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POSITIVE OPERATOR–VALUED TOEPLITZ OPERATORS ON VECTOR–VALUED GENERALIZED FOCK SPACES

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Abstract. In this paper, we obtain several equivalent descriptions of Carleson conditions to characterize the boundedness and compactness of positive vector-valued Toeplitz operators on the vector-valued generalized Fock spaces $F^p_\varphi(\mathcal{H})$.

1. Introduction

Let $\mathbb{C}^n$ be the n-dimensional complex Euclidean space and dv denote the Lebesgue volume measure on $\mathbb{C}^n$. Given a separable Hilbert space $\mathcal{H}$ with the norm $\|\cdot\|_{\mathcal{H}}$, and a function $\varphi \in C^2(\mathbb{C}^n)$ satisfing $0 < mdd^c|z|^2 \leqslant dd^c\varphi \leqslant Mdd^c|z|^2$ for two positive constants m and M, where $d^c = \frac{\sqrt{-1}}{4}(\overline{\partial} - \partial)$. For $1 \leqslant p < \infty$, we denote by $L^p_\varphi(\mathcal{H})$ the space of measurable $\mathcal{H}$-valued functions satisfying

$$\|f\|_{p,\varphi} = \left(\int_{\mathbb{C}^n} \|f(z)\|^p_{\mathcal{H}} e^{-p\varphi(z)} dv(z)\right)^{\frac{1}{p}} < \infty.$$

The vector-valued generalized Fock space $F^p_\varphi(\mathcal{H})$ is the closed subspace of $L^p_\varphi(\mathcal{H})$ consisting of holomorphic functions, i.e.

$$F^p_\varphi(\mathcal{H}) = \left\{ f : \mathbb{C}^n \to \mathcal{H} \text{ holomorphic} : \|f\|_{F^p_\varphi(\mathcal{H})} = \left(\int_{\mathbb{C}^n} \|f(z)\|^p_{\mathcal{H}} e^{-p\varphi(z)} dv(z)\right)^{\frac{1}{p}} < \infty \right\}.$$

Let

$$F^\infty_\varphi(\mathcal{H}) = \left\{ f : \mathbb{C}^n \to \mathcal{H} \text{ holomorphic} : \|f\|_{\infty,\varphi} = \sup_{z\in\mathbb{C}^n} \|f(z)\|_{\mathcal{H}} e^{-\varphi(z)} < \infty \right\}.$$

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In particular, $F^2_\varphi(\mathcal{H})$ is a Hilbert space, the orthogonal projection $P : L^2_\varphi(\mathcal{H}) \to F^2_\varphi(\mathcal{H})$ can be represented as

$$Pf(z) = \int_{\mathbb{C}^n} f(w)K_\varphi(z,w)e^{-2\varphi(w)}dv(w), \quad z \in \mathbb{C}^n,$$

where $K_\varphi(z,w)$ denotes the reproducing kernel of the scalar generalized Fock spaces $F^2_\varphi(\mathbb{C}^n)$. We still use $||\cdot||_{p,\varphi}$ to represent the norm of $F^p_\varphi(\mathbb{C}^n)$, because it is easy to distinguish between the scalar and the vector case. For more information about generalized Fock spaces $F^p_\varphi(\mathbb{C}^n)$ one may refer to [14]. For $p \geqslant 1$ and $z \in \mathbb{C}^n$, set $k_{p,z}(w) = \frac{K_\varphi(w,z)}{\|K_\varphi(w,z)\|_{p,\varphi}}$, and let $k_\varphi(z,w) = K_\varphi(z,w)/||K_\varphi(\cdot,w)||_{2,\varphi}$. Again, this formula is easily deduced from the reproducing formula of the scalar Fock space $F^2_\varphi(\mathbb{C}^n)$ applied to $z \to \langle Pf(z), h\rangle$, where $h \in \mathcal{H}$ is arbitrary.

