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ON WEAVING FRAMES IN HILBERT SPACES

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Abstract. In this paper, we obtain some new properties of weaving frames and present some conditions under which a family of frames is woven in Hilbert spaces. Some characterizations of weaving frames in terms of operators are given. We also give a condition associated with synthesis operators of frames such that the sequence of frames is woven. Finally, for a family of woven frames, we show that they are stable under invertible operators and small perturbations.

1. Introduction

Frames in Hilbert spaces were first introduced by Duffin and Schaeffer [10] for studying some problems in nonharmonic Fourier series, reintroduced in 1986 by Daubechies, Grossman, and Meyer [9] and popularized from then on. Redundancy of frames is one of the key features that are important in both theory and application, and it provides flexibility on constructions of various classes of frames. Just as the nice properties of frames, frames have been applied to wide range of science and technology fields such as signal processing [14], coding theory [4, 13, 15], sampling theory [16], quantum measurements [1, 11] and image processing [6], etc.

Let $\mathcal{H}$ be a separable space and J a countable index set. A sequence $\{f_j\}_{i\in J}$ of elements of $\mathcal{H}$ is a frame for $\mathcal{H}$ if there exist constants $A,\ B>0$ such that

$$A\|f\|^2 \leqslant \sum_{j\in J} \left|\langle f, f_j\rangle\right|^2 \leqslant B\|f\|^2, \quad \forall f\in\mathcal{H}.$$

The numbers $A,\ B$ are called lower and upper frame bounds, respectively. If $A=B$, then this frame is called an A-tight frame, and if $A=B=1$, then it is called a Parseval frame.

Suppose $\{f_j\}_{j\in J}$ is a frame for $\mathcal{H}$, then the frame operator is a self-adjoint positive invertible operator, which is given by

$$S:\mathcal{H}\to\mathcal{H},\ \ Sf=\sum_{j\in J}\langle f,f_j\rangle f_j.$$

The following reconstruction formula holds:

$$f=\sum_{j\in J}\langle f,f_j\rangle S^{-1}f_j=\sum_{j\in J}\langle f,S^{-1}f_j\rangle f_j,$$

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where the family $\{\widetilde{f_j}\}_{j\in J} = \{S^{-1}f_j\}_{j\in J}$ is also a frame for $\mathcal{H}$, which is called the canonical dual frame of $\{f_j\}_{j\in J}$. The frame $\{g_j\}_{j\in J}$ for $\mathcal{H}$ is called an alternate dual frame of $\{f_j\}_{j\in J}$ if the following formula holds:

$$f = \sum_{j\in J}\langle f, f_j\rangle g_j = \sum_{j\in J}\langle f, g_j\rangle f_j$$

for all $f \in \mathcal{H}$ [12].

Weaving frames were introduced in [2] and investigated in [3, 5]. The concept of weaving frames is motivated by distributed signal processing, which have potential applications in wireless sensor networks and quantum communications that require distributed processing under different frames, as well as pre-processing of signals using Gabor frames. For example, in wireless sensor network, let two frames $F = \{f_j\}_{j\in J}$ and $G = \{g_j\}_{j\in J}$ be measure tools. At each sensor, we encode signal f either with f_j or g_j, so the encode coefficients is the set of numbers $\{\langle f, f_j\rangle\}_{j\in\sigma} \cup \{\langle f, g_j\rangle\}_{j\in\sigma^c}$ for some $\sigma \subset J$. If $\{f_j\}_{j\in\sigma} \cup \{g_j\}_{j\in\sigma^c}$ is still a frame, then f can be recovered robustly from these coefficients.

But $\{f_j\}_{j\in\sigma} \cup \{g_j\}_{j\in\sigma^c}$ may or may not be a frame for $\mathcal{H}$ for any $\sigma \subset J$. For example, let $\{e_j\}_{j=1}^3$ be an orthonormal basis for $\mathcal{H}$, $F = \{f_j\}_{j=1}^3 = \{e_1, e_2, e_1 + e_3\}$ and $G = \{g_j\}_{j=1}^3 = \{e_1, e_3, e_1 + e_2\}$, then F and G are two frames for $\mathcal{H}$. If we choose $\sigma = \{1, 2\}$, then $\{f_j\}_{j\in\sigma} \cup \{g_j\}_{j\in\sigma^c} = \{e_1, e_2, e_1 + e_2\}$ is not a frame for $\mathcal{H}$.

What is the condition such that $\{f_j\}_{j\in\sigma} \cup \{g_j\}_{j\in\sigma^c}$ is a frame for $\mathcal{H}$ for any $\sigma \subset J$?

In this paper, we give some sufficient conditions under that a family of frames is woven in Hilbert spaces, and we also consider those perturbations that can be applied to woven frames so as to leaves them woven.

Throughout the paper, let $[m] = \{1, 2, \cdots, m\}$. We first recall some concept and properties of woven frames.

DEFINITION 1. [2] A family of frames $\{F_i = \{f_{ij}\}_{j\in J}\}_{i\in[m]}$ for $\mathcal{H}$ is said to be woven if there are universal constants A and B such that for every partition $\{\sigma_i\}_{i\in[m]}$ of J the family $\{f_{ij}\}_{j\in\sigma_i, i\in[m]}$ is a frame for $\mathcal{H}$ with lower and upper frame bounds A and B, respectively. And $\{f_{ij}\}_{j\in\sigma_i, i\in[m]}$ is called a weaving frame (or a weaving).

The following proposition gives that every weaving automatically has a universal upper frame bound.

PROPOSITION 1. [2] *If each $F_i = \{f_{ij}\}_{j\in J}$ is a Bessel sequence for $\mathcal{H}$ with bound B_i for all $i \in [m]$, then every weaving is a Bessel sequence with $\sum_{i\in[m]} B_i$ as a Bessel bound.*

Let $\{\sigma_i\}_{i\in[m]}$ be any partition of J, now we define the space:

$$\bigoplus_{i\in[m]} \ell^2(\sigma_i) = \left\{ \{c_{ij}\}_{j\in\sigma_i, i\in[m]} | c_{ij} \in \mathbb{C},\ \sigma_i \subset J,\ i \in [m],\ \sum_{j\in\sigma_i, i\in[m]} |c_{ij}|^2 < \infty \right\},$$

with the inner product

$$\left\langle \{c_{ij}\}_{j\in\sigma_i,i\in[m]}, \{d_{ij}\}_{j\in\sigma_i,i\in[m]} \right\rangle = \sum_{j\in\sigma_i,i\in[m]} |c_{ij}\overline{d_{ij}}|,$$

it is clear that $\bigoplus_{i\in[m]} \ell^2(\sigma_i)$ is a Hilbert space.

