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A NORM INEQUALITY FOR THREE REAL MATRICES

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Abstract. In this paper, we prove a norm inequality for three real 2×2 matrices conjectured by L. László [3] recently, which is a generalization of the famous Böttcher-Wenzel inequality.

1. Introduction

Let $M_n(\mathbb{R})$ be the set of real $n\times n$ matrices. For $A,B\in M_n(\mathbb{R})$, Böttcher-Wenzel [1] conjectured

$$\|AB-BA\|_F^2 \leqslant 2\|A\|_F^2\|B\|_F^2, \tag{1.1}$$

where $\|\cdot\|_F$ means the Frobenius norm. This conjecture was proved by many authors, please see the survey [2] and the references therein. Motivated by (1.1), László [3] considered three real $n\times n$ matrices and proved a new norm inequality as follows.

THEOREM 1.1. [3, Theorem 3.1] *If $A,B,C\in M_n(\mathbb{R})$, then*

$$\|ABC-CBA\|_F^2 \leqslant \|A\|_F^2\,\|B\|_F^2\,\|C\|_F^2 - \|B\|_F^2\,\mathrm{tr}^2\left(A^TC\right). \tag{1.2}$$

At the same time, László [3] introduced the commutator for $A,B,C\in M_n(\mathbb{R})$:

$$D := (ABC+BCA+CAB)-(CBA+ACB+BAC),$$

and proposed the following conjecture:

CONJECTURE 1.2. [3] For any $A,B,C\in M_n(\mathbb{R})$, it holds

$$\|D\|_F^2 \leqslant \frac{3}{2}\left(\|A\|_F^2\cdot\|BC-CB\|_F^2+\|B\|_F^2\cdot\|CA-AC\|_F^2+\|C\|_F^2\cdot\|AB-BA\|_F^2\right).$$

László [3] himself verified the conjecture for $n=2$ and A,B,C being upper triangular. In this paper, we will consider the László's conjecture for $n=2$. To be precise, we will prove the following theorem.

THEOREM 1.3. *Let $A,B,C\in M_2(\mathbb{R})$ and one of them be symmetric. Then*

$$\|D\|_F^2 \leqslant \frac{3}{2}\left(\|A\|_F^2\cdot\|BC-CB\|_F^2+\|B\|_F^2\cdot\|CA-AC\|_F^2+\|C\|_F^2\cdot\|AB-BA\|_F^2\right). \tag{1.3}$$

We will also show that the constant $\frac{3}{2}$ is optimal.

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2. Proof of Theorem 1.3

Proof of Theorem 1.3. *Step* 1: Without loss of generality, we can assume

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}, \quad C = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix},$$

where $a_i, b_i, \lambda_j \in \mathbb{R}$ for $1 \leqslant i \leqslant 4$, $1 \leqslant j \leqslant 2$.

Step 2: Computation of $AB - BA$, $BC - CB$, $CA - AC$ and their norms.

$$\begin{aligned} AB - BA &= \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} - \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \\ &= \begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{pmatrix} - \begin{pmatrix} a_1b_1 + a_3b_2 & a_2b_1 + a_4b_2 \\ a_1b_3 + a_3b_4 & a_2b_3 + a_4b_4 \end{pmatrix} \\ &= \begin{pmatrix} a_2b_3 - a_3b_2 & (a_1b_2 - a_2b_1) + (a_2b_4 - a_4b_2) \\ (a_3b_1 - a_1b_3) + (a_4b_3 - a_3b_4) & -(a_2b_3 - a_3b_2) \end{pmatrix}, \end{aligned} \tag{2.1}$$

$$\begin{aligned} BC - CB &= \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} - \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 b_1 & \lambda_2 b_2 \\ \lambda_1 b_3 & \lambda_2 b_4 \end{pmatrix} - \begin{pmatrix} \lambda_1 b_1 & \lambda_1 b_2 \\ \lambda_2 b_3 & \lambda_2 b_4 \end{pmatrix} \\ &= (\lambda_1 - \lambda_2) \begin{pmatrix} 0 & -b_2 \\ b_3 & 0 \end{pmatrix}, \end{aligned} \tag{2.2}$$

$$\begin{aligned} CA - AC &= \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} - \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 a_1 & \lambda_1 a_2 \\ \lambda_2 a_3 & \lambda_2 a_4 \end{pmatrix} - \begin{pmatrix} \lambda_1 a_1 & \lambda_2 a_2 \\ \lambda_1 a_3 & \lambda_2 a_4 \end{pmatrix} \\ &= (\lambda_1 - \lambda_2) \begin{pmatrix} 0 & a_2 \\ -a_3 & 0 \end{pmatrix}. \end{aligned} \tag{2.3}$$

