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THE INFINITE DIMENSIONAL PERFECT–MIRSKY CONJECTURE

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Abstract. The spectrum of an infinite-dimensional doubly stochastic matrix, when considered as a bounded operator on the sequence space ℓ^p with $1 \leqslant p < \infty$, is contained within the closed unit disc $\mathbb{D}$. In our work, we present an infinite doubly stochastic matrix that exhibits the entire closed unit disc as its spectrum. However, we prove that the points $\mathrm{e}^{i\pi r}$, where r is an irrational real number, cannot serve as eigenvalues for any doubly stochastic matrices, be it finite or infinite in size. On the other hand, we show that every other point within the closed unit disc can indeed be an eigenvalue of an infinite-dimensional doubly stochastic matrix. In fact, we construct a specific example of an infinite doubly stochastic matrix whose point spectrum precisely consists of $\mathbb{D} \cup \{\mathrm{e}^{i\pi r} : r \in \mathbb{Q}\}$. Additionally, we investigate the behavior of doubly stochastic matrices in the context of the sequence space ℓ^∞, highlighting the contrasts with the ℓ^p setting for $1 \leqslant p < \infty$.

1. Introduction

A stochastic matrix is a square matrix with non-negative entries whose rows sum up to one. In 1938, in a lecture on Markov chains given at the meeting of the Moscow Mathematical Society, finding the location of eigenvalues of stochastic matrices was proposed by Kolmogorov. This region, i.e., the loci of all eigenvalues of all $n \times n$ stochastic matrices, was first characterized by Dmitriev and Dynkin [7] for $1 \leqslant n \leqslant 5$, and then by Karpelevich [11] for all integers $n \geqslant 1$. His formulation was later simplified by Ito [9]. More recently, Johnson and Paparella provided a matricial approach to Karpelevich theorem [10]. Here, we study this question for infinite dimensional doubly stochastic matrices. To set the stage and explain in more detail, let us start with some general definitions.

Let I be a, finite or infinite (countable or uncountable), index set. Then $\ell^p(I)$, $p \in [1,\infty)$, is the Banach space of (generalized) sequences $\mathbf{x} := (x_i)_{i \in I}$ such that

$$\|\mathbf{x}\|_p := \left(\sum_{i \in I} |x_i|^p\right)^{1/p} < \infty. \tag{1.1}$$

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In some occasions, we write $\mathbf{x}(i)$ for x_i to emphasize further that $\mathbf{x}$ is a function on the index set I. The Banach space $\ell^\infty(I)$ consists of all bounded sequences and is equipped with the uniform norm

$$\|\mathbf{x}\|_\infty := \sup_{i\in I}|x_i|. \tag{1.2}$$

Assume that $T:\ell^p(I)\to\ell^p(I)$ is a bounded linear operator. Then, with respect to the canonical (Riesz) basis $\{\mathbf{e}_i : i\in I\}$, T has the unique matrix representation $[t_{ij}]_{i,j\in I}$, where $t_{ij}=(T\mathbf{e}_j)(i)$. More explicitly, for each $\mathbf{x}\in\ell^p(I)$, we have

$$(T\mathbf{x})(i)=\sum_{j\in I}t_{ij}x_j, \qquad i\in I. \tag{1.3}$$

As is the tradition, we do not distinguish between T and its corresponding matrix $[t_{ij}]$. Note that if I is infinite, in particular if it is uncountable, we should interpret (1.1) and (1.3) as the limit of all finite sums. However, it is more practical to consider them as Lebesgue integrals formed with the counting measure on I.

We say that $\mathbf{x}\in\ell^p(I)$ is non-negative, and write $\mathbf{x}\geqslant 0$, if

$$x_i\geqslant 0, \qquad i\in I.$$

The collection of all $\mathbf{x}\in\ell^p(I)$, $\mathbf{x}\geqslant 0$, will be denoted by $\ell^p(I)^+$. A bounded linear operator $T:\ell^p(I)\to\ell^p(I)$ is said to be non-negative if $T\mathbf{x}\geqslant 0$ for all $\mathbf{x}\in\ell^p(I)^+$. Clearly, T is non-negative if and only if $t_{ij}\geqslant 0$, for all $i,j\in I$. A non-negative operator T is called *stochastic* if

$$\sum_{j\in I}t_{ij}=1, \qquad i\in I.$$

If, moreover,

$$\sum_{j\in I}t_{ij}=1, \quad i\in I, \qquad\qquad \text{and} \qquad\qquad \sum_{i\in I}t_{ij}=1, \quad j\in I,$$

then we say that T is *doubly stochastic*. Finally, T is *doubly sub-stochastic* if it is non-negative and

$$\sum_{j\in I}t_{ij}\leqslant 1, \quad i\in I, \qquad\qquad \text{and} \qquad\qquad \sum_{i\in I}t_{ij}\leqslant 1, \quad j\in I.$$

In the following, we will briefly write d.s. and d.s.s. respectively for doubly stochastic and doubly sub-stochastic. Detailed treatment of this topic is available in textbooks [16, 20]. Se also [3, 5, 15].