Let the family of frames $\{F_i = \{f_{ij}\}_{j\in J}\}_{i\in[m]}$ be woven for $\mathcal{H}$, for any partition $\{\sigma_i\}_{i\in[m]}$ of J, $W = \{f_{ij}\}_{j\in\sigma_i,i\in[m]}$ is a frame for $\mathcal{H}$, the operator $T_W : \bigoplus_{i\in[m]} \ell^2(\sigma_i) \to \mathcal{H}$ defined by

$$T_W(\{c_{ij}\}) = \sum_{i\in[m]} T_{F_i} D_{\sigma_i}(\{c_{ij}\}) = \sum_{i\in[m]} \sum_{j\in\sigma_i} c_{ij} f_{ij},$$

is called the synthesis operator, where T_{F_i} is the synthesis operator of F_i and D_{σ_i} is a $|J|\times|J|$ diagonal matrix with $d_{jj} = 1$ for $j \in \sigma_i$ and otherwise 0. The adjoint operator of T_W is given by:

$$T_W^* : \mathcal{H} \to \bigoplus_{i\in[m]} \ell^2(\sigma_i), \quad T_W^*(f) = \sum_{i\in[m]} D_{\sigma_i} T_{F_i}^{\sigma_i *}(f) = \{\langle f, f_{ij}\rangle\}_{j\in\sigma_i,i\in[m]}, \quad \forall f \in \mathcal{H},$$

and is called the analysis operator. The frame operator S_W is defined as

$$\begin{aligned} S_W : \mathcal{H} \to \mathcal{H}, \quad S_W(f) = T_W T_W^*(f) &= (\sum_{i\in[m]} T_{F_i} D_{\sigma_i})(\sum_{i\in[m]} T_{F_i} D_{\sigma_i})^*(f) \\ &= \sum_{i\in[m]} S_{F_i}^{\sigma_i}(f) = \sum_{j\in\sigma_i,i\in[m]} \langle f, f_{ij}\rangle f_{ij}, \end{aligned}$$

where S_{F_i} is the frame operator of F_i and $S_{F_i}^{\sigma_i}$ is a “truncated form” of S_{F_i}. The operator S_W is positive, self-adjoint and invertible.

2. Main results

We first give some properties of weaving frames. The following result can be direct obtained from Proposition 1.1.4 of [7].

PROPOSITION 2. *Let two frames $F = \{f_j\}_{j\in J}$ and $G = \{g_j\}_{j\in J}$ be woven with synthesis operators T_F and T_G, respectively. For any $\sigma \subset J$, a weaving $\{f_j\}_{j\in\sigma} \cup \{g_j\}_{j\in\sigma^c}$ is an A-tight frame for $\mathcal{H}$ if and only if $T_F D_\sigma T_F^* + T_G D_{\sigma^c} T_G^* = AI_{\mathcal{H}}$, which D_σ is a $|J|\times|J|$ diagonal matrix with $d_{jj} = 1$ for $j \in \sigma$ and otherwise 0, D_{σ^c} is a $|J|\times|J|$ diagonal matrix with $d_{jj} = 1$ for $j \in \sigma^c$ and otherwise 0.*

Proof. For any $\sigma \subset J$, then the synthesis of weaving frame $W = \{f_j\}_{j\in\sigma} \cup \{g_j\}_{j\in\sigma^c}$ is $T_F D_\sigma + T_G D_{\sigma^c}$. Then the frame operator

$$\begin{aligned} S_W &= (T_F D_\sigma + T_G D_{\sigma^c})(T_F D_\sigma + T_G D_{\sigma^c})^* \\ &= T_F D_\sigma T_F^* + T_F D_\sigma D_{\sigma^c} T_G^* + T_G D_{\sigma^c} D_\sigma T_F^* + T_G D_{\sigma^c} T_G^* \\ &= T_F D_\sigma T_F^* + T_G D_{\sigma^c} T_G^*. \quad \square \end{aligned}$$

If two frames are woven, one may ask whether their canonical dual frames are also woven. The following result presents a condition such that their canonical dual frames are also woven when two frames are woven.

PROPOSITION 3. *Let two frames* $F=\{f_j\}_{j\in J}$ *and* $G=\{g_j\}_{j\in J}$ *be woven with universal constants* A, B *and frame operators* S_F *and* S_G, *respectively.*

If $\|S_F\|\|S_F^{-1}-S_G^{-1}\|<\frac{A}{B}$ *(or* $\|S_G\|\|S_F^{-1}-S_G^{-1}\|<\frac{A}{B}$ *), then* $S_F^{-1}F=\{S_F^{-1}f_j\}_{j\in J}$ *and* $S_G^{-1}G=\{S_G^{-1}g_j\}_{j\in J}$ *are also woven.*