Therefore,

$$\begin{aligned}\|AB-BA\|_F^2 &= 2\,(a_2b_3-a_3b_2)^2+[(a_1b_2-a_2b_1)+(a_2b_4-a_4b_2)]^2\\ &\quad +[(a_3b_1-a_1b_3)+(a_4b_3-a_3b_4)]^2,\\ \|BC-CB\|_F^2 &= (b_2^2+b_3^2)\,(\lambda_1-\lambda_2)^2,\\ \|CA-AC\|_F^2 &= (a_2^2+a_3^2)\,(\lambda_1-\lambda_2)^2.\end{aligned} \tag{2.4}$$

Step 3: Computation of $C\,(AB-BA)$, $A\,(BC-CB)$, $B\,(CA-AC)$.

By (2.1), we have

$$\begin{aligned}&C(AB-BA)\\ &=\begin{pmatrix}\lambda_1 & \\ & \lambda_2\end{pmatrix}\begin{pmatrix} a_2b_3-a_3b_2 & (a_1b_2-a_2b_1)+(a_2b_4-a_4b_2)\\ (a_3b_1-a_1b_3)+(a_4b_3-a_3b_4) & -(a_2b_3-a_3b_2)\end{pmatrix}\\ &=\begin{pmatrix}\lambda_1(a_2b_3-a_3b_2) & \lambda_1(a_1b_2-a_2b_1)+\lambda_1(a_2b_4-a_4b_2)\\ \lambda_2(a_3b_1-a_1b_3)+\lambda_2(a_4b_3-a_3b_4) & -\lambda_2(a_2b_3-a_3b_2)\end{pmatrix}.\end{aligned} \tag{2.5}$$

By (2.2), we have

$$A(BC-CB)=\begin{pmatrix}a_1 & a_2\\ a_3 & a_4\end{pmatrix}(\lambda_1-\lambda_2)\begin{pmatrix}0 & -b_2\\ b_3 & 0\end{pmatrix}=(\lambda_1-\lambda_2)\begin{pmatrix}a_2b_3 & -a_1b_2\\ a_4b_3 & -a_3b_2\end{pmatrix}. \tag{2.6}$$

By (2.3), we have

$$B(CA-AC)=\begin{pmatrix}b_1 & b_2\\ b_3 & b_4\end{pmatrix}(\lambda_1-\lambda_2)\begin{pmatrix}0 & a_2\\ -a_3 & 0\end{pmatrix}=(\lambda_1-\lambda_2)\begin{pmatrix}-a_3b_2 & a_2b_1\\ -a_3b_4 & a_2b_3\end{pmatrix}. \tag{2.7}$$

Step 4: Computation of $\|D\|_F^2$.

By (2.5)–(2.7), we get

$$\begin{aligned}D&=C(AB-BA)+A(BC-CB)+B(CA-AC)\\ &=\begin{pmatrix}(2\lambda_1-\lambda_2)\,(a_2b_3-a_3b_2) & \lambda_2(a_1b_2-a_2b_1)+\lambda_1(a_2b_4-a_4b_2)\\ \lambda_2(a_3b_1-a_1b_3)+\lambda_1(a_4b_3-a_3b_4) & (\lambda_1-2\lambda_2)\,(a_2b_3-a_3b_2)\end{pmatrix},\end{aligned} \tag{2.8}$$