In what follows, $\mathcal{L}(\mathcal{H})$ will stand for the space of bounded linear operators on $\mathcal{H}$ with the norm $\|\cdot\|_{\mathcal{L}(\mathcal{H})}$, and $B(\mathcal{L}(\mathcal{H}))$ refers to the Banach space of Bochner integral $\mathcal{L}(\mathcal{H})$-valued functions $G : \mathbb{C}^n \to \mathcal{L}(\mathcal{H})$ with the norm defined by

$$\|G\|_{B(\mathcal{L}(\mathcal{H}))} = \int_{\mathbb{C}^n} \|G(z)\|_{\mathcal{L}(\mathcal{H})}dv(z) < \infty,$$

see [6] for more information. Then, we denote by $\mathcal{T}(\mathcal{L}(\mathcal{H}))$ the space of all strongly positive measurable operator-valued functions $G \in \mathcal{T}(\mathcal{L}(\mathcal{H}))$ that satisfies

$$K_\varphi(\cdot,w)\|G(\cdot)\|_{\mathcal{L}(\mathcal{H})} \in \bigcup_{1\leqslant p<\infty} L^p_\varphi(\mathbb{C}^n)$$

for any $w \in \mathbb{C}^n$. For a $G \in \mathcal{T}(\mathcal{L}(\mathcal{H}))$, we define the Toeplitz operator with operator-valued function G on $\mathbb{C}^n$ as symbol by

$$T_G f(z) = \int_{\mathbb{C}^n} G(w)f(w)K_\varphi(z,w)e^{-2\varphi(w)}dv(w)$$

for any $f \in F^p_\varphi(\mathcal{H})$.

Toeplitz operators on Fock spaces have been studied by many mathematicians, see [7, 8, 10, 12, 15]. The current study mainly focuses on their boundedness, compactness, invertibility, Fredholmness, and Schatten-p class on scalar-valued analytic function spaces. For instance, Hu and Lv [10] studied the boundedness and compactness of the Toeplitz operators $T_\mu : F^p_\alpha \to F^q_\alpha, 1 < p,q < \infty$ with positive Borel measures μ by using (p,q)-Fock Carleson measure. The Schatten-p class of Toeplitz operators was introduced in Zhu's book [15]. Schuster and Varolin [14] by using Fock Carleson measure studied the boundedness and compactness of Toeplitz operators T_μ induced by positive Borel measures μ on generalized Bargmann-Fock spaces $F^p_\varphi(\mathbb{C}^n)$ $(1 \leqslant p < \infty)$, which initiated the study of Toeplitz operators on these generalized Fock spaces.

Although the existing literature in the scalar case boundedness and compactness have been studied for Toeplitz operators acting between F^p_φ and F^q_φ with $0 < p,q < \infty$ with reference to [7, 10]. However, we are currently unable to handle this general situation in vector-valued generalized Fock spaces. We provide two conjectures at the end of this manuscript. Bommier-Hato and Constantin [1] provided complete descriptions for the boundedness, compactness and Schatten class membership of big Hankel operators acting on a large class of vector-valued Fock spaces $\mathcal{F}^2_\varphi(\mathcal{H})$ with radial weights subject to a mild smoothness condition, which differs from the weights in $F^2_\varphi(\mathcal{H})$. So, the point evaluations on $\mathcal{F}^2_\varphi(\mathcal{H})$ are considerably different from $F^2_\varphi(\mathcal{H})$. Chen, Xia and Wang [3] generalized the boundedness and compactness results of [14] and the Schatten class results of [11] the Toeplitz opeartors with positive operator-valued symbols to the setting of vectorial-valued generalized Fock spaces $\mathcal{F}^2_\varphi(\mathcal{H})$. Inspired by Chen, Xia and Wang's work, we define vector-valued p-Carleson conditions on $F^p_\varphi(\mathcal{H})$ to characterize the boundedness and compactness of positive operator-valued Toeplitz operators on $F^p_\varphi(\mathcal{H})$. Using different methods to extend the main results of Chen, Wang and Xia [3], for example, we use the spectral mapping theorem to get equivalent descriptions of vector-valued p-Carleson, instead of the approach used to show Theorem 2.1 in [3].

Now we state our main results.

