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ON JOINT NUMERICAL RADIUS OF OPERATORS AND JOINT NUMERICAL INDEX OF A BANACH SPACE

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Abstract. Generalizing the notion of numerical range and numerical radius of an operator on a Banach space, we introduce the notion of joint numerical range and joint numerical radius of tuple of operators on a Banach space. We prove that on a finite-dimensional Banach space, the joint numerical radius can be retrieved from the extreme points. Furthermore, we introduce a notion of joint numerical index of a Banach space. We explore the same for direct sum of Banach spaces. Applying these results, finally we compute the joint numerical index of some classical Banach spaces.

1. Introduction

The purpose of this article is to generalize the notion of numerical range and numerical radius of an operator on a Banach space to k-tuple of operators and explore the joint behavior of operators through these concepts. Moreover, we generalize the notion of numerical index of a Banach space and study the related areas. To proceed further, we introduce the relevant notations and terminologies.

Suppose H is a Hilbert space and X,Y are Banach spaces. Unless otherwise mentioned, we always assume that the scalar field F is either real or complex. Let $S_X = \{x \in X : \|x\| = 1\}$, $B_X = \{x \in X : \|x\| \leqslant 1\}$ and E_X be the set of all extreme points of B_X. X^* denotes the dual space of X. Suppose $L(X,Y)$ denotes the space of all bounded linear operators from X to Y, endowed with the usual operator norm. If $X = Y$, then we simply write $L(X)$ instead of $L(X,X)$. The symbols I and O denote respectively the identity operator and the zero operator on the corresponding space. For $T \in L(H)$, the numerical range $W(T)$ and the numerical radius $w(T)$ associated with the operator T are defined as follows:

$$W(T) = \{\langle Tx,x\rangle : x \in S_H\},\ w(T) = \sup\{|\langle Tx,x\rangle| : x \in S_H\}.$$

The heart of the theory of numerical range lies in the Toeplitz-Hausdorff Theorem, which says that $W(T)$ is always convex for $T \in L(H)$. If H is a complex Hilbert space, then $w(\cdot)$ defines a norm on $L(H)$. On the other hand, if H is a real Hilbert space, then $w(\cdot)$ defines a semi-norm on $L(H)$. For other nice properties of $W(T)$ and

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$w(T)$, we refer the readers to [6, 7]. Lumer [10] and Bauer [1] generalized the notion of the numerical range of an operator on a Banach space. For $T \in L(X)$, the numerical range $W(T)$ and the numerical radius $w(T)$ are defined as

$$W(T) = \{x^*(Tx) : (x,x^*) \in \Pi_X\},\ w(T) = \sup\{|x^*(Tx)| : (x,x^*) \in \Pi_X\},$$

where $\Pi_X = \{(x,x^*) \in S_X \times S_{X^*} : x^*(x) = 1\}$. In contrast to the Hilbert space, if $T \in L(X)$, $W(T)$ may not be always convex. Note that, $w(\cdot)$ is always a semi-norm on $L(X)$. In particular, if X is a complex Banach space, then it defines a norm. However, there are some real Banach spaces X where $w(\cdot)$ is a norm on $L(X)$. Classical references for the theory of numerical range on a Banach space and related areas are [2, 3]. There is a constant on a Banach space, known as the numerical index of the space, which relates the behavior of the numerical radius with the usual norm of an operator. The numerical index $n(X)$ of a space X is defined as follows:

$$n(X) = \inf\{w(T) : T \in L(X), \|T\| = 1\}.$$

Let us recall a few facts on $n(X)$, which we will use in the sequel without mentioning further. If X is a complex Banach space, then $\frac{1}{e} \leqslant n(X) \leqslant 1$ and if X is a real Banach space, then $0 \leqslant n(X) \leqslant 1$. If H is a complex Hilbert space, then $n(H) = \frac{1}{2}$ and for a real Hilbert space H, $n(H) = 0$. Moreover, $n(\ell_1) = n(\ell_1^m) = n(\ell_\infty) = n(\ell_\infty^m) = 1$, where $m \in \mathbb{N}$. All these results can be found in [2, 8]. Some recent references for the study of the numerical index are [11, 14, 19].

For a k-tuple $\mathscr{T} = (T_1, \ldots, T_k)$ of operators on H, the joint numerical range $W(\mathscr{T})$ of $\mathscr{T}$ is a subset of F^k, defined as follows:

$$W(\mathscr{T}) = \{(\langle T_1x,x\rangle, \ldots, \langle T_kx,x\rangle) : x \in S_H\}.$$

In general, the joint numerical range is not convex (see [16, Ex. 3.1]). To get an overview of other interesting properties of the joint numerical range of Hilbert space operators, the readers may follow [9, 17], the survey article [16] and the references therein. In [18] Popescu defined the joint numerical radius $w(\mathscr{T})$ of $\mathscr{T}$, also known as the euclidean operator radius $w_e(\mathscr{T})$, by

$$w(\mathscr{T}) = \sup\left\{\left(\sum_{i=1}^{k} |\langle T_ix,x\rangle|^2\right)^{1/2} : x \in S_H\right\}.$$

In [15], Moslehian et al. extended the notion of euclidean operator radius in the following way:

$$w_p(\mathscr{T}) = \sup\left\{\left(\sum_{i=1}^{k} |\langle T_ix,x\rangle|^p\right)^{1/p} : x \in S_H\right\},\ \text{where } p \geqslant 1.$$

In [15, 20] the authors studied inequalities related to $w_p(\mathscr{T})$ for $\mathscr{T} \in L(H)^k$. Motivated by the concept of joint numerical range (radius) of Hilbert space operators, we generalize these concepts on a Banach space. Suppose $T_1, T_2, \dots, T_k \in L(X,Y)$ and $\mathscr{T} = (T_1, T_2, \dots, T_k) \in L(X,Y)^k$. For $1 \leqslant p < \infty$, we define

$$\|\mathscr{T}\|_p = \sup\left\{ \Big(\sum_{i=1}^{k} \|T_i x\|^p\Big)^{1/p} : x \in S_X \right\}.$$

Using the Minkowski's inequality, it is easy to observe that $\|\cdot\|_p$ actually defines a norm on $L(X,Y)^k$. We call $\|\mathscr{T}\|_p$ as the p-th joint operator norm of $\mathscr{T}$. Similarly, we define the joint numerical range $W(\mathscr{T})$ of $\mathscr{T} \in L(X)^k$ as follows:

$$W(\mathscr{T}) = \Big\{ (x^*(T_1 x), x^*(T_2 x), \dots, x^*(T_k x)) : (x, x^*) \in \Pi_X \Big\}.$$

For $1 \leqslant p < \infty$, we define the p-th joint numerical radius $w_p(\mathscr{T})$ of $\mathscr{T} \in L(X)^k$ in the following way:

$$w_p(\mathscr{T}) = \sup\left\{ \Big(\sum_{i=1}^{k} |x^*(T_i x)|^p\Big)^{1/p} : (x, x^*) \in \Pi_X \right\}.$$

Once again, using the Minkowski's inequality, it is straightforward to check that $w_p(\cdot)$ defines a norm (resp. semi-norm) on $L(X)^k$ if and only if $w(\cdot)$ defines a norm (resp. semi-norm) on $L(X)$. Next, we generalize the notion of the numerical index of a Banach space. For $1 \leqslant p < \infty$, and $k \in \mathbb{N}$, we define the (p,k)-th joint numerical index of X as follows:

$$n_{(p,k)}(X) = \inf\{ w_p(\mathscr{T}) : \mathscr{T} \in L(X)^k, \|\mathscr{T}\|_p = 1 \}.$$

Observe that, the (p,k)-th joint numerical index of a Banach space is monotonically decreasing with respect to k. Indeed, if $k_1 < k_2$, then $L(X)^{k_1} \subset L(X)^{k_2}$, which implies that $n_{(p,k_2)}(X) \leqslant n_{(p,k_1)}(X)$.