Proof. We only consider the case of $\|S_F\|\|S_F^{-1}-S_G^{-1}\|<\frac{A}{B}$. Now for every $\sigma\subset J$ and each $f\in\mathcal{H}$, we have

$$
\begin{aligned}
&\Big(\sum_{j\in\sigma}\big|\langle f,S_F^{-1}f_j\rangle\big|^2+\sum_{j\in\sigma^c}\big|\langle f,S_G^{-1}g_j\rangle\big|^2\Big)^{1/2}\\
&=\Big(\sum_{j\in\sigma}\big|\langle S_F^{-1}f,f_j\rangle\big|^2+\sum_{j\in\sigma^c}\big|\langle S_G^{-1}f,g_j\rangle\big|^2\Big)^{1/2}\\
&=\Big(\sum_{j\in\sigma}\big|\langle S_F^{-1}f,f_j\rangle\big|^2+\sum_{j\in\sigma^c}\big|\langle S_F^{-1}f+(S_G^{-1}-S_F^{-1})f,g_j\rangle\big|^2\Big)^{1/2}\\
&\geqslant\Big(\sum_{j\in\sigma}\big|\langle S_F^{-1}f,f_j\rangle\big|^2+\sum_{j\in\sigma^c}\big|\langle S_F^{-1}f,g_j\rangle\big|^2\Big)^{1/2}-\Big(\sum_{j\in\sigma^c}\big|\langle (S_G^{-1}-S_F^{-1})f,g_j\rangle\big|^2\Big)^{1/2}\\
&\geqslant\sqrt{A}\|S_F^{-1}f\|-\Big(\sum_{j\in J}\big|\langle (S_G^{-1}-S_F^{-1})f,g_j\rangle\big|^2\Big)^{1/2}\\
&\geqslant\sqrt{A}\|S_F^{-1}f\|-\sqrt{B}\|S_G^{-1}-S_F^{-1}\|\|f\|\\
&\geqslant\left(\frac{\sqrt{A}}{\|S_F\|}-\sqrt{B}\|S_G^{-1}-S_F^{-1}\|\right)\|f\|,
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{j\in\sigma}\big|\langle f,S_F^{-1}f_j\rangle\big|^2+\sum_{j\in\sigma^c}\big|\langle f,S_G^{-1}g_j\rangle\big|^2&\leqslant\sum_{j\in J}\big|\langle f,S_F^{-1}f_j\rangle\big|^2+\sum_{j\in J}\big|\langle f,S_G^{-1}g_j\rangle\big|^2\\
&=\sum_{j\in J}\big|\langle S_F^{-1}f,f_j\rangle\big|^2+\sum_{j\in J}\big|\langle S_G^{-1}f,g_j\rangle\big|^2\\
&\leqslant B(\|S_F^{-1}\|^2+\|S_G^{-1}\|^2)\|f\|^2.\quad\Box
\end{aligned}
$$

In Proposition 3, for any $\sigma\subset J$, $\{f_j\}_{j\in\sigma}\cup\{g_j\}_{j\in\sigma^c}$ is a frame for $\mathcal{H}$, and $\{S_F^{-1}f_j\}_{j\in\sigma}\cup\{S_G^{-1}g_j\}_{j\in\sigma^c}$ is also a frame, it should be noted that $\{S_F^{-1}f_j\}_{j\in\sigma}\cup\{S_G^{-1}g_j\}_{j\in\sigma^c}$ is not a dual frame of $\{f_j\}_{j\in\sigma}\cup\{g_j\}_{j\in\sigma^c}$ in general.

EXAMPLE 1. For two given frames $F=\{f_j\}_{j=1}^3$, $G=\{g_j\}_{j=1}^3$,

$$
F=\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right\},\quad G=\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\}.
$$

It is easy to verify that F and G are woven, and $S_F^{-1}F$ and $S_G^{-1}G$ are woven. Suppose $\sigma = \{1,2\}$, then the weaving is given by

$$W = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\},$$

$$\widetilde{W} = \{S_F^{-1}f_1, S_F^{-1}f_2, S_G^{-1}g_3\} = \left\{ \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{2} \end{bmatrix} \right\}.$$

We compute $T_W T_{\widetilde{W}}^* = \begin{bmatrix} 1 & -\frac{5}{6} \\ -\frac{2}{3} & \frac{7}{6} \end{bmatrix} \neq I_{2\times 2}$, thus $\widetilde{W}$ is not a dual frame of W.

The following result give a simple characterization of dual frames of a weaving.

PROPOSITION 4. *Suppose that the family of frames $\{F_i = \{f_{ij}\}_{j\in J}\}_{i\in[m]}$ is woven. Let $\{\sigma_i\}_{i\in[m]}$ be any partition of J, then the synthesis operator of dual frame of weaving frame $W = \{f_{ij}\}_{j\in\sigma_i, i\in[m]}$ is given by $S_W^{-1}T_W + U$, where $\sum_{i\in[m]} T_{F_i} D_{\sigma_i} U^* = 0$.*

Proof. A simple calculation yields this. □

We next give a characterization of weaving frames in terms of bounded linear operators.

THEOREM 2. *For $i \in [m]$, let $F_i = \{f_{ij}\}_{j\in J}$ be a sequence for $\mathcal{H}$. The following conditions are equivalent:*

(i) The family of sequences $\{F_i\}_{i\in[m]}$ is woven frames for $\mathcal{H}$.

(ii) For any partition $\{\sigma_i\}_{i\in[m]}$ of J, there exists $A > 0$ such that there exists a bounded linear operator $T : \bigoplus_{i\in[m]} \ell^2(\sigma_i) \to \mathcal{H}$ such that $T(u_{ij}) = f_{ij}$ for all $j \in \sigma_i, i \in [m]$, and $AI_{\mathcal{H}} \leqslant TT^*$, where $\{u_{ij}\}_{j\in\sigma_i, i\in[m]}$ is the standard orthonormal basis for $\bigoplus_{i\in[m]} \ell^2(\sigma_i)$.

Proof. (i)⇒(ii): Suppose A is a universal lower frame bound for the family of sequences $\{F_i\}_{i\in[m]}$, let T_W be the synthesis operator associated with $W = \{f_{ij}\}_{j\in\sigma_i, i\in[m]}$.

Choose $T = T_W$, then

$$T(u_{ij}) = T_W(u_{ij}) = \sum_{i\in[m]} T_{F_i} D_{\sigma_i}(u_{ij}) = f_{ij}, \quad j \in \sigma_i, i \in [m],$$

where $\{u_{ij}\}_{j\in\sigma_i, i\in[m]}$ is the standard orthonormal basis for $\bigoplus_{i\in[m]} \ell^2(\sigma_i)$.

Furthermore, for all $f \in \mathcal{H}$, we have

$$A\langle f, f\rangle = A\|f\|^2 \leqslant \sum_{j\in\sigma_i, i\in[m]} \left|\langle f, f_{ij}\rangle\right|^2 = \|T_W^*(f)\|^2 = \|T^*(f)\|^2 = \langle TT^*f, f\rangle.$$

This gives $AI_{\mathcal{H}} \leqslant TT^*$.