THEOREM 1.1. *Suppose that $G \in \mathcal{T}(\mathcal{L}(\mathcal{H}))$ and $1 < p < \infty$. The following conditions are equivalent:*

(a) The Toeplitz operator T_G is bounded on $F^p_\varphi(\mathcal{H})$;

(b) G satisfies the p-Fock Carleson condition;

(c) For any $z \in \mathbb{C}^n$, G satisfies

$$\int_{\mathbb{C}^n} \|G(w)\|_{\mathcal{L}(\mathcal{H})} |k_\varphi(w,z)|^2 e^{-2\varphi(w)} dv(w) < \infty.$$

Moreover,

$$\begin{aligned} ||T_G|| &\simeq \sup_{z\in\mathbb{C}^n} \frac{1}{|B(z,r)|} \int_{B(z,r)} \|G(w)\|_{\mathcal{L}(\mathcal{H})} dv(w) \\ &\simeq \sup_{z\in\mathbb{C}^n} \int_{\mathbb{C}^n} \|G(w)\|_{\mathcal{L}(\mathcal{H})} |k_\varphi(w,z)|^2 e^{-2\varphi(w)} dv(w). \end{aligned}$$

THEOREM 1.2. *Suppose that $G \in \mathcal{T}(\mathcal{L}(\mathcal{H}))$ and $1 < p < \infty$. The following conditions are equivalent:*

(a) The Toeplitz operator T_G is compact on $F^p_\varphi(\mathcal{H})$;

(b) G satisfies the vanishing p-Fock Carleson condition;

(c) The function G satisfies

$$\lim_{z\to\infty} \int_{\mathbb{C}^n} \|G(w)\|_{\mathcal{L}(\mathcal{H})} |k_\varphi(w,z)|^2 e^{-2\varphi(w)} dv(w) = 0.$$

REMARK 1.3. Let $N \in \mathbb{N}$ and $\mathcal{H} = \mathbb{C}^N$, then $F^p_\varphi(\mathcal{H}) = F^p_{\varphi,N}$, in view of [9]. Let $(f_{jk})_{1 \leqslant j,k \leqslant N}$ be the set of $N \times N$ matrix. If f in $(f_{jk})_{1 \leqslant j,k \leqslant N}$ is not positive definite, then our results don't apply to block Toeplitz operator T_f. In fact, Theorem 2.7 used the positive definiteness of G when characterizing the Fock-Carleson condition.

As we all know, in the scalar setting, if f is a nonnegative measurable function, then there is a nonnegative measure μ so that $d\mu = f dv$. Unfortunately, an important fact about the Bochner integral is that the Radon-Nikodym theorem fails to hold in general, see [14]. In other words, for an operator-valued function G generally, we can not find a measure μ such that $d\mu = G dv$. Therefore, our results don't extend to the case of general measure symbols except when the Radon-Nikodym property holds.

The organization of this paper is as follows. In Section 2, we first give the boundedness of Bergman projection P from $L^p_\varphi(\mathcal{H})$ onto $F^p_\varphi(\mathcal{H})$ for $1 \leqslant p \leqslant \infty$, and give the duality of the spaces $F^p_\varphi(\mathcal{H})$. We also give the definitions and the equivalent characterizations of the vector-valued p-Fock Carleson condition on $F^p_\varphi(\mathcal{H})$. The main proof of the boundedness and compactness of Toeplitz operators T_G with positive operator-valued symbols G are given in Section 3.

Note that we will write $A \lesssim B$ for two quantities A and B if there exists an unimportant constant $C > 0$ such that $A \leqslant CB$. Furthermore, $B \lesssim A$ is defined similarly and we will write $A \simeq B$ if $A \lesssim B$ and $B \lesssim A$.

2. Bergman projection, duality and Fock Carleson conditions

Firstly, we show that the Bergman projection P is a bounded projection from $L^p_\varphi(\mathcal{H})$ onto $F^p_\varphi(\mathcal{H})$ when $1 \leqslant p \leqslant \infty$ and obtain the dual of the vector-valued generalized Fock spaces. Secondly, We give some equivalent characterizations of the p-Fock Carleson condition. We are going to present some basic conclusions that will be used in the following sections.