therefore,

$$
\begin{aligned}
\|D\|_F^2 &= [(2\lambda_1-\lambda_2)(a_2b_3-a_3b_2)]^2+[(\lambda_1-2\lambda_2)(a_2b_3-a_3b_2)]^2\\
&\quad+[\lambda_2(a_1b_2-a_2b_1)+\lambda_1(a_2b_4-a_4b_2)]^2\\
&\quad+[\lambda_2(a_3b_1-a_1b_3)+\lambda_1(a_4b_3-a_3b_4)]^2\\
&=\left[(2\lambda_1-\lambda_2)^2+(\lambda_1-2\lambda_2)^2\right](a_2b_3-a_3b_2)^2\\
&\quad+[\lambda_2(a_1b_2-a_2b_1)+\lambda_1(a_2b_4-a_4b_2)]^2\\
&\quad+[\lambda_2(a_3b_1-a_1b_3)+\lambda_1(a_4b_3-a_3b_4)]^2\\
&=\left(5\lambda_1^2+5\lambda_2^2-8\lambda_1\lambda_2\right)(a_2b_3-a_3b_2)^2\\
&\quad+\left[(a_2b_4-a_4b_2)^2+(a_4b_3-a_3b_4)^2\right]\lambda_1^2\\
&\quad+\left[(a_1b_2-a_2b_1)^2+(a_3b_1-a_1b_3)^2\right]\lambda_2^2\\
&\quad+[2(a_1b_2-a_2b_1)(a_2b_4-a_4b_2)+2(a_3b_1-a_1b_3)(a_4b_3-a_3b_4)]\lambda_1\lambda_2\\
&=\left[5(a_2b_3-a_3b_2)^2+(a_2b_4-a_4b_2)^2+(a_4b_3-a_3b_4)^2\right]\lambda_1^2\\
&\quad+\left[5(a_2b_3-a_3b_2)^2+(a_1b_2-a_2b_1)^2+(a_3b_1-a_1b_3)^2\right]\lambda_2^2\\
&\quad+\left[2(a_1b_2-a_2b_1)(a_2b_4-a_4b_2)+2(a_3b_1-a_1b_3)(a_4b_3-a_3b_4)\right.\\
&\quad\left.-8(a_2b_3-a_3b_2)^2\right]\lambda_1\lambda_2
\end{aligned}
\tag{2.9}
$$

Step 5: Computation of

$$\|A\|_F^2\cdot\|BC-CB\|_F^2+\|B\|_F^2\cdot\|CA-AC\|_F^2+\|C\|_F^2\cdot\|AB-BA\|_F^2.$$

By (2.4), we get

$$
\begin{aligned}
&\|A\|_F^2\cdot\|BC-CB\|_F^2+\|B\|_F^2\cdot\|CA-AC\|_F^2+\|C\|_F^2\cdot\|AB-BA\|_F^2\\
&=\|A\|_F^2\left(b_2^2+b_3^2\right)(\lambda_1-\lambda_2)^2+\|B\|_F^2\left(a_2^2+a_3^2\right)(\lambda_1-\lambda_2)^2\\
&\quad+\left(\lambda_1^2+\lambda_2^2\right)\|AB-BA\|_F^2\\
&=\left[\|A\|_F^2\left(b_2^2+b_3^2\right)+\|B\|_F^2\left(a_2^2+a_3^2\right)+\|AB-BA\|_F^2\right]\lambda_1^2\\
&\quad+\left[\|A\|_F^2\left(b_2^2+b_3^2\right)+\|B\|_F^2\left(a_2^2+a_3^2\right)+\|AB-BA\|_F^2\right]\lambda_2^2\\
&\quad-2\left[\|A\|_F^2\left(b_2^2+b_3^2\right)+\|B\|_F^2\left(a_2^2+a_3^2\right)\right]\lambda_1\lambda_2.
\end{aligned}
\tag{2.10}
$$

Step 6: Computation of

$$3\left(\|A\|_F^2\cdot\|BC-CB\|_F^2+\|B\|_F^2\cdot\|CA-AC\|_F^2+\|C\|_F^2\cdot\|AB-BA\|_F^2\right)-2\|D\|_F^2.$$

Denote

$$\mathbf{I}_1 = 3\left[\|A\|_F^2\left(b_2^2+b_3^2\right)+\|B\|_F^2\left(a_2^2+a_3^2\right)\right],$$
$$\mathbf{J}_1 = 5(a_2b_3-a_3b_2)^2+(a_2b_4-a_4b_2)^2+(a_4b_3-a_3b_4)^2,$$
$$\mathbf{J}_2 = 5(a_2b_3-a_3b_2)^2+(a_1b_2-a_2b_1)^2+(a_3b_1-a_1b_3)^2,$$
$$\mathbf{J}_3 = 2(a_1b_2-a_2b_1)(a_2b_4-a_4b_2)+2(a_3b_1-a_1b_3)(a_4b_3-a_3b_4)-8(a_2b_3-a_3b_2)^2. \tag{2.11}$$