An $I\times I$ d.s.s. matrix T always acts boundedly on $\ell^p(I)$. In fact, the classical Schur's test immediately implies

$$\|T\|_{\ell^p(I)\to\ell^p(I)}\leqslant 1. \tag{1.4}$$

See [2, Theorem 2.2] and [8, Chapter 3].

It is clear that every d.s. operator is a priori a d.s.s. operator. Hence, by the estimation (1.4), the norm of any d.s. operator is also less than or equal to one. We denote the family of all d.s. operators on $\ell^p(I)$ by $\mathcal{DS}(\ell^p(I))$. According to [1, Theorem 2.4], $\mathcal{DS}(\ell^p(I))$ is closed under composition (infinite matrix multiplication). The same property holds for the family of d.s.s. operators on $\ell^p(I)$.

The spectrum and point spectrum (eigenvalues) of a bounded operator T are respectively denoted by $\sigma(T)$ and $\sigma_\wp(T)$. There are two sets which are our main concerns in this work. We define the loci of eigenvalues of all d.s. operators on $\ell^p(I)$ by

$$\Omega_{p,I} := \bigcup_{T\in\mathcal{DS}(\ell^p(I))} \sigma_\wp(T),$$

and the loci of spectrum of all d.s. operators on $\ell^p(I)$ by

$$\widetilde{\Omega}_{p,I} := \bigcup_{T\in\mathcal{DS}(\ell^p(I))} \sigma(T).$$

Despite clearly $\Omega_{p,I} \subset \widetilde{\Omega}_{p,I}$, we did not use $\overline{\Omega_{p,I}}$ for the loci spectrum since, at this point, it is not even known if $\widetilde{\Omega}_{p,I}$ is closed. However, after stating the main results the relation between the two sets is clarified. See Theorem 2.1 and the comment after it.

If $I = \{1,2,\ldots,n\}$, the above two sets coincide and are independent of p. In the literature, the common set is denoted by Ω_n. In other words, Ω_n is the loci of all eigenvalues of all $n\times n$ doubly stochastic matrices. In 1956, Perfect and Mirsky [19] studied Ω_n and showed that

$$\cup_{k=2}^{n}\Pi_k \subseteq \Omega_n \subset \overline{\mathbb{D}}, \qquad n \geqslant 2, \tag{1.5}$$

where

$$\Pi_k := \operatorname{co}\{e^{i2\pi\frac{s}{k}} : 1 \leqslant s \leqslant k\},$$

the convex hull of the kth roots of unity, $\mathbb{D} := \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ is the open unit disc and $\overline{\mathbb{D}}$ denotes its closure, the closed unit disc. Furthermore, they also conjectured that $\Omega_n = \cup_{k=2}^{n}\Pi_k$. The conjecture trivially holds for $n = 2$ and $n = 3$. In 2007, J. Mashreghi and R. Rivard disproved the conjecture for $n = 5$ [17]. Surprisingly, in 2014 and using novel techniques, J. Levick, R. Pereira, and D. W. Kribs established the conjecture for $n = 4$ [14]. They also formulated new variations of the conjecture. However, in 2022, B. Kim and J. Kim showed the invalidity of the proposed variations [13]. Up to now, the status of Perfect–Mirsky conjecture is not known for $n \geqslant 6$. In this note, we completely settle this question when I is an infinite set.

2. Main results

In this section, we characterize the eigenvalues location and spectrum location of infinite d.s. operators on $\ell^p(I)$. Contrary to the finite dimensional case, the following two theorems show that the location of eigenvalues and the spectrum values are both independent of the cardinality of I whenever I is an infinite (countable or uncountable) set.