For $z \in \mathbb{C}^n$ and $r > 0$, let $B(z,r) = \{w \in \mathbb{C}^n : |w - z| < r\}$. As we all know, there exist positive constants θ and C, depending only on n such that for all $z, w \in \mathbb{C}^n$,

$$|K_\varphi(z,w)| e^{-\varphi(z)} e^{-\varphi(w)} \leqslant C e^{-\theta |z-w|}, \tag{2.1}$$

see [4, 5]; and also, there exists a positive constant r_0 such that

$$|K_\varphi(z,w)| e^{-\varphi(z)} e^{-\varphi(w)} \geqslant C K_\varphi(z,z) e^{-2\varphi(z)} \geqslant C \tag{2.2}$$

for $z \in \mathbb{C}^n$ and $w \in B(z, r_0)$, see [14].

Given some $r > 0$, a sequence $\{a_k\}_{k=1}^\infty$ in $\mathbb{C}^n$ is called an r-lattice if the balls $\{B(a_k, r)\}_{k=1}^\infty$ cover $\mathbb{C}^n$ and $\{B(a_k, \frac{r}{2})\}_{k=1}^\infty$ are pairwise disjoint, and there exists some integer N such that each $z \in \mathbb{C}^n$ can be in at most N balls of $\{B(a_k, 2r)\}$, that is,

$$1 \leqslant \sum_{k=1}^{\infty} \chi_{B(a_k, 2r)}(z) \leqslant N, \quad z \in \mathbb{C}^n.$$

In order to prove the boundedness of Bergman projection, we need the following lemma, see [13].

LEMMA 2.1. *If $Q:\mathbb{C}^n\to\mathcal{H}$ is a dv-Bochner integrable vector-valued function, then the following inequality holds*

$$\left\|\int_{\mathbb{C}^n} Q(w)dv(w)\right\|_{\mathcal{H}} \leqslant \int_{\mathbb{C}^n}\|Q(w)\|_{\mathcal{H}}dv(w).$$

PROPOSITION 2.2. *Let $1\leqslant p\leqslant\infty$. Then the Bergman projection P is bounded as a map from $L^p_\varphi(\mathcal{H})$ to $F^p_\varphi(\mathcal{H})$.*

Proof. For $f\in L^p_\varphi(\mathcal{H})$ with $1<p<\infty$, by Hölder's inequality, Lemma 2.1 and (2.1), we have

$$\begin{aligned}
\|Pf\|^p_{p,\varphi} &= \int_{\mathbb{C}^n}\|Pf(z)\|^p_{\mathcal{H}}e^{-p\varphi(z)}dv(z)\\
&= \int_{\mathbb{C}^n}\left\|\int_{\mathbb{C}^n} f(w)K_\varphi(z,w)e^{-2\varphi(w)}dv(w)\right\|^p_{\mathcal{H}} e^{-p\varphi(z)}dv(z)\\
&\leqslant \int_{\mathbb{C}^n}\left(\int_{\mathbb{C}^n}\|f(w)\|_{\mathcal{H}}|K_\varphi(z,w)|e^{-2\varphi(w)}dv(w)\right)^p e^{-p\varphi(z)}dv(z)\\
&= \int_{\mathbb{C}^n}\left(\int_{\mathbb{C}^n}\|f(w)\|_{\mathcal{H}}|K_\varphi(z,w)|^{\frac{1}{p}}e^{-(1+\frac{1}{p})\varphi(w)}\right.\\
&\qquad \left.\times|K_\varphi(z,w)|^{\frac{1}{q}}e^{-\frac{1}{q}\varphi(w)}dv(w)\right)^p e^{-p\varphi(z)}dv(z)\\
&\leqslant \int_{\mathbb{C}^n}\int_{\mathbb{C}^n}\|f(w)\|^p_{\mathcal{H}}|K_\varphi(z,w)|e^{-(p+1)\varphi(w)}dv(w)\\
&\qquad \times\left(\int_{\mathbb{C}^n}|K_\varphi(z,w)|e^{-\varphi(w)}dv(w)\right)^{p-1} e^{-p\varphi(z)}dv(z)\\
&= \int_{\mathbb{C}^n}\int_{\mathbb{C}^n}\|f(w)\|^p_{\mathcal{H}}e^{-p\varphi(w)}|K_\varphi(z,w)|e^{-\varphi(w)-\varphi(z)}dv(w)\\
&\qquad \times\left(\int_{\mathbb{C}^n}|K_\varphi(z,w)|e^{-\varphi(w)-\varphi(z)}dv(w)\right)^{p-1} dv(z)\\
&\lesssim \int_{\mathbb{C}^n}\int_{\mathbb{C}^n}\|f(w)\|^p_{\mathcal{H}}e^{-p\varphi(w)}|K_\varphi(z,w)|e^{-\varphi(w)-\varphi(z)}dv(w)dv(z)\\
&\lesssim \|f\|^p_{p,\varphi}.
\end{aligned}$$

