Operators and Matrices
Volume 17, Number 3 (2023), 767–778
doi:10.7153/oam-2023-17-50

FURTHER REFINEMENTS OF DAVIS–WIELANDT RADIUS INEQUALITIES

PINTU BHUNIA, KALLOL PAUL AND SOMDATTA BARIK

(*Communicated by F. Kittaneh*)

Abstract. Suppose T,S are bounded linear operators on a complex Hilbert space. We show that the Davis-Wielandt radius $dw(\cdot)$ satisfies the following inequalities

$$\begin{aligned} dw(T+S) &\leqslant \sqrt{2\big(dw^2(T)+dw^2(S)\big)+6\big\||T|^4+|S|^4\big\|} \\ &\leqslant 2\sqrt{2}\sqrt{dw^2(T)+dw^2(S)} \\ &\leqslant 2\sqrt{2}\big(dw(T)+dw(S)\big). \end{aligned}$$

From the third inequality we obtain the following lower and upper bounds for the Davis-Wielandt radius $dw(T)$ of the operator T:

$$\begin{aligned} dw(T) &\geqslant \frac{1}{4\sqrt{2}}\max\big\{dw\big(2Re(T)\big), dw\big(2Im(T)\big)\big\}, \\ dw(T) &\leqslant 2\sqrt{2}\big(dw\big(Re(T)\big)+dw\big(Im(T)\big)\big). \end{aligned}$$

Further, we develop several new lower and upper bounds for the Davis-Wielandt radius of the operator T which improve the existing ones. Application of these bounds are also provided.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ induced by the inner product. Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators acting on $\mathcal{H}$. For $T\in\mathcal{B}(\mathcal{H})$, T^* denotes the adjoint of T, and $|T|=(T^*T)^{1/2}$, $|T^*|=(TT^*)^{1/2}$. The real part and the imaginary part of $T\in\mathcal{B}(\mathcal{H})$ are denoted by $Re(T)$ and $Im(T)$, respectively. Therefore, $Re(T)=(T+T^*)/2$ and $Im(T)=(T-T^*)/2i$. The operator norm of T, denoted by $\|T\|$, is defined as $\|T\|=\sup\{\|Tx\|: x\in\mathcal{H}, \|x\|=1\}$. The numerical radius of T, denoted by $w(T)$, is defined as $w(T)=\sup\{|\langle Tx,x\rangle| : x\in\mathcal{H}, \|x\|=1\}$. It is well known that $w(\cdot)$ defines a norm on $\mathcal{B}(\mathcal{H})$, and it satisfies the following inequality $w(T)\leqslant\|T\|\leqslant 2w(T)$ for every $T\in\mathcal{B}(\mathcal{H})$. For more details about the numerical radius and related inequalities, we refer the reader to see the books [3, 9,

Mathematics subject classification (2020): 47A12, 47A30, 15A60, 47A50.
Keywords and phrases: Davis-Wielandt radius, numerical radius, operator norm, inequality.
Dr. Pintu Bhunia would like to thank SERB, Govt. of India for the financial support in the form of National Post Doctoral Fellowship (N-PDF, File No. PDF/2022/000325) under the mentorship of Prof. Apoorva Khare. Miss Somdatta Barik would like to thank UGC, Govt. of India for the financial support in the form of Junior Research Fellowship under the mentorship of Prof. Kallol Paul.

Paper OaM-17-50

18] and the recent articles [8, 16]. The concept of numerical radius is useful in studying the bounded linear operators, and attracted many researchers over the years. Based on the importance of the numerical radius, many generalizations of it has been studied in the literature, see [1, 7, 14, 17]. One such generalization is the Davis-Wielandt radius, see [7, 17]. The Davis-Wielandt radius of $T \in \mathcal{B}(\mathcal{H})$, denoted by $dw(T)$, is defined as

$$dw(T) = \sup\left\{\sqrt{|\langle Tx,x\rangle|^2 + \|Tx\|^4} : x \in \mathcal{H}, \|x\| = 1\right\}.$$

Clearly, $dw(T) \geqslant 0$ and $dw(T) = 0$ if and only if $T = 0$. Note that for $\lambda \in \mathbb{C}$ and for a non-zero operator $T \in \mathcal{B}(\mathcal{H})$, we have $dw(\lambda T) = |\lambda| dw(T)$, if $|\lambda| = 1$, also $dw(\lambda T) > |\lambda| dw(T)$ if $|\lambda| > 1$ and $dw(\lambda T) < |\lambda| dw(T)$ if $|\lambda| < 1$. This implies that $dw(\cdot)$ does not define a norm on $\mathcal{B}(\mathcal{H})$. Note that the inequality $dw(T+S) \leqslant dw(T) + dw(S)$ does not always hold for arbitrary $T, S \in \mathcal{B}(\mathcal{H})$. The above triangle inequality for the Davis-Wielandt radius holds when $Re(T^*S) = 0$, see [5, Corollary 2.2]. It is not difficult to verify that the Davis-Wielandt radius $dw(\cdot)$ satisfies the following inequality:

$$\max\left\{w(T), \|T\|^2\right\} \leqslant dw(T) \leqslant \sqrt{w^2(T) + \|T\|^4} \tag{1.1}$$