THEOREM 2.1. *Let I be an infinite index set. Then*

(i) $\Omega_{\infty,I} = \overline{\mathbb{D}}$,

(ii) and, for $1 \leqslant p < \infty$,

$$\Omega_{p,I} = \mathbb{D} \cup \{e^{i\pi r} : r \in \mathbb{Q}\}.$$

Theorem 2.1 reveals a dichotomy about the set $\Omega_{p,I}$, as long as the index set I is not finite. On the one hand, $\Omega_{\infty,I}$ is the compact set $\overline{\mathbb{D}}$. On the other hand, $\Omega_{p,I}$, independent of the value of $p < \infty$, is a proper not-closed subset of $\overline{\mathbb{D}}$.

THEOREM 2.2. *Let $1 \leqslant p \leqslant \infty$, and let I be an infinite index set. Then* $\widetilde{\Omega}_{p,I} = \overline{\mathbb{D}}$.

REMARK. Theorems 2.1 and 2.2 ensure that it is now legitimate to write $\widetilde{\Omega}_{p,I} = \overline{\Omega_{p,I}}$ for any value of p and any index set I.

Recalling that $\Omega_{p,I}$ is the collection of all eigenvalues of all d.s. operators on $\ell^p(I)$, there is a more difficult, and yet open, question to ask: which subsets of $\mathbb{D} \cup \{e^{i\pi r} : r \in \mathbb{Q}\}$ can be served as the set of eigenvalues of a fixed (but arbitrary) d.s. operator? We provide a partial answer below. At the same time, we show that there is a d.s. operator whose spectrum is precisely $\overline{\mathbb{D}}$.

THEOREM 2.3. *Let I be an infinite index set. Then the following hold.*

(i) If

$$E \subset \mathbb{D} \cup \{e^{i\pi r} : r \in \mathbb{Q}\} \quad \textit{and} \quad \operatorname{card}(E) \leqslant \operatorname{card}(I),$$

then there is an operator $T \in \mathcal{DS}(\ell^p(I))$ such that

$$E \subset \sigma_{\wp}(T).$$

(ii) There is an operator $T \in \mathcal{DS}(\ell^p(I))$ for which

$$\sigma(T) = \overline{\mathbb{D}}.$$

We highlight two special cases in which the condition $\operatorname{card}(E) \leqslant \operatorname{card}(I)$ trivially holds. First, if I is uncountable, then any subsets of $\mathbb{D} \cup \{e^{i\pi r} : r \in \mathbb{Q}\}$ fulfills the required condition. Second, if E is countable, since I is at least countable, then again E satisfies the requirements. In these two cases, the conclusion of part (i) in Theorem 2.3 necessarily holds. In particular, we have the following interesting result.

COROLLARY 2.4. *Let I be an uncountable index set. Then there is an operator $T \in \mathcal{DS}(\ell^p(I))$ such that*

$$\sigma_{\wp}(T) = \mathbb{D} \cup \{e^{i\pi r} : r \in \mathbb{Q}\}.$$

3. The proof of Theorem 2.1

The following definitions of *formal sum* and *direct sum* are needed in constructions appearing below. See [4] for further detail. Let $\{I_\gamma : \gamma \in \Gamma\}$ be a partition of I. Suppose that $\{\mathbf{x}_\gamma : I_\gamma \to \mathbb{R} : \gamma \in \Gamma\}$ is a family of functions (generalized sequences) and $\{T_\gamma = [t^\gamma_{ij}]_{i,j \in I_\gamma} : \gamma \in \Gamma\}$ is a family of square matrices. Then is

$$\mathbf{x} = \bigoplus_{\gamma \in \Gamma} \mathbf{x}_\gamma : I \to \mathbb{R},$$

the formal sum of $\mathbf{x}_\gamma$, is defined by $\mathbf{x}(i) = \mathbf{x}_\gamma(i)$ whenever $i \in I_\gamma$. Note that there is a unique γ for which $i \in I_\gamma$. The direct sum of matrices T_γ is the $I \times I$ (block) matrix

$$T = \bigoplus_{\gamma \in \Gamma} T_\gamma = [t_{ij}],$$

where t_{ij} is defined by

$$t_{ij} = \begin{cases} t^\gamma_{ij}, & \text{if } i,j \in I_\gamma \text{ for some } \gamma, \\ 0, & \text{otherwise.} \end{cases}$$

Naively speaking, we can say that T is the block matrix with T_γ on its diagonal and zero elsewhere.