(ii)$\Rightarrow$(i): For any partition $\{\sigma_i\}_{i\in[m]}$ of J, for $\{c_{ij}\}\in\bigoplus_{i\in[m]}\ell^2(\sigma_i)$ and $T:\bigoplus_{i\in[m]}\ell^2(\sigma_i)\to\mathcal{H}$, we have

$$\begin{aligned}\langle T(\{c_{ij}\}),f\rangle &= \Big\langle T(\sum_{j\in\sigma_i,i\in[m]} c_{ij}u_{ij}),f\Big\rangle = \Big\langle \sum_{j\in\sigma_i,i\in[m]} c_{ij}Tu_{ij},f\Big\rangle\\ &= \Big\langle \sum_{j\in\sigma_i,i\in[m]} c_{ij}f_{ij},f\Big\rangle = \sum_{j\in\sigma_i,i\in[m]} c_{ij}\overline{\langle f,f_{ij}\rangle}.\end{aligned}$$

This gives

$$T^*(f)=\langle f,f_{ij}\rangle_{j\in\sigma_i,i\in[m]},\quad \forall f\in\mathcal{H}. \tag{2.1}$$

Since $AI_{\mathcal{H}}\leqslant TT^*$, by using (2.1), we have

$$A\|f\|^2\leqslant\langle TT^*f,f\rangle=\|T^*(f)\|^2=\sum_{i\in[m]}\sum_{j\in\sigma_i}\left|\langle f,f_{ij}\rangle\right|^2.$$

On the other hand, for any $f\in\mathcal{H}$, we have

$$\sum_{i\in[m]}\sum_{j\in\sigma_i}\left|\langle f,f_{ij}\rangle\right|^2=\|T^*f\|^2\leqslant\|T^*\|^2\|f\|^2,$$

Hence, $\{f_{ij}\}_{j\in\sigma_i,i\in[m]}$ is a frame for $\mathcal{H}$ and the family of sequences $\{F_i\}_{i\in[m]}$ is woven. □

We know that any pair of Bessel sequences can be extended to a pair of dual frames [8]. In fact, any two Bessel sequences can also be woven on certain condition. The following result gives a sufficient condition such that two Bessel sequences are woven.

THEOREM 3. *Let $F=\{f_j\}_{j\in J}$ and $G=\{g_j\}_{j\in J}$ be two Bessel sequences for $\mathcal{H}$ with synthesis operators T_F and T_G, respectively. For any $\sigma\subset J$, if $I_{\mathcal{H}}=T_FT_G^*=T_GT_F^*$ and $T_F^\sigma T_G^{\sigma*}=T_G^\sigma T_F^{\sigma*}$, then F and G are woven frames for $\mathcal{H}$, which T_F^σ and T_G^σ are "truncated form" of T_F and T_G for $\sigma\subset J$, respectively.*

Proof. Let B_1 and B_2 be Bessel bounds for F and G, respectively. For any $f\in\mathcal{H}$, $\sigma\subset J$, we have $f=\sum_{j\in J}\langle f,f_j\rangle g_j=\sum_{j\in J}\langle f,g_j\rangle f_j$, and $\sum_{j\in\sigma}\langle f,f_j\rangle g_j=\sum_{j\in\sigma}\langle f,g_j\rangle f_j$. By using $(a+b)^2\leqslant 2a^2+2b^2$, we compute

$$\begin{aligned}\|f\|^4&=|\langle f,f\rangle|^2\\ &=\left|\Big\langle\sum_{j\in J}\langle f,f_j\rangle g_j,f\Big\rangle\right|^2\\ &=\left|\Big\langle\sum_{j\in\sigma}\langle f,f_j\rangle g_j+\sum_{j\in\sigma^c}\langle f,f_j\rangle g_j,f\Big\rangle\right|^2\end{aligned}$$

$$
\begin{aligned}
&\leqslant 2\left|\left\langle \sum_{j\in\sigma}\langle f,f_j\rangle g_j,f\right\rangle\right|^2+2\left|\left\langle \sum_{j\in\sigma^c}\langle f,f_j\rangle g_j,f\right\rangle\right|^2\\
&=2\left|\left\langle \sum_{j\in\sigma}\langle f,f_j\rangle g_j,f\right\rangle\right|^2+2\left|\left\langle \sum_{j\in\sigma^c}\langle f,g_j\rangle f_j,f\right\rangle\right|^2\\
&=2\left|\sum_{j\in\sigma}\langle f,f_j\rangle\langle g_j,f\rangle\right|^2+2\left|\sum_{j\in\sigma^c}\langle f,g_j\rangle\langle f_j,f\rangle\right|^2\\
&\leqslant 2\sum_{j\in\sigma}\left|\langle f,f_j\rangle\right|^2\sum_{j\in\sigma}\left|\langle f,g_j\rangle\right|^2+2\sum_{j\in\sigma^c}\left|\langle f,g_j\rangle\right|^2\sum_{j\in\sigma^c}\left|\langle f,f_j\rangle\right|^2\\
&\leqslant 2B_2\|f\|^2\sum_{j\in\sigma}\left|\langle f,f_j\rangle\right|^2+2B_1\|f\|^2\sum_{j\in\sigma^c}\left|\langle f,g_j\rangle\right|^2\\
&\leqslant 2\max\{B_1,B_2\}\|f\|^2\left(\sum_{j\in\sigma}\left|\langle f,f_j\rangle\right|^2+\sum_{j\in\sigma^c}\left|\langle f,g_j\rangle\right|^2\right).
\end{aligned}
$$

Therefore, for all $f\in\mathcal{H}$, we have

$$
\frac{1}{2\max\{B_1,B_2\}}\|f\|^2\leqslant\sum_{j\in\sigma}\left|\langle f,f_j\rangle\right|^2+\sum_{j\in\sigma^c}\left|\langle f,g_j\rangle\right|^2\leqslant(B_1+B_2)\|f\|^2.
$$

Hence, F and G are woven. □

When a set of operators is applied to a frame, the following result gives a sufficient condition such that a set of frames are woven.