The inequalities (1.1) are sharp, see [5]. The second inequality in (1.1) becomes equality, i.e., $dw(T) = \sqrt{w^2(T) + \|T\|^4}$ if and only if T is normaloid (i.e., $w(T) = \|T\|$), see [19, Corollary 3.2]. Zamani and Shebrawi [20, Theorem 2.1] proved that

$$dw(T) \leqslant \sqrt{w^2(T - |T|^2) + 2\|T\|^2 w(T)}. \tag{1.2}$$

Further in [20, Theorem 2.13, Theorem 2.14, Theorem 2.17] it is proved that

$$dw^2(T) \leqslant \max\{\|T\|^2, \|T\|^4\} + \sqrt{2} w(|T|^2 T), \tag{1.3}$$

$$dw^2(T) \leqslant \frac{1}{2}\left(w\left(|T|^4 + |T|^2\right) + w\left(|T|^4 - |T|^2\right)\right) + \sqrt{2} w(|T|^2 T) \tag{1.4}$$

and

$$dw^2(T) \leqslant \|T\| \max\{w(T), w(|T|^2)\}(1 + \|T\|^2 + 2w(T))^{\frac{1}{2}}. \tag{1.5}$$

Recently, Bhunia et al. in [5, Theorem 2.4] developed the upper bound

$$dw(T) \leqslant \sqrt{\||T|^2 + |T|^4\|}. \tag{1.6}$$

Also in [5, Theorem 2.1 (i), (ii)] they developed the lower bounds

$$dw^2(T) \geqslant \max\left\{w^2(T) + c^2(T^*T), \|T\|^4 + c^2(T)\right\} \tag{1.7}$$

and

$$dw^2(T) \geqslant 2\max\left\{w(T)c(T^*T), c(T)\|T\|^2\right\}. \tag{1.8}$$

For more results on the Davis-Wielandt radius and related inequalities we refer the readers to [2, 4, 11, 12, 13, 19].

In this paper, we study Davis-Wielandt radius inequalities for the sum of two bounded linear operators. We also develop several new lower and upper bounds for the Davis-Wielandt radius of bounded linear operators and considering the numerical examples we show that these bounds give better bounds than the existing bounds mentioned above. Applications of some inequalities obtained here are also given.

2. Main results

We begin this section by noting that the Davis-Wielandt radius does not satisfy the triangle inequality, in general. In our first theorem we prove that $dw(T+S) \leqslant 2\sqrt{2}(dw(T)+dw(S))$ holds for all $T,S \in \mathcal{B}(\mathcal{H})$. To do so we need the following lemma, known as Hölder–McCarthy inequality (see [15, p. 20]).

LEMMA 2.1. *If $T \in \mathcal{B}(\mathcal{H})$ is positive and $x \in \mathcal{H}$ with $\|x\| = 1$, then*

$$\langle Tx,x\rangle^r \leqslant \langle T^r x,x\rangle,$$

for all $r \geqslant 1$. The inequality is reversed when $0 < r \leqslant 1$.

Now, we are in a position to prove the following theorem.

THEOREM 2.2. *If $T,S \in \mathcal{B}(\mathcal{H})$, then*

$$\begin{aligned} dw(T+S) &\leqslant \sqrt{2\left(dw^2(T)+dw^2(S)\right)+6\,\||T|^4+|S|^4\|} \\ &\leqslant 2\sqrt{2}\sqrt{dw^2(T)+dw^2(S)} \\ &\leqslant 2\sqrt{2}\big(dw(T)+dw(S)\big). \end{aligned} \tag{2.1}$$