In the light of the above definition, if T_γ is a family of bounded operators on $\ell^p(I_\gamma)$, we may consider the mapping (a direct sum of operators) $T := \bigoplus_{\gamma \in \Gamma} T_\gamma$. Then T is d.s. if and only if each T_γ is d.s., and λ is an eigenvalue of T if and only if λ is an eigenvalue of some T_γ, i.e.,

$$\sigma_\wp(T) = \bigcup_{\gamma \in \Gamma} \sigma_\wp(T_\gamma).$$

More generally, T is a bounded operator on $\ell^p(I)$ if and only if each T_γ is a bounded operator on $\ell^p(I_\gamma)$ and, moreover, their norms are uniformly bounded by a constant independent of γ.

In order to shorten the proof of Theorem 2.1, we present some parts of the proof as independent lemmas, which are interesting in their own right. Let us start with a simple observation. If λ is an eigenvalue of an operator $T \in \mathcal{DS}(\ell^p(I))$, $1 \leqslant p \leqslant \infty$, then $T\mathbf{x} = \lambda\mathbf{x}$ holds for an eigenvector $\mathbf{x} \in \ell^p(I) \setminus \{0\}$. Therefore, we have $|\lambda| \cdot \|\mathbf{x}\| = \|T\mathbf{x}\| \leqslant \|T\| \cdot \|\mathbf{x}\|$. Hence, by (1.4),

$$|\lambda| \leqslant 1. \tag{3.1}$$

The first lemma describes the lattice of $\Omega_{p,I}$, when p ranges over the interval $[1,\infty]$ and the index set I takes different cardinalities. In some sense, we can say that the lattice is increasing with respect to both parameters. However, the final result (Theorem 2.1), provides the complete picture of the lattice.

LEMMA 3.1. *Let I and J be arbitrary index sets and $p,q \in [1,\infty]$. Then the following hold.*

(i) If $\operatorname{card}(I) \leqslant \operatorname{card}(J)$*, then* $\Omega_{p,I} \subseteq \Omega_{p,J}$.

(ii) If $p \leqslant q$*, then* $\Omega_{p,I} \subseteq \Omega_{q,I}$.

Proof. (i): Let $T \in \mathcal{DS}(\ell^p(I))$. Then

$$T \bigoplus \mathrm{id}_{J\setminus I} \in \mathcal{DS}(\ell^p(J)),$$

and thus

$$\sigma_{\wp}(T) \subseteq \sigma_{\wp}(T \bigoplus \mathrm{id}_{J\setminus I}) \subseteq \bigcup_{S\in\mathcal{DS}(\ell^p(J))} \sigma_{\wp}(S) = \Omega_{p,J}.$$

This shows

$$\Omega_{p,I} = \bigcup_{T\in\mathcal{DS}(\ell^p(I))} \sigma_{\wp}(T) \subseteq \Omega_{p,J}.$$

Note that if J is infinite, we may even replace $\mathrm{id}_{J\setminus I}$ by id_J in the above argument.

(ii): If $\lambda \in \Omega_{p,I}$, then there is a d.s. operator $T = [t_{ij}] : \ell^p(I) \to \ell^p(I)$ and a non-zero $\mathbf{x} \in \ell^p(I)$ such that $T\mathbf{x} = \lambda\mathbf{x}$. However, by (1.4), $[t_{ij}]$ is also a d.s. operator on $\ell^q(I)$, which we denote by $\widehat{T}$. Recall that $\ell^p(I) \subset \ell^q(I)$, and since

$$\widehat{T}\mathbf{x} = T\mathbf{x} = \lambda\mathbf{x},$$

we conclude that $\lambda \in \Omega_{q,I}$. □

As an immediate consequence of the previous lemma, if $1 \leqslant p \leqslant \infty$ and I is any infinite index set, then

$$\bigcup_{n\geqslant 1} \Omega_n \subseteq \Omega_{p,I} \subseteq \overline{\mathbb{D}}. \tag{3.2}$$

To continue, the next result states that the kernel of any d.s. operator does not contain any positive non-zero elements.