We use (2.1) and Fubini's theorem similarly to obtain cases $p=1$ and $p=\infty$. □

PROPOSITION 2.3. *Suppose $1<p<\infty$ and let $q=\frac{p}{p-1}$ be the dual exponent. Then*

$$F^p_\varphi(\mathcal{H})^* = F^q_\varphi(\mathcal{H})$$

with equivalent norms and under the integral pairing:

$$\langle f,g\rangle = \int_{\mathbb{C}^n}\langle f(z),g(z)\rangle_{\mathcal{H}}e^{-2\varphi(z)}dv(z)$$

where $f \in F^p_\varphi(\mathcal{H})$ *and* $g \in F^q_\varphi(\mathcal{H})$.

Proof. First given $g \in F^q_\varphi(\mathcal{H})$, we define the linear functional on $F^p_\varphi(\mathcal{H})$ by

$$\Phi_g(f) := \int_{\mathbb{C}^n} \langle f(z), g(z)\rangle_{\mathcal{H}} e^{-2\varphi(z)} dv(z) \tag{2.3}$$

for all $f \in F^p_\varphi(\mathcal{H})$. If (2.3) holds, by Hölder's inequality, we have

$$\begin{aligned}
|\Phi_g(f)| &\leqslant \int_{\mathbb{C}^n} |\langle f(z), g(z)\rangle_{\mathcal{H}}| e^{-2\varphi(z)} dv(z) \\
&\leqslant \int_{\mathbb{C}^n} \|f(z)\|_{\mathcal{H}} \|g(z)\|_{\mathcal{H}} e^{-2\varphi(z)} dv(z) \\
&\leqslant \left(\int_{\mathbb{C}^n} \|f(z)\|^p_{\mathcal{H}} e^{-p\varphi(z)} dv(z)\right)^{\frac{1}{p}} \left(\int_{\mathbb{C}^n} \|f(z)\|^q_{\mathcal{H}} e^{-q\varphi(z)} dv(z)\right)^{\frac{1}{q}} \\
&= \|f\|_{F^p_\varphi(\mathcal{H})} \|g\|_{F^q_\varphi(\mathcal{H})},
\end{aligned}$$

this means $\Phi_g \in F^p_\varphi(\mathcal{H})^*$.

Next, we prove Φ is surjective. Let $F \in F^p_\varphi(\mathcal{H})^*$, by the Hahn-Banach extension Theorem, we can extend F to an element $\widetilde{F} \in L^p_\varphi(\mathcal{H})^*$ such that

$$\|\widetilde{F}\| = \|F\|.$$

By [2, Theorem 1.5.4], there exists $h \in L^q_\varphi(\mathcal{H})$ such that

$$\widetilde{F}(f) = \int_{\mathbb{C}^n} \langle f(z), h(z)\rangle_{\mathcal{H}} e^{-2\varphi(z)} dv(z)$$

for all $f \in L^p_\varphi(\mathcal{H})$, and $\|\widetilde{F}\| = \|h\|_{L^q_\varphi(\mathcal{H})}$.