Proof. Take $x \in \mathcal{H}$ with $\|x\| = 1$. We have

$$\begin{aligned} &|\langle (T+S)x,x\rangle|^2 + \|(T+S)x\|^4 \\ &\leqslant (|\langle Tx,x\rangle| + |\langle Sx,x\rangle|)^2 + (\|Tx\| + \|Sx\|)^4 \\ &\leqslant 2(|\langle Tx,x\rangle|^2 + |\langle Sx,x\rangle|^2) + 4(\|Tx\|^2 + \|Sx\|^2)^2 \\ &\leqslant 2(|\langle Tx,x\rangle|^2 + |\langle Sx,x\rangle|^2) + 8(\|Tx\|^4 + \|Sx\|^4) \\ &\leqslant 2(|\langle Tx,x\rangle|^2 + \|Tx\|^4 + |\langle Sx,x\rangle|^2 + \|Sx\|^4) + 6\langle (|T|^4 + |S|^4)x,x\rangle \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 2.1)} \\ &\leqslant 2\left(dw^2(T)+dw^2(S)\right) + 6\left\||T|^4+|S|^4\right\|. \end{aligned}$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain

$$\begin{aligned}
dw(T+S) &\leqslant \sqrt{2\left(dw^2(T)+dw^2(S)\right)+6\,\||T|^4+|S|^4\|} \\
&\leqslant \sqrt{2\left(dw^2(T)+dw^2(S)\right)+6\left(\|T\|^4+\|S\|^4\right)} \\
&\leqslant \sqrt{8\left(dw^2(T)+dw^2(S)\right)} \text{ (by first inequality in (1.1))} \\
&\leqslant 2\sqrt{2}\big(dw(T)+dw(S)\big),
\end{aligned}$$

as desired. □

Note that the second inequality in Theorem 2.2 is sharp, i.e., the inequality $dw(T+S) \leqslant 2\sqrt{2}\sqrt{dw^2(T)+dw^2(S)}$ is sharp. If we take $\mathcal{H}$ to be an n-dimensional complex Hilbert space $\mathbb{C}^n$ and $T = S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus 0 \in \mathcal{B}(\mathcal{H})$, then by a simple computation we have $dw(T+S) = 4 = 2\sqrt{2}\sqrt{dw^2(T)+dw^2(S)}$. However, the last inequality in Theorem 2.2 is never sharp unless the operators T,S are both zero operators. A natural question that remains to be answered in this connection is "what is the best constant c $\left(2 \leqslant c < 2\sqrt{2}\right)$ available so that $dw(T+S) \leqslant c\big(dw(T)+dw(S)\big)$ holds for all bounded linear operators T and S?"

Next, by employing Theorem 2.2 we derive the following lower and upper bound for the Davis-Wielandt radius of an operator $T \in \mathcal{B}(\mathcal{H})$ in terms of $Re(T)$ and $Im(T)$.

COROLLARY 2.3. *If $T \in \mathcal{B}(\mathcal{H})$, then*

$$\frac{1}{4\sqrt{2}} \max\left\{dw(2Re(T)), dw(2Im(T))\right\} \leqslant dw(T) \leqslant 2\sqrt{2}\big(dw(Re(T))+dw(Im(T))\big).$$

Proof. The first inequality follows from Theorem 2.2 by putting $S = T^*$ and $S = -T^*$, respectively. The second inequality also follows from Theorem 2.2 by replacing T by $Re(T)$ and S by $iIm(T)$. □

Next bound for the Davis-Wielandt radius of an operator T reads as in the following theorem, proof of which follows from [4, Corollary 2.21].

THEOREM 2.4. *If $T \in \mathcal{B}(\mathcal{H})$, then*

$$dw(T) \leqslant \sqrt{\min\{\beta,\gamma,\delta,\mu\}},$$

where

$$\beta = \min_{0 \leqslant \alpha \leqslant 1} \left\{ \frac{\alpha}{2}\,\|Re(|T||T^*|)\| + \left\| \frac{\alpha}{4}|T|^2 + \left(1-\frac{3\alpha}{4}\right)|T^*|^2 + |T|^4 \right\| \right\},$$

$$\gamma = \min_{0 \leqslant \alpha \leqslant 1} \left\{ \frac{\alpha}{2}\,\|Re(|T||T^*|)\| + \left\| \frac{\alpha}{4}|T^*|^2 + \left(1-\frac{3\alpha}{4}\right)|T|^2 + |T|^4 \right\| \right\},$$

$$\delta = \min_{0\leqslant\alpha\leqslant 1}\left\{\frac{\alpha}{2}\,\|Re(|T||T^*|)\| + \left\|\frac{\alpha}{4}|T|^2 + \left(1-\frac{3\alpha}{4}\right)|T^*|^2 + |T^*|^4\right\|\right\},$$

$$\mu = \min_{0\leqslant\alpha\leqslant 1}\left\{\frac{\alpha}{2}\,\|Re(|T||T^*|)\| + \left\|\frac{\alpha}{4}|T^*|^2 + \left(1-\frac{3\alpha}{4}\right)|T|^2 + |T^*|^4\right\|\right\}.$$

REMARK 2.5. Clearly $\gamma \leqslant \left\||T|^2+|T|^4\right\|$ and so

$$\sqrt{\min\{\beta,\gamma,\delta,\mu\}} \leqslant \sqrt{\||T|^2+|T|^4\|}.$$

Therefore, the bound in Theorem 2.4 is stronger than the existing bound (1.6).