Let us now briefly discuss the content of the article. After this introductory part, in Section 2, we study some preliminary properties of the joint numerical range and the p-th joint numerical radius of a k-tuple of operators on a Banach space. Analogous to the numerical radius, we prove that in a finite-dimensional Banach space, $w_p(\mathscr{T})$ can be retrieved from a subset of Π_X, namely from the set $G_X = \{(x, x^*) \in \Pi_X : x \in E_X, x^* \in E_{X^*}\}$. This result will be used in Section 3. Furthermore, we compare some basic properties of the operator norm and the numerical radius with that of the joint operator norm and the p-th joint numerical radius. In Section 3, we explore the (p,k)-th joint numerical index of Banach spaces. In this direction, we first obtain a lower bound and an upper bound of $n_{(p,k)}(X)$ in terms of $n(X)$. Then we study $n_{(p,k)}(X)$, where X is a direct sum of Banach spaces. As a consequence, we can explicitly compute $n_{(p,k)}(X)$, where X is a direct sum of certain class of Banach spaces. Finally, using the results of this section, we compute $n_{(p,k)}(X)$, where $X \in \{\ell_\infty, \ell_1, \ell_2, \ell_\infty^m, \ell_1^m, \ell_2^m : m \in \mathbb{N}\}$.

2. Joint numerical range and the p-th joint numerical radius

We begin this section with a simple observation on the convexity of the joint numerical range. The proof the next proposition follows from the fact that $W(T_i)$ is the projection of $W(\mathscr{T})$ to the i-th coordinate.

PROPOSITION 2.1. *Let* $\mathscr{T} = (T_1, T_2, \ldots, T_k) \in L(X)^k$. *If* $W(\mathscr{T})$ *is convex, then* $W(T_i)$ *is convex for each* $1 \leqslant i \leqslant k$.

The converse of Proposition 2.1 is not true in general. From [3], it is known that $W(T)$ is always connected for $T \in L(X)$. Therefore, if X is a real Banach space, then $W(T)$ is convex. However, it is easy to construct examples on real Banach spaces, where the joint numerical range is not always convex. Next, we exhibit such an example.

EXAMPLE 2.2. Consider the real Banach space $X = \ell_1^2$. Suppose $T, S \in L(X)$ be defined as $T(a,b) = (b,0)$, $S(a,b) = (0,a)$ for all $(a,b) \in X$. Observe that $G_X = \{((1,0),(1,\pm 1)),((0,1),(\pm 1,1))\}$ and

$$\Pi_X = G_X \cup \{((a,b),(sgn\ a, sgn\ b)) : a \neq 0, b \neq 0, |a| + |b| = 1\},$$

where sgn is the usual sign function. Let $\mathscr{T} = (T,S)$. Then it is easy to check that

$$W(\mathscr{T}) = \{\pm(a,b) : 0 \leqslant a, b \leqslant 1, a+b = 1\}.$$

Thus, $(-1,0),(0,1) \in W(\mathscr{T})$. However, $t(-1,0)+(1-t)(0,1) = (-t, 1-t) \notin W(\mathscr{T})$ for all $t \in (0,1)$. This proves that $W(\mathscr{T})$ is not convex.

The following example provides an easy way to construct $\mathscr{T} \in L(X)^k$ such that $W(\mathscr{T})$ is convex.

EXAMPLE 2.3. Choose $T \in L(X)$ such that $W(T)$ is convex. Fix $j \in \{1,2,\ldots,k\}$. Let $T_j = T$ and $T_i = c_i I$ for all $1 \leqslant i \neq j \leqslant k$, where c_i are scalars. Consider $\mathscr{T} = (T_1, T_2, \ldots, T_k)$. Then observe that $W(\mathscr{T}) = \prod_{i=1}^k W(T_i)$. Now, the convexity of $W(\mathscr{T})$ follows from the fact that $\prod_{i=1}^k W(T_i)$ is convex.

Recall from [12, Lem. 2.5] that for $T \in L(X)$,

$$w(T) = \sup\{|x^{**}(T^* x^*)| : x^* \in E_{X^*}, x^{**} \in E_{X^{**}}, x^{**}(x^*) = 1\}.$$

In the following theorem, we obtain an analogous result for the p-th joint numerical radius of operators on a finite-dimensional Banach space.

THEOREM 2.4. *Let* $\dim(X) < \infty$. *Then for* $\mathscr{T} \in L(X)^k$,

$$w_p(\mathscr{T}) = \max\left\{ \left(\sum_{i=1}^k |x^*(T_i x)|^p \right)^{1/p} : (x, x^*) \in G_X \right\},$$

where $G_X = \{(x, x^*) \in \Pi_X : x \in E_X, x^* \in E_{X^*}\}$.

Proof. Since $\dim(X) < \infty$, S_X, S_{X^*} are compact. Hence $S_X \times S_{X^*}$ is compact. Observe that, Π_X is a closed subset of $S_X \times S_{X^*}$. Indeed, if $\{(x_n, x_n^*)\}$ is a sequence of

Π_X converging to (x,x^*), then $x_n \to x$ and $x_n^* \to x^*$. Clearly, $\|x\| = 1, \|x^*\| = 1$. Thus,

$$\begin{aligned}|1 - x^*(x)| &= |x_n^*(x_n) - x^*(x)| \\ &\leqslant |x_n^*(x_n) - x_n^*(x)| + |x_n^*(x) - x^*(x)| \\ &\leqslant \|x_n^*\|\|x_n - x\| + \|x_n^* - x^*\|\|x\| \\ &\to 0,\end{aligned}$$

from which it follows that $x^*(x) = 1$, and so $(x,x^*) \in \Pi_X$. This proves that Π_X is a compact subset of $S_X \times S_{X^*}$. Observe that, the mapping $\phi : X \times X^* \to F$ defined by $\phi(x,x^*) = \left(\sum_{i=1}^k |x^*(T_i x)|^p\right)^{1/p}$ is continuous. Therefore, the set

$$\phi(\Pi_X) = \left\{ \left(\sum_{i=1}^k |x^*(T_i x)|^p \right)^{1/p} : (x,x^*) \in \Pi_X \right\}$$

is compact. Hence, there exists $(x_0, x_0^*) \in \Pi_X$ such that $\phi(x_0, x_0^*) = \sup \phi(\Pi_X)$. Thus, $w_p(\mathscr{T}) = \left(\sum_{i=1}^k |x_0^*(T_i x_0)|^p\right)^{1/p}$. Suppose that $x_0 \notin E_X$. Then there exist $\lambda_i \in (0,1)$ and $x_i \in E_X$ for $1 \leqslant i \leqslant n$ such that $\sum_{i=1}^n \lambda_i = 1$ and $x_0 = \sum_{i=1}^n \lambda_i x_i$. Now,

$$\begin{aligned} T_j x_0 &= \sum_{i=1}^n \lambda_i T_j x_i \ \forall\ 1 \leqslant j \leqslant k \\ \Rightarrow |x_0^*(T_j x_0)| &= |\sum_{i=1}^n \lambda_i x_0^*(T_j x_i)|,\ \forall\ 1 \leqslant j \leqslant k \\ &\leqslant \sum_{i=1}^n \lambda_i |x_0^*(T_j x_i)|,\ \forall\ 1 \leqslant j \leqslant k \\ \Rightarrow |x_0^*(T_j x_0)|^p &\leqslant \left(\sum_{i=1}^n \lambda_i |x_0^*(T_j x_i)| \right)^p,\ \forall\ 1 \leqslant j \leqslant k \\ &\leqslant \sum_{i=1}^n \lambda_i |x_0^*(T_j x_i)|^p,\ \forall\ 1 \leqslant j \leqslant k \\ \Rightarrow \sum_{j=1}^k |x_0^*(T_j x_0)|^p &\leqslant \sum_{i=1}^n \lambda_i \left(\sum_{j=1}^k |x_0^*(T_j x_i)|^p \right) \\ &\leqslant \sum_{i=1}^n \lambda_i w_p(\mathscr{T})^p = w_p(\mathscr{T})^p. \end{aligned}$$