Let $g = Ph$. Then for $f \in F^p_\varphi(\mathcal{H})$,

$$\begin{aligned}
F(f) = \widetilde{F}(f) &= \int_{\mathbb{C}^n} \langle f(z), h(z)\rangle_{\mathcal{H}} e^{-2\varphi(z)} dv(z) \\
&= \int_{\mathbb{C}^n} \left\langle \int_{\mathbb{C}^n} f(w) K_\varphi(z,w) e^{-2\varphi(w)} dv(w), h(z) \right\rangle_{\mathcal{H}} e^{-2\varphi(z)} dv(z) \\
&= \int_{\mathbb{C}^n} \left\langle f(w) \int_{\mathbb{C}^n} K_\varphi(z,w), h(z) e^{-2\varphi(z)} dv(z) \right\rangle_{\mathcal{H}} e^{-2\varphi(w)} dv(w) \\
&= \int_{\mathbb{C}^n} \left\langle f(w), \int_{\mathbb{C}^n} K_\varphi(w,z) h(z) e^{-2\varphi(z)} dv(z) \right\rangle_{\mathcal{H}} e^{-2\varphi(w)} dv(w) \\
&= \int_{\mathbb{C}^n} \langle f(w), g(w)\rangle_{\mathcal{H}} e^{-2\varphi(w)} dv(w) \\
&= \Phi_g(f).
\end{aligned}$$

Proposition 2.2 gives that $\|g\|_{F^q_\varphi(\mathcal{H})} \leqslant C\|h\|_{L^q_\varphi(\mathcal{H})}$. Thus, we get $\|\Phi_g\| \simeq \|g\|_{F^q_\varphi(\mathcal{H})}$. □

LEMMA 2.4. *If $f \in F_\varphi^p(\mathcal{H})$ $(1 \leqslant p < \infty)$, then $\langle f(z), e\rangle_{\mathcal{H}} \in F_\varphi^p(\mathbb{C}^n)$ for any unit element $e \in \mathcal{H}$.*

Proof. By Cauchy-Schwarz inequality, we have

$$\int_{\mathbb{C}^n} |\langle f(z), e\rangle_{\mathcal{H}}|^p e^{-p\varphi(z)} dv(z) \leqslant \int_{\mathbb{C}^n} \|f(z)\|_{\mathcal{H}}^p \|e\|_{\mathcal{H}}^p e^{-p\varphi(z)} dv(z) < \infty. \quad \square$$

For $1 < p < \infty$, let $G(z)$ be an $\mathcal{L}(\mathcal{H})$-valued positive operator on $\mathbb{C}^n$. According to the functional Calculus, $G^{\frac{1}{p}}(z)$ is still an $\mathcal{L}(\mathcal{H})$-valued positive operator on $\mathbb{C}^n$. In order to characterize the boundedness and compactness of Toeplitz operators with positive operator symbols, we are going to introduce (vanishing) p-Fock Carleson condition for $F_\varphi^p(\mathcal{H})$. In particular, $p = 2$ has been characterized, see [3].

DEFINITION 2.5. Let $1 < p < \infty$ and $G(z)$ be an $\mathcal{L}(\mathcal{H})$-valued positive operator on $\mathbb{C}^n$. We say that G satisfies the p-Fock Carleson condition if there exists a constant $C > 0$ such that

$$\int_{\mathbb{C}^n} \|G^{\frac{1}{p}}(z) f(z)\|_{\mathcal{H}}^p e^{-p\varphi(z)} dv(z) \leqslant C \int_{\mathbb{C}^n} \|f(z)\|_{\mathcal{H}}^p e^{-p\varphi(z)} dv(z).$$

And also, we say that G satisfies the vanishing p-Fock Carleson condition if

$$\lim_{m\to\infty} \int_{\mathbb{C}^n} \|G^{\frac{1}{p}}(z) f_m(z)\|_{\mathcal{H}}^p e^{-p\varphi(z)} dv(z) = 0,$$

where $\{f_m\}$ is a bounded sequence in $F_\varphi^p(\mathcal{H})$ that converges to 0 uniformly on compact subsets of $\mathbb{C}^n$.

The following lemma plays an essential role in characterizing the p-Fock Carleson condition.