Next we need the following lemma, known as generalized Cauchy-Schwarz inequality, see [10, Theorem 1].

LEMMA 2.6. *Let $T \in \mathcal{B}(\mathcal{H})$ and let $x,y \in \mathcal{H}$. If f and g are two non-negative continuous functions on $[0,\infty)$ satisfying $f(t)g(t) = t \;\; \forall t \geqslant 0$, then*

$$|\langle Tx,y\rangle|^2 \leqslant \langle f^2(|T|)x,x\rangle\langle g^2(|T^*|)y,y\rangle.$$

In particular, $f(t) = g(t) = \sqrt{t} \;\; \forall t \geqslant 0$,

$$|\langle Tx,y\rangle|^2 \leqslant \langle |T|x,x\rangle\langle |T^*|y,y\rangle. \tag{2.2}$$

By using the above lemma we obtain the following bound for the Davis-Wielandt radius of an operator T.

THEOREM 2.7. *If $T \in \mathcal{B}(\mathcal{H})$, then*

$$dw(T) \leqslant \sqrt[4]{\|f^4(|T|)+f^4(|T|^2)\|\,\|g^4(|T^*|)+g^4(|T|^2)\|},$$

where f and g are as in Lemma 2.6.

Proof. Take $x \in \mathcal{H}$ with $\|x\| = 1$. We have

$$\begin{aligned}
&|\langle Tx,x\rangle|^2 + \|Tx\|^4 \\
&= |\langle Tx,x\rangle|^2 + \langle |T|^2x,x\rangle^2 \\
&\leqslant \langle f^2(|T|)x,x\rangle\langle g^2(|T^*|)x,x\rangle + \langle f^2(|T|^2)x,x\rangle\langle g^2(|T|^2)x,x\rangle \quad \text{(by Lemma 2.6)} \\
&\leqslant \left[\langle f^2(|T|)x,x\rangle^2 + \langle f^2(|T|^2)x,x\rangle^2\right]^{\frac12}\left[\langle g^2(|T^*|)x,x\rangle^2 + \langle g^2(|T|^2)x,x\rangle^2\right]^{\frac12} \\
&\leqslant \left[\langle f^4(|T|)x,x\rangle + \langle f^4(|T|^2)x,x\rangle\right]^{\frac12}\left[\langle g^4(|T^*|)x,x\rangle + \langle g^4(|T|^2)x,x\rangle\right]^{\frac12} \text{(by Lemma 2.1)} \\
&= \langle (f^4(|T|)+f^4(|T|^2))x,x\rangle^{\frac12}\langle (g^4(|T^*|)+g^4(|T|^2))x,x\rangle^{\frac12}.
\end{aligned}$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain

$$dw^2(T) \leqslant \left\| f^4(|T|) + f^4(|T|^2) \right\|^{\frac{1}{2}} \left\| g^4(|T^*|) + g^4(|T|^2) \right\|^{\frac{1}{2}}.$$

This completes the proof.

In particular, considering $f(t) = g(t) = \sqrt{t}$ in Theorem 2.7 we get the following corollary. $\square$

COROLLARY 2.8. *If $T \in \mathcal{B}(\mathcal{H})$, then*

$$dw(T) \leqslant \sqrt[4]{\| |T|^2 + |T|^4 \| \; \| |T^*|^2 + |T|^4 \|}.$$

Next we need the following lemma, known as Buzano's inequality (see [6]).

LEMMA 2.9. *Let $x, y, e \in \mathcal{H}$ and let $\|e\| = 1$. Then*

$$|\langle x, e \rangle \langle e, y \rangle| \leqslant \frac{1}{2} \big(\|x\| \|y\| + |\langle x, y \rangle| \big).$$

By applying the Buzano's inequality we prove the following theorem.

THEOREM 2.10. *If $T \in \mathcal{B}(\mathcal{H})$, then*

$$dw(T) \leqslant \sqrt{\frac{1}{4} w^2 \left(|T| + i|T^*| \right) + \frac{1}{4} w(|T||T^*|) + \frac{1}{8} \min\{\alpha, \beta\}},$$

where $\alpha = \left\| |T|^2 + |T^*|^2 + 8|T|^4 \right\|$ *and* $\beta = \left\| |T|^2 + |T^*|^2 + 8|T^*|^4 \right\|$.