LEMMA 3.2. *Let $T \in \mathcal{DS}(\ell^p(I))$. Then*

$$\ker(T) \cap \ell^p(I)^+ = \{0\}.$$

Proof. Let $\mathbf{x} \in \ell^p(I)^+$ be such that $T\mathbf{x} = 0$. Hence

$$\sum_{j\in I} t_{ij}\mathbf{x}(j) = 0, \qquad i \in I.$$

Since all components are positive, by the Fubini–Tonelli theorem (discrete version) and that T is d.s.,

$$0 = \sum_{i\in I}\left(\sum_{j\in I} t_{ij}\mathbf{x}(j)\right) = \sum_{j\in I}\left(\sum_{i\in I} t_{ij}\right)\mathbf{x}(j) = \sum_{j\in I}\mathbf{x}(j).$$

Therefore,

$$\mathbf{x}(j) = 0, \qquad j \in I. \quad \square$$

A comment is in order: special care is needed in applying Lemma 3.2. While it confirms there is no positive vector in the kernel of a d.s. operator T, it is quite possible that $\ker(T) \neq \{0\}$, For example, let

$$T = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & \cdots \\ 1/2 & 1/2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then $T\mathbf{e}_1 = T\mathbf{e}_2 = \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2)$, and thus $T(\mathbf{e}_1 - \mathbf{e}_2) = 0$. In other words, $\mathbf{e}_1 - \mathbf{e}_2 \in \ker(T)$.

This said, the possibility of $\ker(T) = \{0\}$ is not ruled out. Let us present a concrete example. Let

$$T = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & \cdots \\ 1/2 & 0 & 1/2 & 0 & 0 & \cdots \\ 0 & 1/2 & 0 & 1/2 & 0 & \cdots \\ 0 & 0 & 1/2 & 0 & 1/2 & \\ 0 & 0 & 0 & 1/2 & 0 & \ddots \\ \vdots & \vdots & \vdots & & \ddots & \ddots \end{bmatrix}.$$

Assume that $T\mathbf{x} = \lambda\mathbf{x}$, where $\mathbf{x} = (x_n)_{n \geqslant 1} \in \ell^p$, $1 \leqslant p < \infty$. We know that $\lambda \in \overline{\mathbb{D}}$, and thus we consider three cases.

Case I, $\lambda = 1$: In this case, the equation $\mathbf{x} = T\mathbf{x}$ is written as

$$\begin{aligned} 2x_1 &= x_1 + x_2, \\ 2x_2 &= x_1 + x_3, \\ 2x_3 &= x_2 + x_4, \\ 2x_4 &= x_3 + x_5, \\ &\vdots \end{aligned}$$

and simplifies to $x_1 = x_2 = x_3 = \cdots$. Since $\mathbf{x} \in \ell^p$, we must have $\mathbf{x} = 0$.

Case II, $\lambda = -1$: In this case, the equation $\lambda\mathbf{x} = T\mathbf{x}$ is written as

$$\begin{aligned} -2x_1 &= x_1 + x_2, \\ -2x_2 &= x_1 + x_3, \\ -2x_3 &= x_2 + x_4, \\ -2x_4 &= x_3 + x_5, \\ &\vdots \end{aligned}$$

The first equation simplifies to $x_2 = -3x_1$, and the general solution of the rest of the system is

$$x_n = a(-1)^n + bn(-1)^n, \qquad n \geqslant 1,$$

where a and b are arbitrary constants. The initial requirement $x_2 = -3x_1$ implies that $b = -2a$. Thus we have

$$x_n = (-1)^n(1-2n)a, \qquad n \geqslant 1.$$

But, since $\mathbf{x} \in \ell^p$, we must have $a = 0$, which leads to $\mathbf{x} = 0$.

Case III, $\lambda \in \overline{\mathbb{D}} \setminus \{-1,1\}$: In this case, the equation $\lambda \mathbf{x} = T\mathbf{x}$ is written as

$$\begin{aligned}
2\lambda x_1 &= x_1 + x_2, \\
2\lambda x_2 &= x_1 + x_3, \\
2\lambda x_3 &= x_2 + x_4, \\
2\lambda x_4 &= x_3 + x_5, \\
&\vdots
\end{aligned}$$

The first equation simplifies to $x_2 = (2\lambda - 1)x_1$. Using the general theory difference equations, the general solution of the rest of the system is

$$x_n = a\left(\lambda + \sqrt{\lambda^2 - 1}\right)^n + b\left(\lambda - \sqrt{\lambda^2 - 1}\right)^n, \qquad n \geqslant 1,$$

where a and b are arbitrary constants. However, since $\lambda \neq 1$, the above equation for $n = 1$ along with $x_2 = (2\lambda - 1)x_1$ implies $b = ca$, where c is a complex number which depends on λ and $c \neq 0$. The precise value of c is not important. That $c \neq 0$ is enough for the rest. Thus we have

$$x_n = a\left[\left(\lambda + \sqrt{\lambda^2 - 1}\right)^n + c\left(\lambda - \sqrt{\lambda^2 - 1}\right)^n\right], \qquad n \geqslant 1.$$