Thus, from

$$w_p(\mathscr{T})^p = \sum_{j=1}^k |x_0^*(T_j x_0)|^p \leqslant \sum_{i=1}^n \lambda_i \left(\sum_{j=1}^k |x_0^*(T_j x_i)|^p \right) \leqslant w_p(\mathscr{T})^p,$$

it follows that for all $1 \leqslant i \leqslant n$, $w_p(\mathscr{T})^p = \left(\sum_{j=1}^k |x_0^*(T_j x_i)|^p \right)$. Moreover, from

$$1 = x_0^*(x_0) = \sum_{i=1}^n \lambda_i x_0^*(x_i) = |\sum_{i=1}^n \lambda_i x_0^*(x_i)| \leqslant \sum_{i=1}^n \lambda_i |x_0^*(x_i)| \leqslant \sum_{i=1}^n \lambda_i = 1,$$

we get $x_0^*(x_i) = 1$ for all $1 \leqslant i \leqslant n$, i.e., $(x_i, x_0^*) \in \Pi_X$. Now, suppose that $x_0^* \notin E_{X^*}$. Then there exist $\mu_i \in (0,1)$ and $x_i^* \in E_{X^*}$ for $1 \leqslant i \leqslant m$, such that $\sum_{i=1}^m \mu_i = 1$ and $x_0^* = \sum_{i=1}^m \mu_i x_i^*$. Following similar arguments, we can show that for all $1 \leqslant i \leqslant n, 1 \leqslant r \leqslant m$, $w_p(\mathscr{T}) = \left(\sum_{j=1}^k |x_r^*(T_j x_i)|^p \right)^{1/p}$, where $(x_i, x_r^*) \in \Pi_X$. Moreover, since $x_i \in E_X, x_r^* \in E_{X^*}$, we have $(x_i, x_r^*) \in G_X$. This completes the proof. □

We can obtain an analogous result of Theorem 2.4 for $\|\mathscr{T}\|_p$, where $\mathscr{T} \in L(X,Y)^k$. To avoid monotonicity, we simply state the theorem.

THEOREM 2.5. *Let* $\dim(X) < \infty$. *Then for* $\mathscr{T} \in L(X,Y)^k$,

$$\|\mathscr{T}\|_p = \max\left\{ \left(\sum_{i=1}^k \|T_i x\|^p \right)^{1/p} : x \in E_X \right\}.$$

For $\mathscr{T} = (T_1, \ldots, T_k) \in L(X,Y)^k$, suppose $\mathscr{T}^* = (T_1^*, \ldots, T_k^*) \in L(Y^*, X^*)^k$. In the next example, we exhibit a basic difference between the operator norm and the p-th joint operator norm. More precisely, we show that in general, for $k > 1$, $\|\mathscr{T}\|_p \neq \|\mathscr{T}^*\|_p$.

EXAMPLE 2.6. Suppose $m > 2$. Choose $1 < k \leqslant m$. Consider the operators $T_i \in L(\ell_\infty^m)$ for $1 \leqslant i \leqslant k$ defined as follows:

$$T_i e_i = e_i, \; T_i e_j = \theta, \text{ for } j \in \{1, \ldots, m\} \setminus \{i\},$$

where $\{e_i : 1 \leqslant i \leqslant m\}$ is the standard ordered basis of ℓ_∞^m. Suppose $\mathscr{T} = (T_1, \ldots, T_k) \in L(\ell_\infty^m)^k$. Then $\|\mathscr{T}\|_p = k^{1/p}$ (see Theorem 3.10 (i) for details).

Now, observe that, $\mathscr{T}^* = (T_1, \ldots, T_k) \in L(\ell_1^m)^k$, since $T_i^* = T_i$ for all $1 \leqslant i \leqslant k$. Therefore, using Theorem 2.5, we get

$$\begin{aligned} \|\mathscr{T}^*\|_p &= \sup\left\{ \Big(\sum_{i=1}^k \|T_i e_j\|^p \Big)^{1/p} : 1 \leqslant j \leqslant m \right\} \\ &= 1 \neq \|\mathscr{T}\|_p. \end{aligned}$$

However, for $\mathscr{T} \in L(X)^k$, a slight modification in the argument of [3, Th. 2, Cor. 3, p. 11–12] proves that $W(\mathscr{T}) \subset W(\mathscr{T}^*) \subset \overline{W(\mathscr{T})}$. Thus, we always have $w_p(\mathscr{T}) = w_p(\mathscr{T}^*)$. We note this in the following proposition.

PROPOSITION 2.7. *For* $\mathscr{T} = (T_1, \ldots, T_k) \in L(X)^k$,
(i) $W(\mathscr{T}) \subset W(\mathscr{T}^*) \subset \overline{W(\mathscr{T})}$.
(ii) $w_p(\mathscr{T}) = w_p(\mathscr{T}^*)$.

3. The (p,k)-th joint numerical index

Let us begin this section with a bound of the (p,k)-th joint numerical index of a Banach space. We follow the idea of [5, Th. 9] in the proof.

THEOREM 3.1. *For all* $1 \leqslant p < \infty$ *and* $k \in \mathbb{N}$, *the following is true.*

$$\frac{n(X)}{k^{1/p}} \leqslant n_{(p,k)}(X) \leqslant n(X).$$

Proof. To prove $n_{(p,k)}(X) \leqslant n(X)$, consider an arbitrary non-zero operator $T \in L(X)$. Let $\mathscr{T} = (T, O, \ldots, O) \in L(X)^k$. Then observe that, $\|\mathscr{T}\|_p = \|T\|$ and $w_p(\mathscr{T}) = w(T)$. Thus, $n_{(p,k)}(X) \leqslant \frac{1}{\|\mathscr{T}\|_p} w_p(\mathscr{T}) = \frac{1}{\|T\|} w(T)$. Since this is true for each non-zero operator $T \in L(X)$, we get

$$n_{(p,k)}(X) \leqslant n(X). \tag{1}$$

Observe that, if $n(X) = 0$, then from (1), we get the desired result. Therefore, to prove the first inequality, without loss of generality, we may assume that $n(X) > 0$. Now, consider $\mathscr{T} = (T_1, T_2, \ldots, T_k) \in L(X)^k$ such that $\|\mathscr{T}\|_p = 1$. Choose $\varepsilon > 0$. Then there exists $x \in S_X$ such that $(\sum_{i=1}^k \|T_i x\|^p)^{1/p} > 1 - \varepsilon$. Clearly, there exists $j_0 \in \{1, 2, \ldots, k\}$ such that $\|T_{j_0} x\|^p > \frac{(1-\varepsilon)^p}{k}$. Thus,

$$w(T_{j_0}) \geqslant n(X)\|T_{j_0}\| \geqslant n(X)\|T_{j_0}x\| > \frac{(1-\varepsilon)}{k^{1/p}} n(X).$$