Proof. Take $x \in \mathcal{H}$ with $\|x\| = 1$. We have

$$\begin{aligned}
& |\langle Tx, x \rangle|^2 + \|Tx\|^4 \\
\leqslant & \langle |T|x, x \rangle \langle |T^*|x, x \rangle + \langle |T|^2 x, x \rangle^2 \quad \text{(by (2.2))} \\
\leqslant & \frac{1}{4} (\langle |T|x, x \rangle + \langle |T^*|x, x \rangle)^2 + \langle |T|^2 x, x \rangle^2 \\
= & \frac{1}{4} (\langle |T|x, x \rangle^2 + \langle |T^*|x, x \rangle^2 + 2 \langle |T|x, x \rangle \langle |T^*|x, x \rangle) + \langle |T|^2 x, x \rangle^2 \\
\leqslant & \frac{1}{4} (|\langle |T|x, x \rangle + i \langle |T^*|x, x \rangle|^2 + \left\| |T|x \right\| \left\| |T^*|x \right\| + |\langle |T|x, |T^*|x \rangle|) + \langle |T|^2 x, x \rangle^2 \\
& \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \text{(by Lemma 2.9)} \\
\leqslant & \frac{1}{4} |\langle |T|x, x \rangle + i \langle |T^*|x, x \rangle|^2 + \frac{1}{8} (\left\| |T|x \right\|^2 + \left\| |T^*|x \right\|^2) + \frac{1}{4} |\langle |T^*||T|x, x \rangle| \\
& \qquad \qquad \qquad \qquad \qquad + \langle |T|^4 x, x \rangle \quad \text{(by Lemma 2.1)} \\
= & \frac{1}{4} |\langle (|T| + i|T^*|)x, x \rangle|^2 + \frac{1}{8} \langle (|T|^2 + |T^*|^2 + 8|T|^4)x, x \rangle + \frac{1}{4} |\langle |T^*||T|x, x \rangle.
\end{aligned}$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain

$$dw^2(T) \leqslant \frac{1}{4}w^2\left(|T| + i|T^*|\right) + \frac{1}{8}\left\||T|^2 + |T^*|^2 + 8|T|^4\right\| + \frac{1}{4}w(|T||T^*|). \tag{2.3}$$

Replacing T by T^*, we also obtain

$$dw^2(T) \leqslant \frac{1}{4}w^2\left(|T| + i|T^*|\right) + \frac{1}{8}\left\||T|^2 + |T^*|^2 + 8|T^*|^4\right\| + \frac{1}{4}w(|T||T^*|). \tag{2.4}$$

Therefore, the required inequality follows from (2.3) together with (2.4). □

Next upper bound reads as follows:

THEOREM 2.11. *If $T \in \mathcal{B}(\mathcal{H})$, then*

$$dw(T) \leqslant \sqrt{w^2\left(|T|^2 + e^{i\theta}T\right) + 2\|T\|^2\|Re(e^{i\theta}T)\|},$$

for all $\theta \in \mathbb{R}$.

Proof. Take $x \in \mathcal{H}$ with $\|x\| = 1$. We have

$$\begin{aligned}
|\langle Tx, x\rangle|^2 + \|Tx\|^4 &= \left|\langle Tx, x\rangle + \|Tx\|^2\right|^2 - 2Re\left(\langle Tx, x\rangle\|Tx\|^2\right) \\
&= |\langle (T + |T|^2)x, x\rangle|^2 - 2\|Tx\|^2\langle Re(T)x, x\rangle \\
&\leqslant |\langle (T + |T|^2)x, x\rangle|^2 + 2\|Tx\|^2|\langle Re(T)x, x\rangle| \\
&\leqslant w^2(T + |T|^2) + 2\|T\|^2\|Re(T)\|.
\end{aligned}$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain

$$dw^2(T) \leqslant w^2(T + |T|^2) + 2\|T\|^2\|Re(T)\|. \tag{2.5}$$

Now replacing T by $e^{i\theta}T$, we get

$$dw^2(T) \leqslant w^2\left(|T|^2 + e^{i\theta}T\right) + 2\|T\|^2\|Re(e^{i\theta}T)\|,$$

as desired. □

REMARK 2.12. (i) From Theorem 2.11 we can easily derive the following new bound

$$dw(T) \leqslant \sqrt{w^2(T \pm |T|^2) + 2\|T\|^2\|Re(T)\|}, \tag{2.6}$$

which is stronger than the existing bound (1.2).