THEOREM 4. *Let $F=\{f_j\}_{j\in J}$ be a frame for $\mathcal{H}$ with bounds A, B, and $\{U_i\}_{i\in[m]}\subset L(\mathcal{H})$. For $k\in I$, if U_k has a left inverse $V\in L(\mathcal{H})$ and $\max\limits_{i\neq k}\|U_k-U_i\|<\sqrt{\frac{A}{(m-1)B}}\cdot\frac{1}{\|V\|}$, then the family of frames $\{U_iF\}_{i\in[m]}$ is woven.*

Proof. Since $VU_k=I_{\mathcal{H}}$, then $\|I_{\mathcal{H}}-VU_i\|=\|V(U_k-U_i)\|<\sqrt{\frac{A}{(m-1)B}}<\sqrt{\frac{A}{B}}\leqslant 1$. Therefore, VU_i is invertible for $i\in[m]$. Consequently, U_i has a left inverse. Hence, for any $i\in[m]$, U_i is bounded below by λ, i.e., $\lambda\|f\|\leqslant\|U_if\|$ for any $f\in\mathcal{H}$. Since $\lambda^2\cdot I_{\mathcal{H}}\leqslant U_i^*U_i$ and $A\cdot I_{\mathcal{H}}\leqslant S_F\leqslant B\cdot I_{\mathcal{H}}$, then

$$
A\lambda^2\cdot I_{\mathcal{H}}\leqslant A\cdot U_i^*U_i\leqslant S_{U_iF}\leqslant B\cdot U_i^*U_i\leqslant B\|U_i\|^2\cdot I_{\mathcal{H}}.
$$

Hence $\{U_if_j\}_{j\in J}$ is a frame for any $i\in[m]$. Let $\{\sigma_i\}_{i\in[m]}$ be any partition of J. Then, for every $f\in\mathcal{H}$ we have

$$
\begin{aligned}
\sum_{i\in[m]}\sum_{j\in\sigma_i}\left|\langle f,U_if_j\rangle\right|^2&=\sum_{i\in[m]}\sum_{j\in\sigma_i}\left|\langle U_i^*f,f_j\rangle\right|^2\leqslant\sum_{i\in[m]}\sum_{j\in J}\left|\langle f,U_if_j\rangle\right|^2\\
&\leqslant B\sum_{i\in[m]}\|U_i\|^2\|f\|^2.
\end{aligned}
$$

On the other hand,

$$\begin{aligned}
&\sum_{i\in[m]}\sum_{j\in\sigma_i}\left|\left\langle f,U_i f_j\right\rangle\right|^2\\
&=\sum_{j\in\sigma_1}\left|\left\langle f,U_1 f_j\right\rangle\right|^2+\cdots+\sum_{j\in\sigma_i}\left|\left\langle f,U_i f_j\right\rangle\right|^2+\cdots+\sum_{j\in\sigma_k}\left|\left\langle f,U_k f_j\right\rangle\right|^2\\
&\quad+\cdots+\sum_{j\in\sigma_m}\left|\left\langle f,U_m f_j\right\rangle\right|^2\\
&=\sum_{j\in\sigma_1}\left|\left\langle f,(U_1-U_k+U_k) f_j\right\rangle\right|^2+\cdots+\sum_{j\in\sigma_i}\left|\left\langle f,(U_i-U_k+U_k) f_j\right\rangle\right|^2\\
&\quad+\cdots+\sum_{j\in\sigma_k}\left|\left\langle f,U_k f_j\right\rangle\right|^2+\cdots+\sum_{j\in\sigma_m}\left|\left\langle f,(U_m-U_k+U_k) f_j\right\rangle\right|^2\\
&\geqslant\sum_{j\in\sigma_k}\left|\left\langle f,U_k f_j\right\rangle\right|^2+\sum_{j\in\sigma_i}\left|\left\langle f,U_k f_j\right\rangle\right|^2-\sum_{j\in\sigma_i}\left|\left\langle f,(U_i-U_k) f_j\right\rangle\right|^2\\
&\quad+\sum_{j\in\sigma_1}\left|\left\langle f,U_k f_j\right\rangle\right|^2-\sum_{j\in\sigma_1}\left|\left\langle f,(U_1-U_k) f_j\right\rangle\right|^2\\
&\quad+\cdots+\sum_{j\in\sigma_m}\left|\left\langle f,U_k f_j\right\rangle\right|^2-\sum_{j\in\sigma_m}\left|\left\langle f,(U_m-U_k) f_j\right\rangle\right|^2\\
&\geqslant\sum_{j\in J}\left|\left\langle f,U_k f_j\right\rangle\right|^2-\sum_{i\neq k}\sum_{j\in\sigma_i}\left|\left\langle f,(U_i-U_k) f_j\right\rangle\right|^2\\
&=\sum_{j\in J}\left|\left\langle U_k^* f,f_j\right\rangle\right|^2-\sum_{i\neq k}\sum_{j\in\sigma_i}\left|\left\langle f,(U_i-U_k) f_j\right\rangle\right|^2\\
&\geqslant A\|U_k^* f\|^2-\sum_{i\neq k}\sum_{j\in J}\left|\left\langle f,(U_i-U_k) f_j\right\rangle\right|^2\\
&\geqslant \frac{A}{\|V\|^2}\|f\|^2-B\sum_{i\neq k}\|U_i-U_k\|^2\|f\|^2\\
&\geqslant\left(\frac{A}{\|V\|^2}-(m-1)B\max_{i\neq k}\|U_i-U_k\|^2\right)\|f\|^2.
\end{aligned}$$

This completes the proof. □

Next, we present a characterization of weaving frames by synthesis operators of frames.