Then, from (2.9)–(2.11), we obtain that

$$\begin{aligned}&3\left(\|A\|_F^2\cdot\|BC-CB\|_F^2+\|B\|_F^2\cdot\|CA-AC\|_F^2+\|C\|_F^2\cdot\|AB-BA\|_F^2\right)-2\|D\|_F^2\\ =&\left(\mathbf{I}_1+3\|AB-BA\|_F^2-2\mathbf{J}_1\right)\lambda_1^2-2\left(\mathbf{I}_1+\mathbf{J}_3\right)\lambda_1\lambda_2+\left(\mathbf{I}_1+3\|AB-BA\|_F^2-2\mathbf{J}_2\right)\lambda_2^2\\ =&\,p\lambda_1^2+q\lambda_1\lambda_2+r\lambda_2^2,\end{aligned} \tag{2.12}$$

where

$$\begin{aligned}p&=\mathbf{I}_1+3\|AB-BA\|_F^2-2\mathbf{J}_1,\\ q&=-2\left(\mathbf{I}_1+\mathbf{J}_3\right),\\ r&=\mathbf{I}_1+3\|AB-BA\|_F^2-2\mathbf{J}_2.\end{aligned} \tag{2.13}$$

Step 7: Estimation of p and computation of $\Delta = q^2-4pr$.
(1) Estimation of p. By (2.4), (2.11) and (2.13) , we have

$$\begin{aligned}p&=\mathbf{I}_1+3\|AB-BA\|_F^2-2\mathbf{J}_1\\
&=3\left[\left(a_1^2+a_2^2+a_3^2+a_4^2\right)\left(b_2^2+b_3^2\right)+\left(b_1^2+b_2^2+b_3^2+b_4^2\right)\left(a_2^2+a_3^2\right)\right]\\
&\quad+3\big[2(a_2b_3-a_3b_2)^2+(a_1b_2-a_2b_1)^2+(a_2b_4-a_4b_2)^2+(a_3b_1-a_1b_3)^2\\
&\quad+(a_4b_3-a_3b_4)^2+2(a_1b_2-a_2b_1)(a_2b_4-a_4b_2)+2(a_3b_1-a_1b_3)(a_4b_3-a_3b_4)\big]\\
&\quad-2\left[5(a_2b_3-a_3b_2)^2+(a_2b_4-a_4b_2)^2+(a_4b_3-a_3b_4)^2\right]\\
&=6\left(a_2^2+a_3^2\right)\left(b_2^2+b_3^2\right)-4(a_2b_3-a_3b_2)^2\\
&\quad+3\left(a_1^2b_2^2+a_2^2b_1^2\right)+3(a_1b_2-a_2b_1)^2+3\left(a_2^2b_4^2+a_4^2b_2^2\right)+(a_2b_4-a_4b_2)^2\\
&\quad+6(a_1b_2-a_2b_1)(a_2b_4-a_4b_2)+3\left(a_1^2b_3^2+a_3^2b_1^2\right)+3(a_1b_3-a_3b_1)^2\\
&\quad+3\left(a_3^2b_4^2+a_4^2b_3^2\right)+(a_3b_4-a_4b_3)^2+6(a_1b_3-a_3b_1)(a_3b_4-a_4b_3)\\
&\geqslant 2\left(a_2^2+a_3^2\right)\left(b_2^2+b_3^2\right)\\
&\quad+\frac{9}{2}(a_1b_2-a_2b_1)^2+\frac{5}{2}(a_2b_4-a_4b_2)^2+6(a_1b_2-a_2b_1)(a_2b_4-a_4b_2)\\
&\quad+\frac{9}{2}(a_1b_3-a_3b_1)^2+\frac{5}{2}(a_3b_4-a_4b_3)^2+6(a_1b_3-a_3b_1)(a_3b_4-a_4b_3)\end{aligned}$$

$$
\begin{aligned}
&\geqslant 2\left(a_2^2+a_3^2\right)\left(b_2^2+b_3^2\right)\\
&\quad +2\cdot\sqrt{\frac{9}{2}(a_1b_2-a_2b_1)^2\cdot\frac{5}{2}(a_2b_4-a_4b_2)^2}+6(a_1b_2-a_2b_1)(a_2b_4-a_4b_2)\\
&\quad +2\cdot\sqrt{\frac{9}{2}(a_1b_3-a_3b_1)^2\cdot\frac{5}{2}(a_3b_4-a_4b_3)^2}+6(a_1b_3-a_3b_1)(a_3b_4-a_4b_3)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by AM-GM Inequality)}\\
&=2\left(a_2^2+a_3^2\right)\left(b_2^2+b_3^2\right)\\
&\quad +\sqrt{45}\,|(a_1b_2-a_2b_1)(a_2b_4-a_4b_2)|+6(a_1b_2-a_2b_1)(a_2b_4-a_4b_2)\\
&\quad +\sqrt{45}\,|(a_1b_3-a_3b_1)(a_3b_4-a_4b_3)|+6(a_1b_3-a_3b_1)(a_3b_4-a_4b_3)\\
&\geqslant 0,
\end{aligned}
$$