(ii) Also, from Theorem 2.11 we can easily derive the following existing bound

$$dw(T) \leqslant \sqrt{\min_{0 \leqslant \theta \leqslant 2\pi} w^2\left(|T|^2 + e^{i\theta}T\right) + 2\|T\|^2 w(T)},$$

see [4, Theorem 2.6].

In the following theorem we obtain new lower and upper bounds for the Davis-Wielandt radius of an operator T.

THEOREM 2.13. *If $T \in \mathcal{B}(\mathcal{H})$, then*

$$\max\left\{w(Re(T)+i|T|^2), w(Im(T)+i|T|^2)\right\} \leqslant dw(T)$$
$$\leqslant \min\left\{\sqrt{w^2(Re(T)+i|T|^2)+\|Im(T)\|^2}, \sqrt{w^2(Im(T)+i|T|^2)+\|Re(T)\|^2}\right\}.$$

Proof. Take $x \in \mathcal{H}$ with $\|x\| = 1$. From the Cartesian decomposition of T (that is, $T = Re(T) + iIm(T)$) we have,

$$\begin{aligned} |\langle Tx,x\rangle|^2 + \|Tx\|^4 &= |\langle Re(T)x,x\rangle|^2 + |\langle Im(T)x,x\rangle|^2 + \langle |T|^2x,x\rangle^2 \\ &= |\langle (Re(T)+i|T|^2)x,x\rangle|^2 + |\langle Im(T)x,x\rangle|^2 \\ &\leqslant w^2(Re(T)+i|T|^2) + \|Im(T)\|^2. \end{aligned} \tag{2.7}$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain

$$dw^2(T) \leqslant w^2(Re(T)+i|T|^2) + \|Im(T)\|^2. \tag{2.8}$$

Similarly, we also obtain

$$dw^2(T) \leqslant w^2(Im(T)+i|T|^2) + \|Re(T)\|^2. \tag{2.9}$$

Combining (2.8) and (2.9) we have the desired second bound. Now, it follows from (2.7) that

$$|\langle Tx,x\rangle|^2 + \|Tx\|^4 \geqslant |\langle (Re(T)+i|T|^2)x,x\rangle|^2.$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\| = 1$, we have

$$dw(T) \geqslant w(Re(T)+i|T|^2). \tag{2.10}$$

Similarly, we also have

$$dw(T) \geqslant w(Im(T)+i|T|^2). \tag{2.11}$$

Therefore, the desired first inequality follows by combining (2.10) and (2.11). □

By employing Theorem 2.13 we obtain the following corollary that gives an equality for the numerical radius $w(T)$ of an operator T with $Im(T) = (Re(T))^2$.

COROLLARY 2.14. *Let $T \in \mathcal{B}(\mathcal{H})$ be such that $Im(T) = (Re(T))^2$. Then*

$$w(T) = \|Re(T)\|\sqrt{1+\|Re(T)\|^2}.$$

Proof. Suppose $S \in \mathcal{B}(\mathcal{H})$ is self-adjoint. Then it follows from Theorem 2.13 that $dw(S) = w(S + iS^2)$. Also, $dw(S) = \sqrt{\|S\|^2 + \|S\|^4}$.

Thus, $w(S+iS^2) = \|S\|\sqrt{1+\|S\|^2}$. Taking $S = Re(T)$, the proof follows. □

Now, we consider an example to show that the bounds obtained in Theorem 2.7, Theorem 2.10 and Theorem 2.13 are better than the existing ones. The bounds

(a) $dw^2(T) \leqslant \frac{1}{2}\{w^2(T+T^*T) + w^2(T-T^*T)\}$,

(b) $dw^2(T) \leqslant \left\||T|^2 + |T|^4\right\|$,

(c) $dw^2(T) \leqslant \frac{1}{2}\left(w(T^2) + \|T\|^2\right) + \|T\|^4$,

obtained in [5, Theorem 2.2, Theorem 2.4 (i), Theorem 2.4 (ii)]. If we take

$$T = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix},$$

then from Corollary 2.8, Theorem 2.10 and Theorem 2.13, we get $dw(T) \leqslant 4.294, 4.286$ and 4.301, respectively, whereas the bounds in (a),(b) and (c) respectively give $dw(T) \leqslant 4.621, 4.472$ and 4.358. Thus, for this example, the upper bounds of $dw(T)$ in Theorem 2.7, Theorem 2.10 and Theorem 2.13 are better than the existing bounds mentioned above.

Next bound reads as follows:

THEOREM 2.15. *If $T \in \mathcal{B}(\mathcal{H})$, then*

$$dw(T) \leqslant \sqrt{w^2(|T|^2 + e^{i\theta}T) + \frac{1}{2}\left\||T|^4 + |T^*|^2\right\| + w(T|T|^2)},$$

for all $\theta \in \mathbb{R}$.