Now,

$$\begin{aligned} w_p(\mathscr{T}) &\geqslant \Big(\sum_{i=1}^k |x^*(T_i x)|^p\Big)^{1/p}, \ \forall\, (x, x^*) \in \Pi_X \\ &\geqslant |x^*(T_{j_0} x)|, \ \forall\, (x, x^*) \in \Pi_X \\ \Rightarrow w_p(\mathscr{T}) &\geqslant w(T_{j_0}) > \frac{(1-\varepsilon)}{k^{1/p}} n(X). \end{aligned}$$

Since the above inequality is true for each $\varepsilon > 0$, we have $w_p(\mathscr{T}) \geqslant \frac{n(X)}{k^{1/p}}$. Thus, $n_{p,k}(X) \geqslant \frac{n(X)}{k^{1/p}}$. This completes the proof. $\square$

Note that, if $n(X) > 0$, then for each non-zero $T \in L(X)$, $w(T) > 0$. However, from [13, Ex. 3b], it follows that the converse is not true. In particular, there is a Banach space X, for which $n(X) = 0$ but $w(T) > 0$ for all non-zero operator T. Using this fact and the last theorem, we immediately get an analogous result for the (p,k)-th joint numerical index of a Banach space.

COROLLARY 3.2. *There exists a Banach space* X *such that* $n_{(p,k)}(X) = 0$, *whereas* $w_p(\mathscr{T}) \neq 0$ *for all non-zero* $\mathscr{T} \in L(X)^k$.

Proof. Using [13, Ex. 3b], we can choose a Banach space X such that $n(X) = 0$, whereas $w(T) \neq 0$ for all non-zero $T \in L(X)$. Now, suppose $\mathscr{T} = (T_1, \dots, T_k) \in L(X)^k$ and $\mathscr{T}$ is non-zero. Then T_{i_0} is non-zero for some $1 \leqslant i_0 \leqslant k$. Therefore, $w_p(\mathscr{T}) \geqslant w(T_{i_0}) > 0$. Whereas, from Theorem 3.1, we get

$$0 = \frac{n(X)}{k^{1/p}} \leqslant n_{(p,k)}(X) \leqslant n(X) = 0,$$

which implies that $n_{(p,k)}(X) = 0$. $\square$

Our next goal is to study the (p,k)-th joint numerical index of direct sum of Banach spaces. Suppose $\{X_\lambda : \lambda \in \Lambda\}$ is a family of Banach spaces. By the symbol $[\oplus_{\lambda \in \Lambda} X_\lambda]_{\ell_q}$, where $1 \leqslant q \leqslant \infty$ (resp. $[\oplus_{\lambda \in \Lambda} X_\lambda]_{c_0}$), we denote the ℓ_q-sum (resp. c_0-sum) of the family. In what follows, we use the notation $J(x)$ to denote the set $\{x^* \in S_{X^*} : x^*(x) = \|x\|\}$ for all non-zero $x \in X$. To serve our purpose, the next proposition from [4, Prop. 5.1] is required.

PROPOSITION 3.3. [4, Prop. 5.1] *Let X, Y be Banach spaces and $Z = X \oplus_p Y$, where $1 \leqslant p \leqslant \infty$.*

(1) *Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Suppose $x \in X \setminus \{\theta\}$ and $y \in Y \setminus \theta$. Then*

$$(a)\ J((x,y)) = \left\{ \left(\frac{\|x\|^{p-1}}{(\|x\|^p + \|y\|^p)^{\frac{1}{q}}} f, \frac{\|y\|^{p-1}}{(\|x\|^p + \|y\|^p)^{\frac{1}{q}}} g \right) \in S_{X^* \oplus_q Y^*} : f \in J(x), g \in J(y) \right\},$$

$$(b)\ J(x,\theta) = \{(f,\theta) \in S_{X^* \oplus_q Y^*} : f \in J(x)\},$$

$$(c)\ J(\theta,y) = \{(\theta,g) \in S_{X^* \oplus_q Y^*} : g \in J(y)\}.$$

(2) *Let $p = 1$, $x \in X \setminus \{\theta\}, y \in Y \setminus \{\theta\}$. Then*

$$(a)\ J((x,\theta)) = \{(f,g) \in S_{X^* \oplus_\infty Y^*} : f \in J(x),\ g \in B_{Y^*}\},$$

$$(b)\ J((\theta,y)) = \{(f,g) \in S_{X^* \oplus_\infty Y^*} : f \in B_{X^*},\ g \in J(y)\},$$

$$(c)\ J((x,y)) = \{(f,g) \in S_{X^* \oplus_\infty Y^*} : f \in J(x),\ g \in J(y)\}.$$

(3) *Let $p = \infty$, $(x,y) \in Z \setminus \{\theta\}$.*

(a) *If $\|x\| > \|y\|$, then $J((x,y)) = \{(f,\theta) \in S_{X^* \oplus_1 Y^*} : f \in J(x)\}$.*

(b) *If $\|x\| < \|y\|$, then $J((x,y)) = \{(\theta,g) \in S_{X^* \oplus_1 Y^*} : g \in J(y)\}$.*

(c) *If $\|x\| = \|y\|$, then*

$$J((x,y)) = \{(tf,(1-t)g) \in S_{X^* \oplus_1 Y^*} : f \in J(x), g \in J(y), 0 \leqslant t \leqslant 1\}.$$

Proof. (1), (2), (3a) and (3b) follow from [4, Prop. 5.1]. For (3c), note that in [4, Prop. 5.1], it is proved that

$$\{(tf,(1-t)g) \in S_{X^*\oplus_1 Y^*} : f \in J(x), g \in J(y), 0 \leqslant t \leqslant 1\} \subseteq J((x,y)).$$

For the reverse inclusion, let $A = \{(tf,(1-t)g) \in S_{X^*\oplus_1 Y^*} : f \in J(x), g \in J(y), 0 \leqslant t \leqslant 1\}$. Suppose that $(x^*,y^*) \in J((x,y))$. Then $\|x^*\| + \|y^*\| = 1$. Moreover,

$$\begin{aligned} 1 = (x^*,y^*)(x,y) &= x^*(x) + y^*(y) \\ &= \Re(x^*(x) + y^*(y)) \\ &\leqslant \|x^*\|\|x\| + \|y^*\|\|y\| \\ &= (\|x^*\| + \|y^*\|)\|x\|, \text{ (since } \|x\| = \|y\|) \\ &= \|x\| \leqslant 1, \end{aligned}$$

where $\Re(\lambda)$ denotes the real part of λ. Thus, the above inequality is actually an equality and $x^*(x) = \|x^*\|\|x\|$, $y^*(y) = \|y^*\|\|y\|$. If $x^* = \theta$, then $\|y^*\| = 1$, i.e., $y^* \in J(y)$ and so $(x^*,y^*) = (\theta,y^*) \in A$. Similarly, if $y^* = \theta$, then $\|x^*\| = 1$, i.e., $x^* \in J(x)$ and $(x^*,\theta) \in A$. Let $x^* \neq \theta, y^* \neq \theta$. Then clearly, $\frac{1}{\|x^*\|}x^* \in J(x)$ and $\frac{1}{\|y^*\|}y^* \in J(y)$. Let $t = \|x^*\|$. Then $1-t = 1-\|x^*\| = \|y^*\|$. Thus, $t \in (0,1)$ and $(x^*,y^*) = \left(t\frac{1}{\|x^*\|}x^*, (1-t)\frac{1}{\|y^*\|}y^*\right) \in A$. Therefore, we get $J((x,y)) \subseteq A$, completing the proof of (3c). □