THEOREM 5. *For* $i\in[m]$, *let* $F_i=\{f_{ij}\}_{j\in J}$ *be a frame for* $\mathcal{H}$ *with bounds* A_i, B_i. *Assume for any* $k\in I$, $\|T_{F_i}-T_{F_k}\|<\frac{A_k}{(m-1)(\sqrt{B_i}+\sqrt{B_k})}$, *then the family of frames* $\{F_i\}_{i\in[m]}$ *is woven.*

Proof. It is clear that the family $\{f_{ij}\}_{j\in\sigma_i,i\in[m]}$ is a Bessel sequence with Bessel bound $m\max\limits_{i\in[m]}B_i$. For any $f\in\mathcal{H}$,

$$\|T_{F_i}^{\sigma_i}f\|^2=\sum_{j\in\sigma_i}\left|\left\langle f,f_{ij}\right\rangle\right|^2\leqslant\sum_{j\in J}\left|\left\langle f,f_{ij}\right\rangle\right|^2=\|T_{F_i}f\|^2\leqslant\|T_{F_i}\|^2\|f\|^2,$$

hence $\|T_{F_i}^{\sigma_i}\| \leqslant \|T_{F_i}\|$. And then

$$\begin{aligned}\|T_{F_i}^{\sigma_i}T_{F_i}^{\sigma_i *} - T_{F_k}^{\sigma_i}T_{F_k}^{\sigma_i *}\| &= \|T_{F_i}^{\sigma_i}T_{F_i}^{\sigma_i *} - T_{F_i}^{\sigma_i}T_{F_k}^{\sigma_i *} + T_{F_i}^{\sigma_i}T_{F_k}^{\sigma_i *} - T_{F_k}^{\sigma_i}T_{F_k}^{\sigma_i *}\| \\ &\leqslant \|T_{F_i}^{\sigma_i}(T_{F_i}^{\sigma_i *} - T_{F_k}^{\sigma_i *})\| + \|(T_{F_i}^{\sigma_i} - T_{F_k}^{\sigma_i})T_{F_k}^{\sigma_i *}\| \\ &\leqslant \|T_{F_i}^{\sigma_i}\|\|T_{F_i} - T_{F_k}\| + \|T_{F_i} - T_{F_k}\|\|T_{F_k}\| \\ &\leqslant (\sqrt{B_i} + \sqrt{B_k})\|T_{F_i} - T_{F_k}\|.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{i\in[m]} S_{F_i}^{\sigma_i} &= \sum_{i\in[m]\setminus\{k\}} S_{F_i}^{\sigma_i} + S_{F_k}^{\sigma_k} \\ &= \sum_{i\in[m]\setminus\{k\}} S_{F_i}^{\sigma_i} + \left(S_{F_k} - \sum_{i\in[m]\setminus\{k\}} S_{F_k}^{\sigma_i}\right) \\ &= S_{F_k} + \sum_{i\in[m]\setminus\{k\}} (S_{F_i}^{\sigma_i} - S_{F_k}^{\sigma_i}) \\ &\geqslant A_k \cdot I_{\mathcal{H}} - \sum_{i\in[m]\setminus\{k\}} \|S_{F_i}^{\sigma_i} - S_{F_k}^{\sigma_i}\| \cdot I_{\mathcal{H}} \\ &\geqslant (A_k - \sum_{i\in[m]\setminus\{k\}} (\sqrt{B_i} + \sqrt{B_k})\|T_{F_i} - T_{F_k}\|) \cdot I_{\mathcal{H}}.\end{aligned}$$

Hence, the sequence $\{f_{ij}\}_{j\in\sigma_i, i\in[m]}$ is a frame for $\mathcal{H}$, and the family of frames $\{F_i\}_{i\in[m]}$ is woven. □

THEOREM 6. *For $i \in [m]$, let $F_i = \{f_{ij}\}_{j\in J}$ be a frame for $\mathcal{H}$ with bounds A_i, B_i. For any $\sigma \subset J$ and a fixed $k \in I$, let $P_i^{\sigma}(f) = \sum_{j\in\sigma}\langle f, f_{ij}\rangle f_{ij} - \sum_{j\in\sigma}\langle f, f_{kj}\rangle f_{kj}$ for $i \neq k$. If P_i^{σ} is a positive linear operator, then the family of frames $\{F_i\}_{i\in[m]}$ is woven.*

Proof. Let $\{\sigma_i\}_{i\in[m]}$ be any partition of J. Then, for every $f \in \mathcal{H}$, we have

$$\begin{aligned}A_k\|f\|^2 &\leqslant \sum_{j\in J} \left|\langle f, f_{kj}\rangle\right|^2 = \sum_{i\in[m]}\sum_{j\in\sigma_i} |\langle f, f_{kj}\rangle|^2 \\ &= \sum_{i\in[m]} \langle \sum_{j\in\sigma_i} \langle f, f_{kj}\rangle f_{kj}, f\rangle = \sum_{i\in[m]} \langle \sum_{j\in\sigma_i} \langle f, f_{ij}\rangle f_{ij} - P_i^{\sigma_i}(f), f\rangle \\ &= \sum_{i\in[m]} \langle \sum_{j\in\sigma_i} \langle f, f_{ij}\rangle f_{ij}, f\rangle - \sum_{i=1}^{m} \langle P_i^{\sigma_i}(f), f\rangle \\ &\leqslant \sum_{i\in[m]} \langle \sum_{j\in\sigma_i} \langle f, f_{ij}\rangle f_{ij}, f\rangle = \sum_{i\in[m]}\sum_{j\in\sigma_i} |\langle f, f_{ij}\rangle|^2 \\ &\leqslant \sum_{i\in[m]}\sum_{j\in J} |\langle f, f_{ij}\rangle|^2 \leqslant \left(\sum_{i\in[m]} B_i\right)\|f\|^2.\end{aligned}$$

thus,

$$A_k\|f\|^2 \leqslant \sum_{i\in[m]}\sum_{j\in\sigma_i}\left|\langle f, f_{ij}\rangle\right|^2 \leqslant \sum_{i\in[m]} B_i\|f\|^2.$$

The proof is completed. □

Finally, we consider the perturbation of a set of frames that is woven. We first need the following definition.

DEFINITION 2. Let $F=\{f_j\}_{j\in J}$ and $G=\{g_j\}_{j\in J}$ be sequences in Hilbert space $\mathcal{H}$, let $0<\mu,\lambda<1$. If

$$\sum_{j\in J}\left|\langle f, f_j-g_j\rangle\right|^2 \leqslant \lambda\sum_{j\in J}\left|\langle f, f_j\rangle\right|^2+\mu\|f\|^2.$$

then we say that G is a (λ,μ)-perturbation of F.