where the first inequality follows from

$$
\begin{aligned}
\left(a_2^2+a_3^2\right)\left(b_2^2+b_3^2\right)-(a_2b_3-a_3b_2)^2&=(a_2b_2+a_3b_3)^2\geqslant 0,\\
2\left(a_1^2b_2^2+a_2^2b_1^2\right)-(a_1b_2-a_2b_1)^2&=(a_1b_2+a_2b_1)^2\geqslant 0,\\
2\left(a_1^2b_3^2+a_3^2b_1^2\right)-(a_1b_3-a_3b_1)^2&=(a_1b_3+a_3b_1)^2\geqslant 0,\\
2\left(a_2^2b_4^2+a_2^2b_4^2\right)-(a_2b_4-a_4b_2)^2&=(a_2b_4+a_4b_2)^2\geqslant 0,\\
2\left(a_3^2b_4^2+a_3^2b_4^2\right)-(a_3b_4-a_4b_3)^2&=(a_3b_4+a_4b_3)^2\geqslant 0.
\end{aligned}
$$

If $p=0$, it is easy to check from all inequalities in the estimation of p that

$$
\begin{aligned}
\left(a_2^2+a_3^2\right)\left(b_2^2+b_3^2\right)&=0,\\
a_1b_2+a_2b_1&=0,\\
a_1b_3+a_3b_1&=0,\\
a_2b_4+a_4b_2&=0,\\
a_3b_4+a_4b_3&=0,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\left(a_2^2+a_3^2\right)\left(b_2^2+b_3^2\right)=0,\\
&\left(b_1^2+b_4^2\right)\left(a_2^2+a_3^2\right)=\left(a_1^2+a_4^2\right)\left(b_2^2+b_3^2\right).
\end{aligned}
$$

Then it follows that $A=O$ or $B=O$ or $a_2=a_3=b_2=b_3=0$. Whatever, $q=r=0$.

(2) Computation of $\Delta = q^2 - 4pr$. By (2.13), we have

$$\begin{aligned}\frac{1}{4}\Delta &= \frac{1}{4}\left(q^2 - 4pr\right)\\ &= (\mathbf{I}_1 + \mathbf{J}_3)^2 - \left(\mathbf{I}_1 + 3\|AB - BA\|_F^2 - 2\mathbf{J}_1\right)\cdot\left(\mathbf{I}_1 + 3\|AB - BA\|_F^2 - 2\mathbf{J}_2\right)\\ &= \mathbf{I}_1 \cdot \left(2\mathbf{J}_1 + 2\mathbf{J}_2 + 2\mathbf{J}_3 - 6\|AB - BA\|_F^2\right) - 9\|AB - BA\|^4\\ &\quad + 6\|AB - BA\|_F^2(\mathbf{J}_1 + \mathbf{J}_2) + \mathbf{J}_3^2 - 4\mathbf{J}_1\mathbf{J}_2.\end{aligned} \tag{2.14}$$

Denote

$$\begin{aligned}\mathbf{I}_2 &= 2\mathbf{J}_1 + 2\mathbf{J}_2 + 2\mathbf{J}_3 - 6\|AB - BA\|_F^2\\ \mathbf{II} &= 9\|AB - BA\|^4\\ \mathbf{III} &= 6\|AB - BA\|_F^2(\mathbf{J}_1 + \mathbf{J}_2)\\ \mathbf{IV} &= \mathbf{J}_3^2\\ \mathbf{V} &= 4\mathbf{J}_1\mathbf{J}_2,\end{aligned} \tag{2.15}$$

then

$$\frac{1}{4}\Delta = (\mathbf{I}_1 \cdot \mathbf{I}_2 - \mathbf{II} + \mathbf{III}) + (\mathbf{IV} - \mathbf{V}). \tag{2.16}$$

Step 8: Estimation of $\mathbf{I}_1 \cdot \mathbf{I}_2 - \mathbf{II} + \mathbf{III}$.