THEOREM 3.4. *Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of Banach spaces. Suppose X is either $[\oplus_{\lambda\in\Lambda} X_\lambda]_{\ell_q}$, where $1 \leqslant q \leqslant \infty$ or $[\oplus_{\lambda\in\Lambda} X_\lambda]_{c_0}$. Then*

$$n_{(p,k)}(X) \leqslant \inf\{n_{(p,k)}(X_\lambda) : \lambda \in \Lambda\}.$$

Proof. Suppose that $Z = [Y \oplus V]_{\ell_q}$ for the Banach spaces Y, V, where $1 \leqslant q \leqslant \infty$. We first claim that $n_{(p,k)}(Z) \leqslant n_{(p,k)}(Y)$. Choose $\mathscr{T}_Y = (A_1, A_2, \ldots, A_k) \in L(Y)^k$ such that $\|\mathscr{T}_Y\|_p = 1$. For $1 \leqslant i \leqslant k$, define $T_i \in L(Z)$ by $T_i(y,v) = (A_i y, \theta)$, where $y \in Y, v \in V$. Let $\mathscr{T}_Z = (T_1, T_2, \ldots, T_k) \in L(Z)^k$. Observe that,

$$\begin{aligned} \|\mathscr{T}_Z\|_p &= \sup_{(y,v)\in S_Z} \left(\sum_{i=1}^k \|T_i(y,v)\|^p\right)^{1/p} \\ &= \sup_{(y,v)\in S_Z} \left(\sum_{i=1}^k \|A_i y\|^p\right)^{1/p} \\ &\geqslant \sup_{(y,\theta)\in S_Z} \left(\sum_{i=1}^k \|A_i y\|^p\right)^{1/p} \\ &= \sup_{y\in S_Y} \left(\sum_{i=1}^k \|A_i y\|^p\right)^{1/p} \\ &= \|\mathscr{T}_Y\|_p. \end{aligned}$$

Thus,

$$\|\mathscr{T}_Z\|_p \geqslant \|\mathscr{T}_Y\|_p. \tag{2}$$

On the other hand, suppose $(y,v) \in S_Z$. Then $\|y\| \leqslant 1$. Now, for $y = \theta$, we have, $\left(\sum_{i=1}^k \|A_i y\|^p\right)^{1/p} = 0 \leqslant \|\mathscr{T}_Y\|_p$ and for $y \neq \theta$,

$$\left(\sum_{i=1}^k \|A_i y\|^p\right)^{1/p} = \|y\| \left(\sum_{i=1}^k \|A_i \frac{y}{\|y\|}\|^p\right)^{1/p} \leqslant \left(\sum_{i=1}^k \|A_i \frac{y}{\|y\|}\|^p\right)^{1/p} \leqslant \|\mathscr{T}_Y\|_p.$$

Thus, for each $(y,v) \in S_Z$, $\left(\sum_{i=1}^k \|T_i(y,v)\|^p\right)^{1/p} = \left(\sum_{i=1}^k \|A_i y\|^p\right)^{1/p} \leqslant \|\mathscr{T}_Y\|_p$. This implies that

$$\|\mathscr{T}_Z\|_p \leqslant \|\mathscr{T}_Y\|_p. \tag{3}$$

Now, from (2) and (3), it follows that $\|\mathscr{T}_Z\|_p = \|\mathscr{T}_Y\|_p = 1$.

Next, we show that $w_p(\mathscr{T}_Z) = w_p(\mathscr{T}_Y)$. Note that, $(y,y^*) \in \Pi_Y$ is equivalent to $((y,\theta),(y^*,\theta)) \in \Pi_Z$. Now,

$$\begin{aligned} w_p(\mathscr{T}_Z) &= \sup_{((y,v),(y^*,v^*))\in\Pi_Z} \left(\sum_{i=1}^k |(y^*,v^*)(T_i(y,v))|^p\right)^{1/p} \\ &= \sup_{((y,v),(y^*,v^*))\in\Pi_Z} \left(\sum_{i=1}^k |y^*(A_i y)|^p\right)^{1/p} \\ &\geqslant \sup_{((y,\theta),(y^*,\theta))\in\Pi_Z} \left(\sum_{i=1}^k |y^*(A_i y)|^p\right)^{1/p} \\ &= \sup_{(y,y^*)\in\Pi_Y} \left(\sum_{i=1}^k |y^*(A_i y)|^p\right)^{1/p} \\ &= w_p(\mathscr{T}_Y). \end{aligned}$$

Thus,

$$w_p(\mathscr{T}_Z) \geqslant w_p(\mathscr{T}_Y). \tag{4}$$

On the other hand, let $((y,v),(y^*,v^*)) \in \Pi_Z$. Then $\|y\| \leqslant 1$, $\|y^*\| \leqslant 1$. Now, suppose that $q = \infty$. Then from Proposition 3.3, it follows that $\|y^*\| + \|v^*\| = 1$ and the following three cases may hold.

(i) $\|y\| < \|v\|$, (ii) $\|y\| = \|v\|$, (iii) $\|y\| > \|v\|$.

For case (i), by Proposition 3.3, we get $y^* = \theta$. Hence,

$$\left(\sum_{i=1}^k |(y^*,v^*)(T_i(y,v))|^p\right)^{1/p} = \left(\sum_{i=1}^k |y^*(A_i y)|^p\right)^{1/p} = 0 \leqslant w_p(\mathscr{T}_Y).$$

Note that, for cases (ii) and (iii), if $y^* = \theta$ holds, then similarly, we get the above inequality. Therefore, consider $y^* \neq \theta$. For case (ii), i.e., if $\|y\| = \|v\|$ holds, then clearly, $\|y\| \neq 0$ and $y^*(y) = \|y^*\|\|y\|$. For case (iii), clearly, $\|y\| \neq 0$ and once again using Proposition 3.3, we get $y^* \in J(y)$, i.e., $y^*(y) = \|y^*\|\|y\|$. Therefore, if case (ii) or case (iii) holds, then

$$\begin{aligned}\Big(\sum_{i=1}^{k} |(y^*,v^*)(T_i(y,v))|^p\Big)^{1/p} &= \Big(\sum_{i=1}^{k} |y^*(A_i y)|^p\Big)^{1/p}\\ &= \|y^*\|\|y\|\Big(\sum_{i=1}^{k}\Big|\frac{y^*}{\|y^*\|}\Big(A_i\frac{y}{\|y\|}\Big)\Big|^p\Big)^{1/p}\\ &\leqslant w_p(\mathscr{T}_Y).\end{aligned}$$

Thus, for each case, $\Big(\sum_{i=1}^{k} |(y^*,v^*)(T_i(y,v))|^p\Big)^{1/p} \leqslant w_p(\mathscr{T}_Y)$. Since the last inequality holds for each $((y,v),(y^*,v^*)) \in \Pi_Z$, we get

$$w_p(\mathscr{T}_Z) \leqslant w_p(\mathscr{T}_Y). \tag{5}$$

Thus, from (4) and (5), it follows that if $q=\infty$, then $w_p(\mathscr{T}_Z) = w_p(\mathscr{T}_Y)$. Similarly, for $1 \leqslant q < \infty$, we can show that if $((y,v),(y^*,v^*)) \in \Pi_Z$, then either $y = \theta$ or $0 < y^*(y) = \|y^*\|\|y\|$. Now, using similar arguments, we can prove that

$$w_p(\mathscr{T}_Z) = w_p(\mathscr{T}_Y).$$

Thus, for each $\mathscr{T}_Y \in L(Y)^k$ with $\|\mathscr{T}_Y\|_p = 1$,

$$n_{(p,k)}(Z) \leqslant w_p(\mathscr{T}_Z) = w_p(\mathscr{T}_Y).$$

Therefore, it follows that $n_{(p,k)}(Z) \leqslant n_{(p,k)}(Y)$ and our claim is proved.