THEOREM 7. *For $i\in[m]$, let $F_i=\{f_{ij}\}_{j\in J}$ be a frame for $\mathcal{H}$ with bounds A_i, B_i. For a fixed $k\in I$, let F_i be the (λ_i,μ_i)-perturbation of F_k for all $i\in[m]$. If*

$$\sum_{i\neq k}\lambda_i<1 \text{ and } A_k>\frac{\sum_{i\neq k}\mu_i}{1-\sum_{i\neq k}\lambda_i},$$

then the family of frames $\{F_i\}_{i\in[m]}$ is woven.

Proof. Let $\{\sigma_i\}_{i\in[m]}$ be any partition of J, observe that

$$\sum_{j\in\sigma_i}\left|\langle f, f_{kj}-f_{ij}\rangle\right|^2 \leqslant \sum_{j\in J}\left|\langle f, f_{kj}-f_{ij}\rangle\right|^2.$$

For every $f\in\mathcal{H}$, we have

$$\begin{aligned}\left(\sum_{i\in[m]}B_i\right)\|f\|^2 &\geqslant \sum_{i=1}^{m}\sum_{j\in\sigma_i}\left|\langle f, f_{ij}\rangle\right|^2\\ &= \sum_{j\in\sigma_k}\left|\langle f, f_{kj}\rangle\right|^2+\sum_{i\neq k}\sum_{j\in\sigma_i}\left|\langle f, f_{ij}\rangle\right|^2\\ &= \sum_{j\in\sigma_k}\left|\langle f, f_{kj}\rangle\right|^2+\sum_{i\neq k}\sum_{j\in\sigma_i}\left|\langle f, f_{ij}-f_{kj}+f_{kj}\rangle\right|^2\\ &\geqslant \sum_{j\in\sigma_k}\left|\langle f, f_{kj}\rangle\right|^2+\sum_{i\neq k}\sum_{j\in\sigma_i}\left|\langle f, f_{kj}\rangle\right|^2-\sum_{i\neq k}\sum_{j\in\sigma_i}\left|\langle f, f_{ij}-f_{kj}\rangle\right|^2\\ &= \sum_{i\in[m]}\sum_{j\in\sigma_k}\left|\langle f, f_{kj}\rangle\right|^2-\sum_{i\neq k}\sum_{j\in\sigma_i}\left|\langle f, f_{ij}-f_{kj}\rangle\right|^2\end{aligned}$$

$$\geqslant \sum_{j\in J}\left|\langle f,f_{kj}\rangle\right|^2-\sum_{i\neq k}\sum_{j\in J}\left|\langle f,f_{ij}-f_{kj}\rangle\right|^2$$

$$\geqslant \sum_{j\in J}\left|\langle f,f_{kj}\rangle\right|^2-\sum_{i\neq k}\left[\lambda_i\sum_{j\in J}\left|\langle f,f_{kj}\rangle\right|^2+\mu_i\|f\|^2\right]$$

$$\geqslant \left[\left(1-\sum_{i\neq k}\lambda_i\right)A_k-\sum_{i\neq k}\mu_i\right]\|f\|^2.$$

Hence the family of frames $\{F_i\}_{i\in[m]}$ is woven with bounds $(1-\sum_{i\neq k}\lambda_1)A_k-\sum_{i\neq k}\mu_i$ and $\sum_{i\in[m]}B_i$. □

EXAMPLE 8. Let $\{e_j\}_{j=1}^3$ be an orthonormal basis for $\mathbb{R}^3$, let $g_1=e_1$, $g_2=e_2, g_3=e_1+e_2$, and let $f_j=\frac{3}{2}g_j$, $h_j=\frac{1}{2}g_j$ for all $j=1,2,3$. Then $F=\{f_j\}_{j=1}^3$, $G=\{g_j\}_{j=1}^3$ and $H=\{h_j\}_{j=1}^3$ are frames for $\mathcal{H}$ with frame bounds $(\frac{9}{4},\frac{27}{4})$, $(1,3)$ and $(\frac{1}{4},\frac{3}{4})$, respectively.

Choose $\lambda_1=\frac{1}{9}$, $\mu_1=\frac{1}{9}$ and $\lambda_2=\frac{4}{9}$, $\mu_2=\frac{2}{9}$. Then $\lambda_1+\lambda_2<1$ and $A_1>\frac{\mu_1+\mu_2}{1-(\lambda_1+\lambda_2)}$. For any $f\in\mathcal{H}$, we compute

$$\sum_{j=1}^3\left|\langle f,f_j-g_j\rangle\right|^2=\frac{1}{4}\sum_{j=1}^3\left|\langle f,g_j\rangle\right|^2=\frac{1}{9}\sum_{j=1}^3\left|\langle f,f_j\rangle\right|^2$$
$$\leqslant \lambda_1\sum_{j=1}^3\left|\langle f,f_j\rangle\right|^2+\mu_1\|f\|^2,$$

and

$$\sum_{j=1}^3\left|\langle f,f_j-h_j\rangle\right|^2=\sum_{j=1}^3\left|\langle f,g_j\rangle\right|^2=\frac{4}{9}\sum_{j=1}^3\left|\langle f,f_j\rangle\right|^2$$
$$\leqslant \lambda_2\sum_{j=1}^3\left|\langle f,f_j\rangle\right|^2+\mu_2\|f\|^2.$$

Hence by Theorem 7, F, G and H are woven. In fact, for any partition $\{\sigma_1,\sigma_2,\sigma_3\}$ of $\mathbb{N}$,

$$\sum_{j\in\sigma_1}\left|\langle f,f_j\rangle\right|^2+\sum_{j\in\sigma_2}\left|\langle f,g_j\rangle\right|^2+\sum_{j\in\sigma_3}\left|\langle f,h_j\rangle\right|^2$$
$$=\sum_{j\in\mathbb{N}}\left|\left\langle f,d_j^{\sigma_1}f_j+d_j^{\sigma_2}g_j+d_j^{\sigma_3}h_j\right\rangle\right|^2$$
$$=\sum_{j\in\mathbb{N}}\left|\left\langle f,\left(\frac{3d_j^{\sigma_1}}{2}+d_j^{\sigma_2}+\frac{d_j^{\sigma_3}}{2}\right)(g_j)\right\rangle\right|^2,$$

where $d_j^{\sigma_i}=1$ for $j\in\sigma_i$ $(i=1,2,3)$ and otherwise 0. Thus $\{f_j\}_{j\in\sigma_1}\cup\{g_j\}_{j\in\sigma_2}\cup\{h_j\}_{j\in\sigma_3}$ is a frame for $\mathcal{H}$.