At this point, we again use the fact that $\mathbf{x} \in \ell^p$. Since

$$\left(\lambda + \sqrt{\lambda^2 - 1}\right) \times \left(\lambda - \sqrt{\lambda^2 - 1}\right) = 1,$$

at least one of the sums

$$\sum_{n=1}^{\infty} \left(\lambda + \sqrt{\lambda^2 - 1}\right)^{pn} \quad \text{or} \quad \sum_{n=1}^{\infty} \left(\lambda - \sqrt{\lambda^2 - 1}\right)^{pn}$$

diverges. Hence, in order to ensure

$$\sum_{n=1}^{\infty} |x_n|^p < \infty,$$

we must have $a = 0$, which again leads to $\mathbf{x} = 0$. The above three cases show that $\ker(T) = \{0\}$.

We continue with the proof of Theorem 2.1. Part (i) is straightforward. By (3.1), we know that $\Omega_{\infty,I} \subseteq \overline{\mathbb{D}}$. To verify $\overline{\mathbb{D}} \subseteq \Omega_{\infty,I}$, let $\lambda \in \overline{\mathbb{D}}$ and consider two cases.

Case I, $|\lambda| < 1$: It is clear that

$$\{e^{i\frac{2\pi k}{n}} : n \geqslant 1, 0 \leqslant k \leqslant n-1\} = \{e^{i\pi r} : r \in \mathbb{Q}\}.$$

Hence, by (1.5),

$$\cup_{n\geqslant 1}\Pi_n = \mathbb{D} \cup \{e^{i\pi r} : r \in \mathbb{Q}\} \subset \cup_{n\geqslant 1}\Omega_n.$$

Therefore, by (3.2), we conclude that

$$\mathbb{D} \cup \{e^{i\pi r} : r \in \mathbb{Q}\} \subseteq \Omega_{p,I} \subseteq \overline{\mathbb{D}}, \qquad 1 \leqslant p \leqslant \infty. \tag{3.3}$$

In particular, we have $\mathbb{D} \subset \Omega_{\infty,I}$.

Case II, $|\lambda| = 1$: Since I is infinite, without loss of generality, we assume that $I = \mathbb{Z} \sqcup I_0$, where $\sqcup$ denotes the disjoint union. Let $T : \ell^\infty(I) \to \ell^\infty(I)$ be the right shift on $\mathbb{Z}$ and the identity operator on I_0, i.e.,

$$(T\mathbf{x})(k) = \begin{cases} \mathbf{x}(k+1) & \text{if } k \in \mathbb{Z}, \\ \mathbf{x}(k) & \text{if } k \in I_0. \end{cases}$$

It is easy to see that T is a d.s. operator. We claim that

$$\sigma_{\wp}(T) = \partial\mathbb{T}. \tag{3.4}$$

To verify this, fix $\lambda \in \partial\mathbb{D}$ and let $\mathbf{x} : I \to \mathbb{R}$ be defined by

$$\mathbf{x}(k) = \begin{cases} \lambda^k & \text{if } k \in \mathbb{Z}, \\ 0 & \text{if } k \in I_0. \end{cases}$$

Clearly, $\mathbf{x} \in \ell^\infty(I)$, and

$$(T\mathbf{x})(k) = \mathbf{x}(k+1) = \lambda^{k+1} = \lambda\mathbf{x}(k), \qquad k \in \mathbb{Z},$$

and

$$(T\mathbf{x})(k) = \mathbf{x}(k) = 0 = \lambda\mathbf{x}(k), \qquad k \in I_0.$$

Therefore, $T\mathbf{x} = \lambda\mathbf{x}$, i.e., λ is even an eigenvalue of T. In particular, $\partial\mathbb{D} \subset \Omega_{\infty,I}$.