Now, fix $\lambda_0 \in \Lambda$. For $X = [\oplus_{\lambda\in\Lambda} X_\lambda]_{c_0}$, assume $V = [\oplus_{\lambda\in\Lambda,\lambda\neq\lambda_0} X_\lambda]_{c_0}$. Then we can write

$$X = [X_{\lambda_0} \oplus V]_{\ell_\infty},$$

and so $n_{(p,k)}(X) \leqslant n_{(p,k)}(X_{\lambda_0})$. Similarly, if $X = [\oplus_{\lambda\in\Lambda} X_\lambda]_{\ell_q}$, where $1 \leqslant q \leqslant \infty$, then assuming $V = [\oplus_{\lambda\in\Lambda,\lambda\neq\lambda_0} X_\lambda]_{\ell_q}$, and writing

$$X = [X_{\lambda_0} \oplus V]_{\ell_q},$$

we get $n_{(p,k)}(X) \leqslant n_{(p,k)}(X_{\lambda_0})$. This completes the proof of the theorem. □

We would like to remark here that in [13], Martín and Payá studied the numerical index of direct sum of Banach spaces. They proved the following result.

THEOREM 3.5. [13, Prop. 1 & Rem. 2a] *Let* $\{X_\lambda : \lambda \in \Lambda\}$ *be a family of Banach spaces. Then*

$$n([\oplus_{\lambda\in\Lambda} X_\lambda]_{c_0}) = n([\oplus_{\lambda\in\Lambda} X_\lambda]_{\ell_1}) = n([\oplus_{\lambda\in\Lambda} X_\lambda]_{\ell_\infty}) = \inf_{\lambda\in\Lambda} n(X_\lambda).$$

If $1 < q < \infty$, *then*

$$n([\oplus_{\lambda\in\Lambda} X_\lambda]_{\ell_q}) \leqslant \inf_{\lambda\in\Lambda} n(X_\lambda).$$

As an immediate consequence of Theorem 3.4 and Theorem 3.5, we can compute the (p,k)-th joint numerical index of a direct sum of certain class of Banach spaces.

COROLLARY 3.6. *Suppose* $\{X_\lambda : \lambda \in \Lambda\}$ *is a family of Banach spaces. Let*

$$X \in \left\{ [\oplus_{\lambda\in\Lambda} X_\lambda]_{\ell_1}, [\oplus_{\lambda\in\Lambda} X_\lambda]_{\ell_\infty}, [\oplus_{\lambda\in\Lambda} X_\lambda]_{c_0} \right\}.$$

Then the following are true.

(i) If $n(X) = 1$ *and* $n_{(p,k)}(X_{\lambda_0}) = \frac{1}{k^{1/p}}$ *for some* $\lambda_0 \in \Lambda$, *then* $n_{(p,k)}(X) = \frac{1}{k^{1/p}}$.

(ii) If $n_{(p,k)}(X_\lambda) = \frac{n(X_\lambda)}{k^{1/p}}$ *for each* $\lambda \in \Lambda$, *then* $n_{(p,k)}(X) = \frac{n(X)}{k^{1/p}}$.

Proof. (i) Note that,

$$\begin{aligned}\frac{1}{k^{1/p}} = \frac{n(X)}{k^{1/p}} &\leqslant n_{(p,k)}(X) \leqslant \inf\{n_{(p,k)}(X_\lambda) : \lambda \in \Lambda\} \\ &\leqslant n_{(p,k)}(X_{\lambda_0}) = \frac{1}{k^{1/p}},\end{aligned}$$

where the first inequality follows from Theorem 3.1 and the second inequality follows from Theorem 3.4. Thus, the above inequalities are actually equalities and so $n_{(p,k)}(X) = \frac{1}{k^{1/p}}$.

(ii) Note that,

$$\begin{aligned}\frac{n(X)}{k^{1/p}} &\leqslant n_{(p,k)}(X) \leqslant \inf\{n_{(p,k)}(X_\lambda) : \lambda \in \Lambda\} \\ &= \inf\left\{\frac{n(X_\lambda)}{k^{1/p}} : \lambda \in \Lambda\right\} = \frac{n(X)}{k^{1/p}},\end{aligned}$$

where the last equality follows from Theorem 3.5. This completes the proof of (ii). □

REMARK 3.7. From Corollary 3.6 (i), it follows that the class of Banach spaces with the (p,k)-th joint numerical index $\frac{1}{k^{1/p}}$ is stable under ℓ_∞-sum (or ℓ_1-sum or c_0-sum), if the sum has numerical index 1.

In the next theorem, we get another upper bound for ℓ_q-sum $(1 < q < \infty)$ of Banach spaces.

THEOREM 3.8. *Let* X, Y *be Banach spaces. Suppose* $Z = [X \oplus Y]_{\ell_q}$, *where* $q \in (1,\infty)$ *and* $\frac{1}{q} + \frac{1}{q'} = 1$. *Then* $n_{(p,k)}(Z) \leqslant \max\{\frac{1}{q}, \frac{1}{q'}\}$.

Proof. Choose $\mathscr{S} = (S_1, \ldots, S_k) \in L(Y,X)^k$ such that $\|\mathscr{S}\|_p = 1$. For $1 \leqslant i \leqslant k$, define $T_i \in L(Z)$ by $T_i(x,y) = (S_i y, \theta)$ for all $(x,y) \in Z$. Suppose $\mathscr{T} = (T_1, \ldots, T_k) \in L(Z)^k$. Proceeding as Theorem 3.4, we can show that $\|\mathscr{T}\|_p = \|\mathscr{S}\|_p = 1$. Now, suppose that $((x,y),(x^*,y^*)) \in \Pi_Z$. Then from Proposition 3.3, it follows that either of the following is true.

(i) $x = x^* = \theta$, (ii) $y = y^* = \theta$, (iii) $x \neq \theta, y \neq \theta, x^* = \|x\|^{q-1} f, y^* = \|y\|^{q-1} g$, where $f \in J(x)$ and $g \in J(y)$.

If (i) holds, then

$$\Big(\sum_{i=1}^{k} |(\theta, y^*)(T_i(\theta, y))|^p\Big)^{1/p} = \Big(\sum_{i=1}^{k} |(\theta, y^*)(S_i y, \theta)|^p\Big)^{1/p} = 0.$$

If (ii) holds, then

$$\Big(\sum_{i=1}^{k} |(x^*, \theta)(T_i(x, \theta))|^p\Big)^{1/p} = \Big(\sum_{i=1}^{k} |(x^*, \theta)(S_i \theta, \theta)|^p\Big)^{1/p} = 0.$$