Proof. Take $x \in \mathcal{H}$ with $\|x\| = 1$. We have

$$\begin{aligned}
&|\langle Tx,x\rangle|^2 + \|Tx\|^4 \\
&= \left|\langle Tx,x\rangle + \|Tx\|^2\right|^2 - 2Re\left(\|Tx\|^2\langle Tx,x\rangle\right) \\
&\leqslant \left|\langle (T+|T|^2)x,x\rangle\right|^2 + 2\left|\langle |T|^2x,x\rangle\langle Tx,x\rangle\right| \\
&\leqslant \left|\langle (T+|T|^2)x,x\rangle\right|^2 + \left\||T|^2x\right\|\|T^*x\| + \left|\langle |T|^2x, T^*x\rangle\right| \quad \text{(by Lemma 2.9)} \\
&\leqslant \left|\langle (T+|T|^2)x,x\rangle\right|^2 + \frac{1}{2}\left(\left\||T|^2x\right\|^2 + \|T^*x\|^2\right) + \left|\langle T|T|^2x,x\rangle\right| \\
&= \left|\langle (T+|T|^2)x,x\rangle\right|^2 + \frac{1}{2}\langle (|T|^4 + |T^*|^2)x,x\rangle + \left|\langle T|T|^2x,x\rangle\right| \\
&\leqslant w^2(T+|T|^2) + \frac{1}{2}\left\||T|^4 + |T^*|^2\right\| + w(T|T|^2).
\end{aligned}$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain

$$dw^2(T) \leqslant w^2(|T|^2 + T) + \frac{1}{2}\left\||T|^4 + |T^*|^2\right\| + w(T|T|^2).$$

Now replacing T by $e^{i\theta}T$, we have

$$dw^2(T) \leqslant w^2(|T|^2 + e^{i\theta}T) + \frac{1}{2}\left\||T|^4 + |T^*|^2\right\| + w(T|T|^2),$$

as desired. $\square$

Applying similar arguments as used in Theorem 2.15, we also obtain the following upper bound.

THEOREM 2.16. *If $T \in \mathcal{B}(\mathcal{H})$, then*

$$dw(T) \leqslant \sqrt{w^2(|T|^2 + e^{i\theta}T) + \frac{1}{2}\left\||T|^4 + |T|^2\right\| + w(|T|^2T)},$$

for all $\theta \in \mathbb{R}$.

Now we consider the following numerical example to show the bounds obtained in Theorem 2.15 and Theorem 2.16 are sharper than the existing ones. If we take

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

then Theorem 2.15 and Theorem 2.16 (for $\theta = \pi$) give $dw(T) \leqslant 2.547$ and 2.464, respectively, whereas the bounds in (1.3), (1.4) and (1.5) respectively give $dw(T) \leqslant 2.613$, 2.613 and 2.565. Thus, for this example, the upper bounds of $dw(T)$ obtained in Theorem 2.15 and Theorem 2.16 are better than the existing bounds in (1.3), (1.4) and (1.5).

Finally, we obtain the following inequality.

THEOREM 2.17. *If $T \in \mathcal{B}(\mathcal{H})$, then*

$$dw^2(T) + 2\|T\|^2\|Re(T)\| \geqslant \max\left\{w^2(T + |T|^2), w^2(T - |T|^2)\right\}.$$

Proof. Take $x \in \mathcal{H}$ with $\|x\| = 1$. We have

$$\begin{aligned} |\langle Tx, x\rangle|^2 + \|Tx\|^4 &= |\langle (T + |T|^2)x, x\rangle|^2 - 2\|Tx\|^2\langle Re(T)x, x\rangle \\ &\geqslant |\langle (T + |T|^2)x, x\rangle|^2 - 2\|Tx\|^2|\langle Re(T)x, x\rangle| \\ &\geqslant |\langle (T + |T|^2)x, x\rangle|^2 - 2\|T\|^2\|Re(T)\|. \end{aligned}$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\| = 1$, we obtain

$$dw^2(T) + 2\|T\|^2\|Re(T)\| \geqslant w^2(T + |T|^2). \tag{2.12}$$

Now replacing T by $-T$, we have

$$dw^2(T)+2\|T\|^2\|Re(T)\| \geqslant w^2(T-|T|^2). \tag{2.13}$$

The desired inequality follows from (2.12) together with (2.13). □

Now, we consider

$$T=\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix},$$

then both Theorem 2.13 and Theorem 2.17 give $dw(T) \geqslant 4.472$, whereas the bounds in (1.7) and (1.8) respectively give $dw(T) \geqslant 4$ and 0. Thus, for this example the lower bounds obtained in Theorem 2.13 and Theorem 2.17 are better than the existing lower bounds in (1.7) and (1.8).

