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ADDENDUM TO: ON A REDUCTION PROCEDURE FOR HORN INEQUALITIES IN FINITE VON NEUMANN ALGEBRAS

BENOÎT COLLINS AND KEN DYKEMA

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Abstract. A proof of the assertion $(XZ)^\sharp(p) = X^\sharp(Z^\sharp(p))$ is provided.

A. Addendum

The assertion contained in equation (11) on page 7 of [1] needs some justification, which was not provided in [1]. In this addendum, we give a proof.

We fix a finite von Nuemann algebra $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$ and we let $\mathfrak{A}_f(\mathcal{M})$ denote the $*$-algebra consisting of all closed, possibly unbounded, densely defined operators on $\mathcal{H}$, that are affiliated with $\mathcal{M}$. See [2] for a nice exposition of some topics related to $\mathfrak{A}_f(\mathcal{M})$. As is usual, $\mathcal{M}'$ denotes the commutant algebra of $\mathcal{M}$ in $B(\mathcal{H})$.

LEMMA A.1. *Suppose $T \in \mathfrak{A}_f(\mathcal{M})$ and that $\mathcal{V} \subseteq \text{dom}(T)$ is a vector subspace that is invariant under the action of $\mathcal{M}'$ and is dense in $\mathcal{H}$. Then $\mathcal{V}$ is a core of T, namely, T is the closure of the restriction of T to $\mathcal{V}$.*

Proof. Let T' denote the restriction of T to $\mathcal{V}$. Since T is closed and $T' \subseteq T$, it follows that T' is a closable operator and

$$\overline{T'} \subseteq T. \tag{1}$$

Suppose $\xi \in \mathcal{V}$ and $A \in \mathcal{M}'$. By hypothesis, $A\xi \in \mathcal{V}$. We have

$$T'A\xi = TA\xi = AT\xi = AT'\xi.$$

It follows that the closure $\overline{T'}$ of T' is affiliated to $\mathcal{M}$. Now using the containment (1) and Proposition 6.7 of [2], we have $\overline{T'} = T$. $\square$

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Given $X \in \mathfrak{A}_f(\mathcal{M})$, then, as in [1], we use the notation $\mathrm{kerproj}(X)$, for the projection onto the null space of X, which is the closed subspace

$$N(X) = \{\xi \in \mathrm{dom}(X) \mid X\xi = 0\}$$

of $\mathcal{H}$, as well as the notations

$$\mathrm{domproj}(X) = 1 - \mathrm{kerproj}(X), \quad \mathrm{ranproj}(X) = \mathrm{domproj}(X^*).$$

We have that $\mathrm{ranproj}(X)\mathcal{H}$ is the closure of $\mathrm{ran}(X) = \{X\xi \mid \xi \in \mathrm{dom}(X)\}$.

LEMMA A.2. *Let $X, Y \in \mathfrak{A}_f(\mathcal{M})$ and let $r = \mathrm{domproj}(Y)$. Then*

$$\mathrm{domproj}(YX) = \mathrm{domproj}(rX).$$

Proof. It will suffice to show $N(YX) = N(rX)$. Since $Y = Yr$, by the definition and associativity of the product in $\mathfrak{A}_f(\mathcal{M})$ (see, for example [2]), we have $YX = (Yr)X = Y(rX)$ and $\mathrm{dom}(YX) \subseteq \mathrm{dom}(rX)$. Since $\mathrm{dom}(YX)$ is an $\mathcal{M}'$-invariant and dense subspace of $\mathcal{H}$, it is, by Lemma A.1, a core of rX. Thus, $N(rX)$ is the closure of $N(rX) \cap \mathrm{dom}(YX)$. For $\xi \in \mathrm{dom}(YX)$, we have

$$\begin{aligned}
YX\xi = 0 \quad &\iff \quad \forall \eta \in \mathcal{H},\ \langle YX\xi, \eta\rangle = 0 \\
&\iff \quad \forall \eta \in \mathrm{dom}(Y^*),\ \langle YX\xi, \eta\rangle = 0 \qquad (2)\\
&\iff \quad \forall \eta \in \mathrm{dom}(Y^*),\ \langle X\xi, Y^*\eta\rangle = 0 \\
&\iff \quad \forall \zeta \in \mathcal{H},\ \langle X\xi, r\zeta\rangle = 0 \qquad (3)\\
&\iff \quad \forall \zeta \in \mathcal{H},\ \langle rX\xi, \zeta\rangle = 0 \\
&\iff \quad rX\xi = 0,
\end{aligned}$$

where the equivalence (2) follows because $\mathrm{dom}(Y^*)$ is dense in $\mathcal{H}$, and (3) follows because $r = \mathrm{ranproj}(Y^*)$ and the range of Y^* is dense in $r\mathcal{H}$. We have shown

$$N(YX) = N(rX) \cap \mathrm{dom}(YX).$$

Since $N(YX)$ is closed and $N(rX) \cap \mathrm{dom}(YX)$ is dense in $N(rX)$, we have $N(YX) = N(rX)$, as required. $\square$

As in [1] for $X \in \mathfrak{A}_f(\mathcal{M})$ and a projection $p \in \mathcal{M}$, we let $X^\sharp(p) = \mathrm{ranproj}(Xp)$. The next result is precisely the assertion of equation (11) of [1].

PROPOSITION A.3. *Let $X, Z \in \mathfrak{A}_f(\mathcal{M})$ and let p be a projection in $\mathcal{M}$. Then*

$$(XZ)^\sharp(p) = X^\sharp(Z^\sharp(p)).$$

Proof. Let $Y = Zp$ and $r = Z^\sharp(p)$. Then $r = \mathrm{ranproj}(Y) = \mathrm{domproj}(Y^*)$ Then we have

$$\begin{aligned}
(XZ)^\sharp(p) &= \mathrm{ranproj}(XZp) = \mathrm{ranproj}(XY) = \mathrm{domproj}(Y^*X^*) \\
&= \mathrm{domproj}(rX^*) = \mathrm{ranproj}(Xr) = X^\sharp(r),
\end{aligned}$$

where for the fourth equality we used Lemma A.2. $\square$

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Benoît Collins
Department of Mathematics
Kyoto University
Kyoto, Japan
e-mail: collins@math.kyoto-u.ac.jp

Ken Dykema
Department of Mathematics
Texas A&M University
College Station, TX 77843-3368, USA
e-mail: kdykema@math.tamu.edu