If (iii) holds, then

$$\begin{aligned}
\Big(\sum_{i=1}^{k} |(x^*, y^*)(T_i(x, y))|^p\Big)^{1/p} &= \Big(\sum_{i=1}^{k} |(\|x\|^{q-1} f, \|y\|^{q-1} g)(S_i y, \theta)|^p\Big)^{1/p} \\
&= \Big(\sum_{i=1}^{k} \|x\|^{p(q-1)} |f(S_i y)|^p\Big)^{1/p} \\
&\leqslant \|x\|^{q-1} \Big(\sum_{i=1}^{k} \|S_i y\|^p\Big)^{1/p} \\
&\leqslant \|x\|^{q-1} \|\mathscr{S}\|_p \|y\| \\
&= \|x\|^{q-1} \|y\| \\
&\leqslant \frac{\|y\|^q}{q} + \frac{\|x\|^{q'(q-1)}}{q'} \quad \text{(using Young's inequality)} \\
&= \frac{\|y\|^q}{q} + \frac{\|x\|^q}{q'} \\
&\leqslant \max\Big\{\frac{1}{q}, \frac{1}{q'}\Big\} (\|x\|^q + \|y\|^q) \\
&= \max\Big\{\frac{1}{q}, \frac{1}{q'}\Big\}, \ \text{(since } (x, y) \in S_Z).
\end{aligned}$$

Therefore, $w_p(\mathscr{T}) \leqslant \max\{\frac{1}{q}, \frac{1}{q'}\}$. Now, it follows that

$$n_{(p,k)}(Z) \leqslant w_p(\mathscr{T}) \leqslant \max\Big\{\frac{1}{q}, \frac{1}{q'}\Big\},$$

completing the proof of the theorem. □

Combining Theorem 3.4 and Theorem 3.8, we get the following inequality for the (p,k)-th joint numerical index of ℓ_q-sum $(1 < q < \infty)$ of Banach spaces.

COROLLARY 3.9. *Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of Banach spaces. Suppose $X = [\oplus_{\lambda \in \Lambda} X_\lambda]_{\ell_q}$, where $1 < q < \infty$. Let $\frac{1}{q} + \frac{1}{q'} = 1$. Then*

$$n_{(p,k)}(X) \leqslant \inf\Big\{\max\Big\{\frac{1}{q}, \frac{1}{q'}\Big\}, n_{(p,k)}(X_\lambda) : \lambda \in \Lambda\Big\}.$$

Note that, if $k=1$, then for all $1\leqslant p<\infty$, $n_{(p,k)}(X)=n(X)$. Therefore, in Corollary 3.9, if we choose X_λ such that $\max\{\frac{1}{q},\frac{1}{q'}\}<\inf\{n(X_\lambda):\lambda\in\Lambda\}$, then Corollary 3.9 improves on both Theorem 3.4 and Theorem 3.5. Finally, our aim is to use the results of this section to explicitly compute the (p,k)-th joint numerical index of some classical Banach spaces.

THEOREM 3.10. *Suppose* $m\in\mathbb{N}\setminus\{1\}$. *Then*
(i) $n_{(p,k)}(\ell_\infty^m)=\frac{1}{k^{1/p}}$, *if* $1\leqslant k\leqslant m$.
(ii) $n_{(p,k)}(\ell_\infty)=\frac{1}{k^{1/p}}$.
(iii) $n_{(p,2)}(\ell_1^m)=\frac{1}{2^{1/p}}$, *for real scalars.*
(iv) $n_{(p,2)}(\ell_1)=\frac{1}{2^{1/p}}$, *for real scalars.*

Proof. (i) Let $1\leqslant k\leqslant m$. First we prove that $n_{(p,k)}(\ell_\infty^m)=\frac{1}{k^{1/p}}$. It is well-known that $n(\ell_\infty^m)=1$. Now, from Theorem 3.1, it follows that

$$\frac{1}{k^{1/p}}\leqslant n_{(p,k)}(\ell_\infty^m). \tag{6}$$

To prove the reverse inequality, define $T_i\in L(\ell_\infty^m)$ for $1\leqslant i\leqslant k$ as follows:

$$T_ie_i=e_i,\ T_ie_j=\theta,\ \text{for } j\in\{1,\dots,m\}\setminus\{i\},$$

where $\{e_i:1\leqslant i\leqslant m\}$ is the standard ordered basis of ℓ_∞^m. Suppose $\mathscr{T}=(T_1,\dots,T_k)\in L(\ell_\infty^m)^k$. Clearly, $x=(a_1,a_2,\dots,a_m)\in E_{\ell_\infty^m}$ if and only if $|a_i|=1$, for all $1\leqslant i\leqslant m$. Note that, $(x,x^*)\in G_{\ell_\infty^m}$ if and only if $x^*\in\{a_1e_1,a_2e_2,\dots,a_me_m\}$. Thus, for $1\leqslant i\leqslant k, 1\leqslant j\leqslant m$, and $x^*=a_je_j$,

$$x^*(T_ix)=a_je_j(a_ie_i)=\begin{cases}0, & j\neq i\\ 1, & j=i.\end{cases}$$

Now, using Theorem 2.5, we get

$$\begin{aligned}\|\mathscr{T}\|_p&=\max\Big\{\Big(\sum_{i=1}^k\|T_i(a_1,a_2,\dots,a_m)\|^p\Big)^{1/p}:(a_1,a_2,\dots,a_m)\in E_{\ell_\infty^m}\Big\}\\&=\max\Big\{\Big(\sum_{i=1}^k|a_i|^p\Big)^{1/p}:|a_i|=1,\ \forall\,1\leqslant i\leqslant k\Big\}\\&=k^{1/p}.\end{aligned}$$

On the other hand, using Theorem 2.4, we get

$$\begin{aligned}w_p(\mathscr{T})&=\max_{(x,x^*)\in G_{\ell_\infty^m}}\Big(\sum_{i=1}^k|x^*(T_ix)|^p\Big)^{1/p}\\&=\max\Big\{\Big(\sum_{i=1}^k|a_je_j(a_ie_i))|^p\Big)^{1/p}:|a_j|=1,\ \forall\,1\leqslant j\leqslant m\Big\}\\&=1.\end{aligned}$$

Therefore,

$$n_{(p,k)}(\ell_\infty^m) \leqslant \frac{w_p(\mathscr{T})}{\|\mathscr{T}\|_p} = \frac{1}{k^{1/p}}. \tag{7}$$

Now, from (6) and (7), it follows that $n_{(p,k)}(\ell_\infty^m) = \frac{1}{k^{1/p}}$.

(ii) It is well-known that $n(\ell_\infty) = 1$. Hence, by Theorem 3.1,

$$\frac{1}{k^{1/p}} = \frac{n(\ell_\infty)}{k^{1/p}} \leqslant n_{(p,k)}(\ell_\infty).$$

For the reverse inequality, choose $m \geqslant k$. Then we can write $\ell_\infty = [\ell_\infty^m \oplus \ell_\infty]_{\ell_\infty}$. Therefore, by (i) and Theorem 3.4, we get

$$n_{(p,k)}(\ell_\infty) \leqslant n_{(p,k)}(\ell_\infty^m) = \frac{1}{k^{1/p}}.$$

Thus, $n_{(p,k)}(\ell_\infty) = \frac{1}{k^{1/p}}$.

(iii) Observe that, for $m = 2$, the result follows from the fact that ℓ_∞^2 is isometrically isomorphic to ℓ_1^2. Suppose that $m > 2$. Then $\ell_1^m = [\ell_1^2 \oplus \ell_1^{m-2}]_{\ell_1}$. Therefore, from Theorem 3.4, we get

$$n_{(p,2)}(\ell_1^m) \leqslant n_{(p,2)}(\ell_1^2) = \frac{1}{2^{1/p}}.$$

On the other hand, from Theorem 3.1, it follows that

$$\frac{1}{2^{1/p}} = \frac{n(\ell_1^m)}{2^{1/p}} \leqslant n_{(p,2)}(\ell_1^m).$$

This completes the proof of (iii).

(iv) Note that, $\ell_1 = [\ell_1^2 \oplus \ell_1]_{\ell_1}$. Now, proceeding similarly as (iii), we get the desired result. □

We end the section by computing the (p,k)-th joint numerical index of Hilbert spaces.