First, we compute $\mathbf{I}_2$. From (2.15), (2.11) and (2.4), we obtain

$$\begin{aligned}\mathbf{I}_2 &= 2\mathbf{J}_1 + 2\mathbf{J}_2 + 2\mathbf{J}_3 - 6\|AB - BA\|_F^2\\ &= 2\big[2(a_1b_2 - a_2b_1)(a_2b_4 - a_4b_2)\\ &\quad + 2(a_3b_1 - a_1b_3)(a_4b_3 - a_3b_4) - 8(a_2b_3 - a_3b_2)^2\big]\\ &\quad + 2\left[5(a_2b_3 - a_3b_2)^2 + (a_2b_4 - a_4b_2)^2 + (a_4b_3 - a_3b_4)^2\right]\\ &\quad + 2\left[5(a_2b_3 - a_3b_2)^2 + (a_1b_2 - a_2b_1)^2 + (a_3b_1 - a_1b_3)^2\right] - 6\|AB - BA\|_F^2\\ &= 4(a_2b_3 - a_3b_2)^2 + 2(a_2b_4 - a_4b_2)^2 + 2(a_1b_2 - a_2b_1)^2\\ &\quad + 4(a_1b_2 - a_2b_1)(a_2b_4 - a_4b_2) + 2(a_4b_3 - a_3b_4)^2 + 2(a_3b_1 - a_1b_3)^2\\ &\quad + 4(a_3b_1 - a_1b_3)(a_4b_3 - a_3b_4) - 6\|AB - BA\|_F^2\\ &= 2\|AB - BA\|_F^2 - 6\|AB - BA\|_F^2\\ &= -4\|AB - BA\|_F^2.\end{aligned} \tag{2.17}$$

Next, we give the estimation of $\mathbf{I}_1 \cdot \mathbf{I}_2 - \mathbf{II} + \mathbf{III}$. From (2.11), (2.15) and (2.17), we obtain

$$\begin{aligned}
&\mathbf{I}_1 \cdot \mathbf{I}_2 - \mathbf{II} + \mathbf{III} \\
&= 3\left[\|A\|_F^2\left(b_2^2+b_3^2\right)+\|B\|_F^2\left(a_2^2+a_3^2\right)\right]\cdot\left(-4\|AB-BA\|_F^2\right) \\
&\quad -9\|AB-BA\|^4+6\|AB-BA\|_F^2\left(\mathbf{J}_1+\mathbf{J}_2\right) \\
&= 3\|AB-BA\|_F^2\cdot\left[-4\|A\|_F^2\left(b_2^2+b_3^2\right)-4\|B\|_F^2\left(a_2^2+a_3^2\right)\right. \\
&\quad \left.-3\|AB-BA\|_F^2+2\left(\mathbf{J}_1+\mathbf{J}_2\right)\right].
\end{aligned} \tag{2.18}$$

$$\begin{aligned}
&-4\|A\|_F^2\left(b_2^2+b_3^2\right)-4\|B\|_F^2\left(a_2^2+a_3^2\right)-3\|AB-BA\|_F^2+2\left(\mathbf{J}_1+\mathbf{J}_2\right) \\
=\ &-4\left(a_1^2+a_2^2+a_3^2+a_4^2\right)\left(b_2^2+b_3^2\right)-4\left(b_1^2+b_2^2+b_3^2+b_4^2\right)\left(a_2^2+a_3^2\right) \\
&-3\|AB-BA\|_F^2+20(a_2b_3-a_3b_2)^2+2(a_2b_4-a_4b_2)^2 \\
&+2(a_4b_3-a_3b_4)^2+2(a_1b_2-a_2b_1)^2+2(a_3b_1-a_1b_3)^2 \\
=\ &\left[-4\left(a_1^2+a_4^2\right)\left(b_2^2+b_3^2\right)-4\left(b_1^2+b_4^2\right)\left(a_2^2+a_3^2\right)+2(a_2b_4-a_4b_2)^2\right. \\
&\left.+2(a_4b_3-a_3b_4)^2+2(a_1b_2-a_2b_1)^2+2(a_3b_1-a_1b_3)^2\right] \\
&+\left[-3\|AB-BA\|_F^2-8\left(a_2^2+a_3^2\right)\left(b_2^2+b_3^2\right)+20(a_2b_3-a_3b_2)^2\right] \\
=\ &:\mathbf{K}_1+\mathbf{K}_2,
\end{aligned} \tag{2.19}$$