Now suppose that $\lambda \in \sigma_{\wp}(T)$ with the corresponding eigenvector $\mathbf{x}$. We decompose the eigenvector as $\mathbf{x} = \mathbf{x}' + \mathbf{x}''$, where $\mathbf{x}' \in \ell^\infty(\mathbb{Z})$ and $\mathbf{x}'' \in \ell^\infty(I_0)$. Therefore, the equation $T\mathbf{x} = \lambda\mathbf{x}$ entails to two sets of equations:

$$\mathbf{x}'(n+1) = \lambda\mathbf{x}'(n), \qquad n \in \mathbb{Z},$$

and

$$\mathbf{x}''(n) = \lambda\mathbf{x}''(n), \qquad n \in I_0.$$

Since $\mathbf{x} \neq 0$, it follows that either $\mathbf{x}' \neq 0$ or $\mathbf{x}'' \neq 0$. In the latter case, $\lambda = 1 \in \partial\mathbb{D}$. In the former, the equation reduces to

$$\mathbf{x}'(n) = \lambda^n \mathbf{x}'(0), \qquad n \in \mathbb{Z},$$

and the requirement $\mathbf{x}' \in \ell^\infty(\mathbb{Z})$ is fulfilled if and only if $|\lambda| = 1$. Hence, (3.4) is established. Cases I and II together show that $\Omega_{\infty,I} = \overline{\mathbb{D}}$.

Part (ii)*:* According to (3.3),

$$\mathbb{D} \cup \{\mathrm{e}^{i\pi r} : r \in \mathbb{Q}\} \subseteq \Omega_{p,I}, \qquad 1 \leqslant p < \infty. \tag{3.5}$$

Recall that $\Omega_{p,I} \subset \overline{\mathbb{D}}$. To show the equality in (3.5), suppose that $\mathrm{e}^{i\theta}$ is an eigenvalue of some d.s. operator $T = [t_{rs}] : \ell^p(I) \to \ell^p(I)$, with the corresponding eigenvector $\mathbf{x} = (x_j)_{j\in I} \in \ell^p(I)$, i.e.,

$$T\mathbf{x} = \mathrm{e}^{i\theta}\mathbf{x}. \tag{3.6}$$

Without loss of generality, assume that $\sup_{j\in I} |x_j| = 1$. Since $\mathbf{x} \in \ell^p(I)$, with $p < \infty$, then $\mathcal{E} := \{j \in I : \ |x_j| = 1\}$ is a non-empty finite subset of I, and

$$c := \max_{j\notin\mathcal{E}} |x_j| < 1.$$

Put $\mathcal{E} = \{i_1, \dots, i_N\}$.

The i_1 th row of (3.6) is

$$\mathrm{e}^{i\theta} x_{i_1} = \sum_{j\in I} t_{i_1 j} x_j = \sum_{j\in\mathcal{E}} t_{i_1 j} x_j + \sum_{j\notin\mathcal{E}} t_{i_1 j} x_j. \tag{3.7}$$

Thus,

$$\begin{aligned} 1 = |\mathrm{e}^{i\theta} x_{i_1}| &\leqslant \sum_{j\in\mathcal{E}} t_{i_1 j} |x_j| + \sum_{j\notin\mathcal{E}} t_{i_1 j} |x_j| \\ &\leqslant \sum_{j\in\mathcal{E}} t_{i_1 j} + c \sum_{j\notin\mathcal{E}} t_{i_1 j}, \end{aligned}$$

which implies

$$\sum_{j\notin\mathcal{E}} t_{i_1 j} \leqslant c \sum_{j\notin\mathcal{E}} t_{i_1 j}.$$

This is only possible if

$$t_{i_1 j} = 0, \qquad j \notin \mathcal{E}.$$

Back to the i_1 th row of (3.6), which now becomes

$$\mathrm{e}^{i\theta} x_{i_1} = \sum_{j\in\mathcal{E}} t_{i_1 j} x_j. \tag{3.8}$$

But the points of unit circle are extreme points of the closed unit disc. Hence, (3.8) implies $\mathrm{e}^{i\theta} x_{i_1} \in \mathcal{E}$. Similarly, repeating the above argument for the row i_k gives

$$\mathrm{e}^{i\theta} x_{i_k} \subseteq \mathcal{E}, \qquad 1 \leqslant k \leqslant N,$$

and thus $e^{i\theta}\mathcal{E} \subseteq \mathcal{E}$. Since both sides have the same cardinality, we deduce the crucial identity $e^{i\theta}\mathcal{E} = \mathcal{E}$, which by induction gives

$$e^{in\theta}\mathcal{E} = \mathcal{E}, \qquad n \in \mathbb{Z}.$$

In return, the above identity immediately implies $\theta = \pi r$ for some rational real number r. For this, we used a well-known result of real and complex analysis: Let $r \in \mathbb{R}$. Then the set $\{e^{in\pi r} : n \in \mathbb{Z}\}$ is a dense subset of $\partial\mathbb{D}$ if and only if r is an irrational number [12]. This finishes the proof of Theorem 2.1.