THEOREM 3.11. *Suppose $m \in \mathbb{N} \setminus \{1,2\}$. Let ℓ_2^m and ℓ_2 be complex Hilbert spaces. Then*

(i) $n_{(2,k)}(\ell_2^m) = \frac{1}{2\sqrt{k}}$, where $1 \leqslant k < m$.

(ii) $n_{(2,k)}(\ell_2) = \frac{1}{2\sqrt{k}}$.

(iii) $n_{(p,2)}(\ell_2^m) = \frac{1}{2^{1+\frac{1}{p}}}$, where $p > 2$.

(iv) $n_{(p,2)}(\ell_2) = \frac{1}{2^{1+\frac{1}{p}}}$, where $p > 2$.

Proof. (i) It follows from Theorem 3.1 that

$$\frac{1}{2\sqrt{k}} = \frac{n(\ell_2^m)}{\sqrt{k}} \leqslant n_{(2,k)}(\ell_2^m). \tag{8}$$

To prove the reverse inequality, we exhibit $\mathscr{T} \in L(\ell_2^m)$ such that $\|\mathscr{T}\|_2 = 1$ and $w_2(\mathscr{T}) = \frac{1}{2\sqrt{k}}$. For this, we follow the idea of [5, Th. 9]. Suppose $\{e_1, e_2, \ldots, e_m\}$ is the standard ordered basis of ℓ_2^m. For $1 \leqslant i \leqslant k$, define $T_i \in L(\ell_2^m)$ as follows.

$$T_i e_j = \begin{cases} \theta, & \text{if } 1 \leqslant j < m, \\ \frac{1}{\sqrt{k}} e_i, & \text{if } j = m. \end{cases}$$

Let $\mathscr{T} = (T_1, T_2, \ldots, T_k) \in L(\ell_2^m)^k$. Then

$$\begin{aligned} \|\mathscr{T}\|_2 &= \sup\Big\{ \big(\sum_{i=1}^{k} \|T_i(x_1, x_2, \ldots, x_m)\|^2 \big)^{1/2} : \sum_{j=1}^{m} |x_j|^2 = 1 \Big\} \\ &= \sup\Big\{ |x_m| : \sum_{j=1}^{m} |x_j|^2 = 1 \Big\} \\ &= 1. \end{aligned}$$

Moreover,

$$\begin{aligned} w_2(\mathscr{T}) &= \sup\Big\{ \big(\sum_{i=1}^{k} |\langle T_i(x_1, \ldots, x_m), (x_1, \ldots, x_m) \rangle|^2 \big)^{1/2} : \sum_{j=1}^{m} |x_j|^2 = 1 \Big\} \\ &= \sup\Big\{ \Big(\sum_{i=1}^{k} \Big| \frac{x_m x_i}{\sqrt{k}} \Big|^2 \Big)^{1/2} : \sum_{j=1}^{m} |x_j|^2 = 1 \Big\} \\ &= \frac{1}{\sqrt{k}} \sup\Big\{ \big(|x_m|^2 \sum_{i=1}^{k} |x_i|^2 \big)^{1/2} : \sum_{j=1}^{m} |x_j|^2 = 1 \Big\} \\ &= \frac{1}{\sqrt{k}} \sup\Big\{ \big(|x_m|^2 \sum_{i=1}^{k} |x_i|^2 \big)^{1/2} : \sum_{j=1}^{k} |x_j|^2 + |x_m|^2 = 1 \Big\} \\ &= \frac{1}{\sqrt{k}} \sup\Big\{ \big(|x_m|^2 (1 - |x_m|^2) \big)^{1/2} : |x_m| \leqslant 1 \Big\} \\ &= \frac{1}{2\sqrt{k}}. \end{aligned}$$

Thus,

$$n_{(2,k)}(\ell_2^m) \leqslant w_2(\mathscr{T}) = \frac{1}{2\sqrt{k}}. \tag{9}$$

Combining (8) and (9), we get the conclusion of (i).

(ii) Note that, $\ell_2 = [\ell_2^{k+1} \oplus \ell_2]_{\ell_2}$. Therefore,

$$\frac{1}{2\sqrt{k}} = \frac{n(\ell_2)}{\sqrt{k}} \leqslant n_{(2,k)}(\ell_2) \leqslant n_{(2,k)}(\ell_2^{k+1}) = \frac{1}{2\sqrt{k}},$$

where the first inequality follows from Theorem 3.1, the second inequality follows from Theorem 3.4 and the last equality follows from (i). This completes the proof of (ii).

(iii) Suppose $p>2$. If we can prove $n_{(p,2)}(\ell_2^3)=\frac{1}{2^{1+\frac{1}{p}}}$, then for $m>3$, writing $\ell_2^m=[\ell_2^3\oplus\ell_2^{m-3}]_{\ell_2}$ and using Theorem 3.1 and Theorem 3.4, we get

$$\frac{1}{2^{1+\frac{1}{p}}}=\frac{n(\ell_2^m)}{2^{1/p}}\leqslant n_{(p,2)}(\ell_2^m)\leqslant n_{(p,2)}(\ell_2^3)=\frac{1}{2^{1+\frac{1}{p}}},$$

proving the result for $m>3$. Thus, we only assume that $m=3$. Clearly,

$$\frac{1}{2^{1+\frac{1}{p}}}=\frac{n(\ell_2^3)}{2^{1/p}}\leqslant n_{(p,2)}(\ell_2^3). \tag{10}$$

To prove the reverse in equality, we use the same operators as in (i), i.e., we consider the operators $T_1,T_2\in L(\ell_2^3)$ defined as follows:

$$T_1(x,y,z)=\Big(\frac{z}{2^{1/p}},0,0\Big),\ T_2(x,y,z)=\Big(0,\frac{z}{2^{1/p}},0\Big),\ \forall\,(x,y,z)\in\ell_2^3.$$

Suppose $\mathscr{T}=(T_1,T_2)\in L(\ell_2^3)^2$. Then

$$\begin{aligned}\|\mathscr{T}\|_p&=\sup\Big\{\big(\|T_1(x,y,z)\|^p+\|T_2(x,y,z)\|^p\big)^{1/p}:|x|^2+|y|^2+|z|^2=1\Big\}\\&=\sup\{|z|:|x|^2+|y|^2+|z|^2=1\}\\&=1.\end{aligned}$$

On the other hand,

$$\begin{aligned}&w_p(\mathscr{T})\\&=\sup\Big\{\big(|\langle T_1(x,y,z),(x,y,z)\rangle|^p+|\langle T_2(x,y,z),(x,y,z)\rangle|^p\big)^{\frac{1}{p}}:|x|^2+|y|^2+|z|^2=1\Big\}\\&=\frac{1}{2^{\frac{1}{p}}}\sup\Big\{\big(|zx|^p+|zy|^p\big)^{\frac{1}{p}}:|x|^2+|y|^2+|z|^2=1\Big\}\\&=\frac{1}{2^{1+\frac{1}{p}}},\ (\text{since }\ p>2).\end{aligned}$$

Therefore,

$$n_{(p,2)}(\ell_2^3)\leqslant w_p(\mathscr{T})=\frac{1}{2^{1+\frac{1}{p}}}. \tag{11}$$

Now, it follows from (10) and (11) that $n_{(p,2)}(\ell_2^3)=\frac{1}{2^{1+\frac{1}{p}}}$. This completes the proof of (iii).

(iv) Proceeding similarly as (ii) and using (iii), we get the desired result. □

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