where

$$\begin{aligned}
\mathbf{K}_1 =\ &-4\left(a_1^2+a_4^2\right)\left(b_2^2+b_3^2\right)-4\left(b_1^2+b_4^2\right)\left(a_2^2+a_3^2\right)+2(a_2b_4-a_4b_2)^2 \\
&+2(a_4b_3-a_3b_4)^2+2(a_1b_2-a_2b_1)^2+2(a_3b_1-a_1b_3)^2 \\
\leqslant\ &-4\left(a_1^2+a_4^2\right)\left(b_2^2+b_3^2\right)-4\left(b_1^2+b_4^2\right)\left(a_2^2+a_3^2\right)+4\left(a_2^2b_4^2+a_4^2b_2^2\right) \\
&+4(a_4^2b_3^2+a_3^2b_4^2)+4(a_1^2b_2^2+a_2^2b_1^2)+4(a_3^2b_1^2+a_1^2b_3^2) \\
=\ &0
\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}
\mathbf{K}_2 =\ &-3\|AB-BA\|_F^2-8\left(a_2^2+a_3^2\right)\left(b_2^2+b_3^2\right)+20(a_2b_3-a_3b_2)^2 \\
\leqslant\ &-6(a_2b_3-a_3b_2)^2-8\left(a_2^2+a_3^2\right)\left(b_2^2+b_3^2\right)+20(a_2b_3-a_3b_2)^2 \\
=\ &-8\left(a_2^2+a_3^2\right)\left(b_2^2+b_3^2\right)+14(a_2b_3-a_3b_2)^2 \\
\leqslant\ &6(a_2b_3-a_3b_2)^2, \quad \text{(by AM-GM Inequality)}
\end{aligned} \tag{2.21}$$

Therefore, by (2.19), (2.20) and (2.21), we get

$$\begin{aligned}
&\mathbf{I}_1 \cdot \mathbf{I}_2 - \mathbf{II} + \mathbf{III} \\
\leqslant\ & 18\|AB - BA\|_F^2 \cdot (a_2b_3 - a_3b_2)^2 \\
=\ & 18\big[2(a_2b_3 - a_3b_2)^2 + (a_1b_2 - a_2b_1)^2 + (a_2b_4 - a_4b_2)^2 \\
& + (a_3b_1 - a_1b_3)^2 + (a_4b_3 - a_3b_4)^2 \\
& + 2(a_1b_2 - a_2b_1)(a_2b_4 - a_4b_2) \\
& + 2(a_3b_1 - a_1b_3)(a_4b_3 - a_3b_4)\big] \cdot (a_2b_3 - a_3b_2)^2 \\
=\ & 36(a_2b_3 - a_3b_2)^4 \\
& + (a_2b_3 - a_3b_2)^2\big[18(a_1b_2 - a_2b_1)^2 + 18(a_2b_4 - a_4b_2)^2 \\
& + 18(a_3b_1 - a_1b_3)^2 + 18(a_4b_3 - a_3b_4)^2 \\
& + 36(a_1b_2 - a_2b_1)(a_2b_4 - a_4b_2) \\
& + 36(a_3b_1 - a_1b_3)(a_4b_3 - a_3b_4)\big].
\end{aligned} \tag{2.22}$$

Step 9: Estimation of $\mathbf{IV} - \mathbf{V}$.

From (2.15) and (2.11), we obtain

$$\begin{aligned}
\mathbf{IV} &= \mathbf{J}_3^2 \\
&= \Big[\big(2(a_1b_2 - a_2b_1)(a_2b_4 - a_4b_2) + 2(a_3b_1 - a_1b_3)(a_4b_3 - a_3b_4) \\
&\quad - 8(a_2b_3 - a_3b_2)^2\big)\Big]^2 \\
&= 64(a_2b_3 - a_3b_2)^4 + 4(a_1b_2 - a_2b_1)^2(a_2b_4 - a_4b_2)^2 \\
&\quad + 4(a_3b_1 - a_1b_3)^2(a_4b_3 - a_3b_4)^2 \\
&\quad + 8(a_1b_2 - a_2b_1)(a_2b_4 - a_4b_2)(a_3b_1 - a_1b_3)(a_4b_3 - a_3b_4) \\
&\quad - 32(a_2b_3 - a_3b_2)^2\big[(a_1b_2 - a_2b_1)(a_2b_4 - a_4b_2) \\
&\quad + (a_3b_1 - a_1b_3)(a_4b_3 - a_3b_4)\big],
\end{aligned} \tag{2.23}$$

and

$$\begin{aligned}
\mathbf{V} &= 4\mathbf{J}_1\mathbf{J}_2 \\
&= 4\left[5(a_2b_3 - a_3b_2)^2 + (a_2b_4 - a_4b_2)^2 + (a_4b_3 - a_3b_4)^2\right] \\
&\quad \cdot \left[5(a_2b_3 - a_3b_2)^2 + (a_1b_2 - a_2b_1)^2 + (a_3b_1 - a_1b_3)^2\right] \\
&= 100(a_2b_3 - a_3b_2)^4 \\
&\quad + 20(a_2b_3 - a_3b_2)^2\big[(a_2b_4 - a_4b_2)^2 + (a_4b_3 - a_3b_4)^2
\end{aligned} \tag{2.24}$$

$$+(a_1b_2-a_2b_1)^2+(a_3b_1-a_1b_3)^2\big]$$
$$+4(a_2b_4-a_4b_2)^2(a_1b_2-a_2b_1)^2+4(a_2b_4-a_4b_2)^2(a_3b_1-a_1b_3)^2$$
$$+4(a_4b_3-a_3b_4)^2(a_1b_2-a_2b_1)^2+4(a_4b_3-a_3b_4)^2(a_3b_1-a_1b_3)^2.$$