In the following result, we give some conditions that under those, perturbation of wovens are woven again. The following definition is needed.

DEFINITION 3. Let $F=\{f_j\}_{j\in J}$ and $G=\{g_j\}_{j\in J}$ be sequences in Hilbert space $\mathcal{H}$, let $0<\lambda<1$. Let $\{c_j\}_{j\in J}$ be an arbitrary sequence of positive numbers such that $\sum_{j\in J} c_j^2<\infty$. If

$$\Big\|\sum_{j\in J} c_j(f_j-g_j)\Big\| \leqslant \lambda\|\{c_j\}_{j\in J}\|, \tag{2.2}$$

then we say that G is a λ-perturbation of F.

THEOREM 9. *Let the family of frames $\{F_i=\{f_{ij}\}_{j\in J}\}_{i\in[m]}$ be woven with bounds A,B, and $F_i^{'}=\{f_{ij}^{'}\}_{j\in J}$ be λ_i-perturbation of F_i. If $\lambda_i<\frac{A}{2\sqrt{mB}}$ for all $i\in[m]$, then the family of frames $\{F_i^{'}\}_{i\in[m]}$ is woven in Hilbert space $\mathcal{H}$.*

Proof. Let T_{F_i} be the synthesis operator of F_i, by (2.2) it follows

$$\|T_{F_i}-T_{F_i'}\|\leqslant\lambda_i.$$

Let $\{\sigma_i\}_{i\in[m]}$ be any partition of J, for every $f\in\mathcal{H}$

$$A\|f\|^2\leqslant\sum_{i\in[m]}\|T_{F_i}^{\sigma_i*}(f)\|^2\leqslant B\|f\|^2.$$

Therefore,

$$\begin{aligned}\sum_{i\in[m]}\Big\|T_{F_i'}^{\sigma_i*}(f)\Big\|^2&=\sum_{i\in[m]}\Big\|T_{F_i'}^{\sigma_i*}(f)-T_{F_i}^{\sigma_i*}(f)+T_{F_i}^{\sigma_i*}(f)\Big\|^2\\&\leqslant 2\sum_{i\in[m]}\Big\|T_{F_i'}^{\sigma_i*}(f)-T_{F_i}^{\sigma_i*}(f)\Big\|^2+2\sum_{i\in[m]}\Big\|T_{F_i}^{\sigma_i*}(f)\Big\|^2\\&\leqslant 2\sum_{i\in[m]}\Big\|\Big(T_{F_i'}^{\sigma_i}-T_{F_i}^{\sigma_i}\Big)^*\Big\|^2\|f\|^2+2\sum_{i\in[m]}\Big\|T_{F_i}^{\sigma_i*}(f)\Big\|^2\\&\leqslant 2\Big(\sum_{i\in[m]}\lambda_i^2\Big)\|f\|^2+2B\|f\|^2\\&=2\Big(B+\sum_{i\in[m]}\lambda_i^2\Big)\|f\|^2.\end{aligned}$$

On the other hand, by using the inequality

$$(a_1+a_2+\cdots+a_m)\leqslant\sqrt{m(a_1^2+a_2^2+\cdots+a_m^2)}\quad(a_i\geqslant 0),$$

for any $f \in \mathcal{H}$, we have

$$\begin{aligned}
&\sum_{i\in[m]} \left\| T_{F_i'}^{\sigma_i *}(f) \right\|^2 \\
&= \sum_{i\in[m]} \left\| T_{F_i'}^{\sigma_i *}(f) - T_{F_i}^{\sigma_i *}(f) + T_{F_i}^{\sigma_i *}(f) \right\|^2 \\
&\geqslant \sum_{i\in[m]} \left(\left\| T_{F_i}^{\sigma_i *}(f) \right\| - \left\| T_{F_i'}^{\sigma_i *}(f) - T_{F_i}^{\sigma_i *}(f) \right\| \right)^2 \\
&= \sum_{i\in[m]} \left(\left\| T_{F_i}^{\sigma_i *}(f) \right\|^2 + \left\| T_{F_i'}^{\sigma_i *}(f) - T_{F_i}^{\sigma_i *}(f) \right\|^2 - 2 \left\| T_{F_i}^{\sigma_i *}(f) \right\| \left\| T_{F_i'}^{\sigma_i *}(f) - T_{F_i}^{\sigma_i *}(f) \right\| \right) \\
&\geqslant \sum_{i\in[m]} \left\| T_{F_i}^{\sigma_i *}(f) \right\|^2 - 2 \sum_{i\in[m]} \left\| T_{F_i}^{\sigma_i *}(f) \right\| \left\| T_{F_i'}^{\sigma_i *}(f) - T_{F_i}^{\sigma_i *}(f) \right\| \\
&\geqslant \sum_{i\in[m]} \left\| T_{F_i}^{\sigma_i *}(f) \right\|^2 - 2 \max_{i\in[m]} \{\lambda_i\} \|f\| \sum_{i\in[m]} \left\| T_{F_i}^{\sigma_i *}(f) \right\| \\
&\geqslant A\|f\|^2 - 2 \max_{i\in[m]} \{\lambda_i\} \|f\| \sqrt{m \sum_{i\in[m]} \|T_{F_i}^{\sigma_i *}(f)\|^2} \\
&\geqslant A\|f\|^2 - 2\sqrt{mB} \max_{i\in[m]} \{\lambda_i\} \|f\|^2.
\end{aligned}$$

This completes the proof. □

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