $\ker(\pi) = I \otimes^h B + M \otimes^h J$. Suppose $a \otimes b \in I \otimes B$, then

$$\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \otimes b \in \mathscr{A}(I) \otimes B \subseteq \ker(\psi) = \ker(\mathscr{A}(\rho)\tilde{\varepsilon}').$$

Thus, $\rho\varepsilon(a \otimes b) = 0$, so $a \otimes b \in \ker(\pi)$. Since $\ker(\pi)$ is closed, so $I \otimes^h B \subseteq \ker(\pi)$. Similarly, $M \otimes^h J \subseteq \ker(\pi)$. Thus, $M \otimes^h J + I \otimes^h B \subseteq \ker(\pi)$. Conversely, let $x \in \ker(\pi)$. Note that the following diagram

$$\begin{array}{ccc} M \otimes^h B & \xrightarrow{\varepsilon'} & M \otimes^{\mathrm{tmax}} B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j'} \\ \mathscr{A}(M) \otimes^h B & \xrightarrow[\tilde{\varepsilon}']{} & \mathscr{A}(M) \otimes^{\max} B \end{array}$$

is commutative i.e. $j'\varepsilon' = \tilde{\varepsilon}' i$. Then as

$$\psi(i(x)) = \mathscr{A}(\rho)\tilde{\varepsilon}'(i(x)) = \mathscr{A}(\rho)(j'(\varepsilon'(x))$$
$$= \mathscr{A}(\rho)\Big(\begin{bmatrix}0 & \varepsilon'(x)\\ 0 & 0\end{bmatrix}\Big) = \begin{bmatrix}0 & \rho(\varepsilon'(x))\\ 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}.$$

So $i(x) \in \ker(\psi) = \mathscr{A}(I)\otimes^h B + \mathscr{A}(M)\otimes^h J$ and therefore $x \in i^{-1}(\mathscr{A}(I)\otimes^h B + \mathscr{A}(M)\otimes^h J) = I\otimes^h B + M\otimes^h J$ by Lemma 4.

(b) Let $P$ be a primitive ideal of $M\otimes^h B$, so $P = \ker(\pi)$ where $\pi = \rho\varepsilon'$ and $\rho$ is an irreducible representation of $M\otimes^{\text{tmax}} B$. Define $\psi = \mathscr{A}(\rho)\tilde{\varepsilon'}$, then $\ker(\psi)$ is a primitive ideal of $\mathscr{A}(M)\otimes^h B$. Thus, by ([1], Theorem 5.13) and Proposition 3, $\ker(\psi) = \mathscr{A}(I)\otimes^h B + \mathscr{A}(M)\otimes^h J$ for some prime ideals $I$ and $J$. By the same argument as in part $(a)$, it is not difficult to see that $\ker(\pi) = I\otimes^h B + M\otimes^h J$.

(c) Follows immediately from Corollary 1 and (b).

(d) Let $I\otimes^h B = \ker(\pi)$, $\pi = \rho\varepsilon'$ and $\rho$ is an irreducible representation of $M\otimes^{\text{tmax}} B$. Let $\psi = \mathscr{A}(\rho)\tilde{\varepsilon}'$, so $\ker(\psi)$ is a primitive ideal of $\mathscr{A}(M)\otimes^h B$. Using ([1], Proposition 2.5) and Proposition 3, it follows that there exist prime ideals $I_1$ and $I_2$ such that $\ker(\psi) = \mathscr{A}(I_1)\otimes^h B + \mathscr{A}(M)\otimes^h I_2$. As in part $(a)$, we can show that $\ker(\pi) = I_1\otimes^h B + M\otimes^h I_2$. Hence by Lemma 10, $I = I_1$ and $I_2 = \{0\}$. Thus, $\ker(\psi) = \mathscr{A}(I)\otimes^h B$. The converse can be proved as in $(a)$. □

An immediate consequence of our results is the following:

COROLLARY 6. *Every maximal ideal of $M\otimes^h B$ is primitive, and every primitive ideal is prime ideal.*

EXAMPLE 4. Let $\mathscr{H}$, $\mathscr{K}$ and $\mathscr{L}$ be infinite dimensional separable Hilbert spaces. It is easy to see that $K(\mathscr{H},\mathscr{K})\otimes^h K(\mathscr{L})$ is not a prime ideal of $B(\mathscr{H},\mathscr{K})\otimes^h B(\mathscr{L})$, and hence not primitive by Corollary 6. Moreover, all other non trivial $\varepsilon$-ideal of $B(\mathscr{H},\mathscr{K})\otimes^h B(\mathscr{L})$ i.e. $B(\mathscr{H},\mathscr{K})\otimes^h K(\mathscr{L})$, $K(\mathscr{H},\mathscr{K})\otimes^h B(\mathscr{L})$, and $B(\mathscr{H},\mathscr{K})\otimes^h K(\mathscr{L}) + K(\mathscr{H},\mathscr{K})\otimes^h B(\mathscr{L})$ are prime ideals by Theorem 3. $B(\mathscr{H},\mathscr{K})\otimes^h K(\mathscr{L}) + K(\mathscr{H},\mathscr{K})\otimes^h B(\mathscr{L})$ is primitive using Corollary 6. By definition of a primitive ideal, one can show that $B(\mathscr{H})\otimes^h K(\mathscr{L})$ and $K(\mathscr{H})\otimes^h B(\mathscr{L})$ are not primitive ideals of $B(\mathscr{H})\otimes^h B(\mathscr{L})$. Using Theorem 4$(d)$, it follows that $B(\mathscr{H},\mathscr{K})\otimes^h K(\mathscr{L})$ and $K(\mathscr{H},\mathscr{K})\otimes^h B(\mathscr{L})$ are not primitive in $B(\mathscr{H},\mathscr{K})\otimes^h B(\mathscr{L})$.

EXAMPLE 5. Let $(\mathscr{K}_n)$ be an increasing sequence of infinite dimensional separable Hilbert spaces and $\mathscr{H}$ and $\mathscr{L}$ be any infinite dimensional separable Hilbert space. For $f \in K(\mathscr{H},\mathscr{K}_n)$, $i_n\circ f \in K(\mathscr{H},\mathscr{K}_{n+1})$ where $i_n : \mathscr{K}_n \to \mathscr{K}_{n+1}$ is inclusion. $\{K(\mathscr{H},\mathscr{K}_n),\alpha_n\}$, $\alpha_n(f) = i_n\circ f$, is an inductive system. Since $K(\mathscr{H},\mathscr{K}_n)$ is simple for all $n$, so by ([9], Corollary 2.23), the inductive limit $\varinjlim K(\mathscr{H},\mathscr{K}_n)$ is also simple. Using ([11], Proposition 4.19), it follows that only non trivial $\varepsilon$-ideal of $\varinjlim(K(\mathscr{H},\mathscr{K}_n))\otimes^h B(\mathscr{L})$ is $\varinjlim(K(\mathscr{H},\mathscr{K}_n))\otimes^h K(\mathscr{L})$. Moreover, since $K(\mathscr{L})$ is prime and maximal ideal of $B(\mathscr{L})$ and $\varinjlim K(\mathscr{H},\mathscr{K}_n)$ is exact by ([9], Corollary 2.18), so $\varinjlim(K(\mathscr{H},\mathscr{K}_n))\otimes^h K(\mathscr{L})$ is the only nontrivial maximal and prime ideal of $\varinjlim(K(\mathscr{H},\mathscr{K}_n))\otimes^h B(\mathscr{L})$.

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