Therefore, from (2.23) and (2.24), we obtain

$$\begin{aligned}
&\mathbf{IV}-\mathbf{V}\\
=&-36(a_2b_3-a_3b_2)^4\\
&-(a_2b_3-a_3b_2)^2\big[20(a_2b_4-a_4b_2)^2+20(a_4b_3-a_3b_4)^2\\
&+20(a_1b_2-a_2b_1)^2+20(a_3b_1-a_1b_3)^2\\
&+32(a_1b_2-a_2b_1)(a_2b_4-a_4b_2)+32(a_3b_1-a_1b_3)(a_4b_3-a_3b_4)\big]\\
&-4\left[(a_2b_4-a_4b_2)(a_3b_1-a_1b_3)-(a_4b_3-a_3b_4)(a_1b_2-a_2b_1)\right]^2\\
\leqslant&-36(a_2b_3-a_3b_2)^4\\
&-(a_2b_3-a_3b_2)^2\big[20(a_2b_4-a_4b_2)^2+20(a_4b_3-a_3b_4)^2\\
&+20(a_1b_2-a_2b_1)^2+20(a_3b_1-a_1b_3)^2\\
&+32(a_1b_2-a_2b_1)(a_2b_4-a_4b_2)+32(a_3b_1-a_1b_3)(a_4b_3-a_3b_4)\big].
\end{aligned}\tag{2.25}$$

Step 10: Estimation of Δ. By (2.16), (2.22) and (2.25), we get

$$\begin{aligned}
\frac{1}{4}\Delta=&\ (\mathbf{I}_1\cdot\mathbf{I}_2-\mathbf{II}+\mathbf{III})+(\mathbf{IV}-\mathbf{V})\\
&\leqslant-(a_2b_3-a_3b_2)^2\big[2(a_2b_4-a_4b_2)^2+2(a_4b_3-a_3b_4)^2\\
&\quad+2(a_1b_2-a_2b_1)^2+2(a_3b_1-a_1b_3)^2\\
&\quad-4(a_1b_2-a_2b_1)(a_2b_4-a_4b_2)-4(a_3b_1-a_1b_3)(a_4b_3-a_3b_4)\big]\\
&=-2(a_2b_3-a_3b_2)^2\Big\{\left[(a_1b_2-a_2b_1)-(a_2b_4-a_4b_2)\right]^2\\
&\quad+\left[(a_3b_1-a_1b_3)-(a_4b_3-a_3b_4)\right]^2\Big\}\\
&\leqslant 0.
\end{aligned}\tag{2.26}$$

Step 11: From Step 7(1), we know $p\geqslant 0$.

- If $p=0$, then from Step 7(1), we know $q=r=0$. Hence
$$p\lambda_1^2+q\lambda_1\lambda_2+r\lambda_2^2=0.$$
- If $p>0$, then from (2.26), we know $\Delta\leqslant 0$. Hence
$$p\lambda_1^2+q\lambda_1\lambda_2+r\lambda_2^2\geqslant 0.$$

By (2.12), it means that

$$\|D\|_F^2 \leqslant \frac{3}{2}\left(\|A\|_F^2 \cdot \|BC-CB\|_F^2 + \|B\|_F^2 \cdot \|CA-AC\|_F^2 + \|C\|_F^2 \cdot \|AB-BA\|_F^2\right).$$

The proof is completed. □

REMARK 2.1. Let

$$A = \begin{pmatrix} & 1 \\ 0 & \end{pmatrix}, \quad B = \begin{pmatrix} & 0 \\ 1 & \end{pmatrix}, \quad C = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix},$$

then it is easy to check that

$$\|D\|_F^2 = \frac{3}{2}\left(\|A\|_F^2 \cdot \|BC-CB\|_F^2 + \|B\|_F^2 \cdot \|CA-AC\|_F^2 + \|C\|_F^2 \cdot \|AB-BA\|_F^2\right) = 18,$$

which shows that the constant $\frac{3}{2}$ in (1.3) is optimal.

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