The above results show the eigenvalues location of d.s. operators on $\ell^p(I)$ is independent of the choice of infinite set I. More explicitly, if A, B are two infinite set and $1 \leqslant p, q < \infty$, then

(i) $\Omega_{p,A} = \Omega_{q,B} = \cup_{n=1}^{\infty}\Omega_n$,

(ii) $\Omega_{\infty,A} = \Omega_{\infty,B} = \overline{\mathbb{D}}$.

4. The proof of Theorem 2.2

If I is a finite index set with n elements, then it is clear that $\widetilde{\Omega}_{p,I} = \Omega_{p,I} = \Omega_n$. In this section, we show when I is an infinite set, then $\widetilde{\Omega}_{p,I} = \overline{\mathbb{D}}$. This is an interesting result, since despite being possible that $\sigma_{\wp}(T) = \emptyset$, due to Gelfand's theorem, $\sigma(T)$ is always a non-empty compact subset of the complex plane. See [18, Theorem 1.2.5].

It is well-known that $\sigma(T)$ is confined in a closed disc of radius $\|T\|$. Hence, by (1.4), we rather easily see that

$$\widetilde{\Omega}_{p,I} \subseteq \overline{\mathbb{D}}.$$

Let $\lambda \in \overline{\mathbb{D}}$. We consider the following two cases.

(i) $|\lambda| < 1$: From 3.3, we have

$$\lambda \in \Omega_{p,I} \subseteq \widetilde{\Omega}_{p,I}.$$

(ii) $|\lambda| = 1$: Let

$$\lambda_k = \frac{k}{k+1}\lambda, \qquad k \geqslant 1.$$

By part (i), $\lambda_k \in \Omega_{p,I}$. More explicitly, λ_k is the eigenvalue of an $n_k \times n_k$ d.s. matrix T_k. Without loss of generality, we may assume that $I = \mathbb{N} \sqcup I_0$. Then let

$$T = \left(T_1 \bigoplus T_2 \bigoplus \cdots\right) \bigoplus \mathrm{id}_{I_0} = \left[\begin{array}{ccc|c} \boxed{T_1} & & & \\ & \boxed{T_2} & & 0 \\ & & \ddots & \\ \hline & 0 & & \mathrm{id}_{I_0} \end{array}\right].$$

It is clear that $T \in \mathcal{DS}(\ell^p(I))$ and that

$$\{\lambda_k : k \geqslant 1\} \subseteq \sigma_{\wp}(T) \subseteq \sigma(T).$$

Since $\sigma(T)$ is compact, we conclude that $\lambda \in \overline{\{\lambda_k : k \geqslant 1\}} \subseteq \sigma(T)$. Therefore, $\lambda \in \widetilde{\Omega}_{p,I}$.

The above two cases show $\widetilde{\Omega}_{p,I} = \overline{\mathbb{D}}$.

5. The proof of Theorem 2.3

(ii): We need a simple fact from set theory. Since $\operatorname{card}(E) \leqslant \operatorname{card}(I)$, there is a partition $\{I_\lambda : \lambda \in E\}$ of I such that

$$\operatorname{card}(I_\lambda) = \operatorname{card}(I), \qquad \lambda \in E.$$

Then, by Theorem 2.1, for each $\lambda \in E$, there is a d.s. operator $T_\lambda : \ell^p(I_\lambda) \to \ell^p(I_\lambda)$ such that

$$\lambda \in \sigma_\wp(T_\lambda).$$

Note that T_λ might have other eigenvalues. However, this does not affect our final conclusion. Now set

$$T = \bigoplus_{\lambda \in E} T_\lambda.$$

Then $T \in \mathcal{DS}(\ell^p(I))$ and

$$\sigma_\wp(T) = \bigcup_{\lambda \in E} \sigma_\wp(T_\lambda) \supseteq \bigcup_{\lambda \in E} \{\lambda\} = E.$$

(ii): Take any countable dense subset $E \subseteq \overline{\mathbb{D}}$. Since $\aleph_0 = \operatorname{card}(E) \leqslant \operatorname{card}(I)$, according to part (i), there is a d.s. operator $T \in \mathcal{DS}(\ell^p(I))$ such that

$$E \subset \sigma_\wp(T).$$

Then the chain of inclusions

$$E \subset \sigma_\wp(T) \subseteq \sigma(T) \subseteq \overline{\mathbb{D}},$$

show that $\overline{E} = \sigma(T) = \overline{\mathbb{D}}$.

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