REFERENCES

[1] A. ABU-OMAR AND F. KITTANEH, *A generalization of the numerical radius*, Linear Algebra Appl. 569 (2019), 323–334.

[2] M. W. ALOMARI, *On the Davis-Wielandt radius inequalities of Hilbert space operators*, Linear Multilinear Algebra 71 (2023), no. 11, 1804–1828.

[3] P. BHUNIA, S. S. DRAGOMIR, M. S. MOSLEHIAN AND K. PAUL, *Lectures on Numerical Radius Inequalities*, Infosys Sci. Found. Ser. Math. Sci., Springer, Cham, (2022), ©2022. xii+209 pp. ISBN: 978-3-031-13669-6; 978-3-031-13670-2.

[4] P. BHUNIA, A. BHANJA AND K. PAUL, *New inequalities for Davis-Wielandt radius of Hilbert space operators*, Bull. Malays. Math. Sci. Soc. 44 (2021), no. 5, 3523–3539.

[5] P. BHUNIA, A. BHANJA, S. BAG AND K. PAUL, *Bounds for the Davis-Wielandt radius of bounded linear operators*, Ann. Funct. Anal. 12 (2021), no. 1, paper no. 18, 23 pp.

[6] M. L. BUZANO, *Generalizzatione della diseguaglianza di Cauchy-Schwarz*, Rend. Sem. Mat. Univ. epolitech. Trimo 31 (1971/73), 405–409.

[7] C. DAVIS, *The shell of a Hilbert-space operator*, Acta Sci. Math. (Szeged) 29 (1968), 69–86.

[8] A. FRAKIS, F. KITTANEH AND S. SOLTANI, *New numerical radius inequalities for operator matrices and a bound for the zeros of polynomials*, Adv. Oper. Theory 8 (2023), no. 1, paper no. 6, 13 pp.

[9] K. E. GUSTAFSON AND D. K. M. RAO, *Numerical Range, The field of values of linear operators and matrices*, Springer, New York, 1997.

[10] F. KITTANEH, *Notes on some inequalities for Hilbert space operators*, Publ. RIMS Kyoto Univ. 24 (1988), 283–293.

[11] C. K. LI AND Y. T. POON, *Spectrum, numerical range and Davis-Wielandt Shell of a normal operator*, Glasgow Math. J. 51 (2009), 91–100.

[12] C. K. LI AND Y. T. POON AND N. S. SZE, *Davis-Wielandt shells of operators*, Oper. Matrices 2 (3) (2008), 341–355.

[13] B. LINS, I. M. SPITKOVSKY AND S. ZHONG, *The normalized numerical range and the Davis-Wielandt shell*, Linear Algebra Appl. 546 (2018), 187–209.

[14] A. SADDI, *A-Normal operators in Semi-Hilbertian spaces*, Aust. J. Math. Anal. Appl. 9 (1) 5 (2012), 1–12.

[15] B. SIMON, *Trace ideals and their applications*, Camrbidge University Press, 1979.

[16] S. SOLTANI AND A. FRAKIS, *Further refinements of some numerical radius inequalities for operators*, Oper. Matrices 17 (1), (2023) 245–257.

[17] H. WIELANDT, *On eigenvalues of sums of normal matrices*, Pac. J. Math. 5 (1955), 633–638.

[18] P. Y. WU AND H.-L. GAU, *Numerical ranges of Hilbert space operators*, Encyclopedia of Mathematics and its Applications, 179. Cambridge University Press, Cambridge, 2021.

[19] A. Zamani, M. S. Moslehian, M. T. Chien and H. Nakazato, *Norm-parallelism and the Davis-Wielandt radius of Hilbert space operators*, Linear Multilinear Algebra 67 (11) (2019), 2147–2158.

[20] A. Zamani and K. Shebrawi, *Some upper bounds for the Davis-Wielandt radius of Hilbert space operators*, Mediterr. J. Math. 17 (2020), no. 1, paper no. 25, 13 pp.

(Received March 16, 2023)

Pintu Bhunia
Department of Mathematics
Indian Institute of Science
Bengaluru 560012, Karnataka, India
e-mail: pintubhunia5206@gmail.com

Kallol Paul
Department of Mathematics
Jadavpur University
Kolkata 700032, West Bengal, India
e-mail: kalloldada@gmail.com

Somdatta Barik
Department of Mathematics
Jadavpur University
Kolkata 700032, West Bengal, India
e-mail: bariksomdatta97@gmail.com