LEMMA 2.6. *Let T be a positive operator on $\mathcal{H}$ and $p > 1$. Then*

$$\|T^{\frac{1}{p}}\|^p = \|T\|.$$

Proof. By the spectral mapping theorem, we get

$$\sigma(T^{\frac{1}{p}}) = \sigma(T)^{\frac{1}{p}},$$

where $\sigma(T)$ denote the spectrum of T. If we use $r(T)$ for the spectral radius of T, then

$$r(T^{\frac{1}{p}}) = r(T)^{\frac{1}{p}}.$$

Hence,

$$\|T^{\frac{1}{p}}\|^p = r(T^{\frac{1}{p}})^p = \left[r(T)^{\frac{1}{p}}\right]^p = r(T) = \|T\|. \quad \square$$

Recall that for a positive Borel measure μ on $\mathbb{C}^n$ and $p \geqslant 1$, μ is a Carleson measure for $F_\varphi^p(\mathbb{C}^n)$ if and only if there exists a constant $C>0$ such that $\mu(B(z,1)) \leqslant C$ for any $z \in \mathbb{C}^n$; μ is vanishing Carleson measure for $F_\varphi^p(\mathbb{C}^n)$ if and only if $\lim_{z\to\infty} \mu(B(z,1)) = 0$, one can see [14] for the detailed proof. We next give some equivalent characterizations of the (vanishing) p-Fock Carleson condition, which helps to characterize the boundedness and the compactness of the operator-valued Toeplitz operators.

THEOREM 2.7. *Suppose that $G \in \mathcal{T}(\mathcal{L}(\mathcal{H}))$, $\{a_k\}_{k=1}^\infty$ is a lattice in $\mathbb{C}^n$ and $1 < p < \infty$. The following conditions are equivalent:*

(a) G satisfies the p-Fock Carleson condition;

(b) For all $r>0$ and $z \in \mathbb{C}^n$, G satisfies

$$\int_{B(z,r)} \|G(w)\|_{\mathcal{L}(\mathcal{H})} dv(w) < \infty;$$

(c) For all $r>0$ and $k>0$, G satisfies

$$\int_{B(a_k,r)} \|G(w)\|_{\mathcal{L}(\mathcal{H})} dv(w) < \infty.$$

Proof. Lemma 2.4 tells us that $z \mapsto \langle f(z), e\rangle_{\mathcal{H}}$ is holomorphic in $F_\varphi^p(\mathbb{C}^n)$ for any $e \in \mathcal{H}$, by Proposition 2.3 in [14],

$$\|f(z)\|_{\mathcal{H}}^p e^{-p\varphi(z)} \lesssim \int_{B(z,r)} \|f(w)\|_{\mathcal{H}}^p e^{-p\varphi(w)} dv(w). \tag{2.4}$$

For any $f \in F_\varphi^p(\mathcal{H})$, and since $G(z)$ is positive operator, using the properties of lattice and (2.4), we get

$$\begin{aligned}
&\int_{\mathbb{C}^n} \|G^{\frac{1}{p}}(z) f(z)\|_{\mathcal{H}}^p e^{-p\varphi(z)} dv(z) \\
&\leqslant \sum_{k=1}^\infty \int_{B(a_k,r)} \|G^{\frac{1}{p}}(z) f(z)\|_{\mathcal{H}}^p e^{-p\varphi(z)} dv(z) \\
&\leqslant \sum_{k=1}^\infty \int_{B(a_k,r)} \|f(z)\|_{\mathcal{H}}^p \|G^{\frac{1}{p}}(z)\|_{\mathcal{L}(\mathcal{H})}^p e^{-p\varphi(z)} dv(z) \\
&\leqslant \sum_{k=1}^\infty \int_{B(a_k,r)} \sup_{\|e\|_{\mathcal{H}}=1} \langle G^{\frac{1}{p}}(z)e, e\rangle_{\mathcal{H}}^p \|f(z)\|_{\mathcal{H}}^p e^{-p\varphi(z)} dv(z) \\
&\lesssim \sum_{k=1}^\infty \int_{B(a_k,r)} \sup_{\|e\|_{\mathcal{H}}=1} \langle G(z)e, e\rangle_{\mathcal{H}} dv(z) \int_{B(a_k,2r)} \|f(w)\|_{\mathcal{H}}^p e^{-p\varphi(w)} dv(w) \\
&= \sum_{k=1}^\infty \int_{B(a_k,r)} \|G(z)\|_{\mathcal{L}(\mathcal{H})} dv(z) \int_{B(a_k,2r)} \|f(w)\|_{\mathcal{H}}^p e^{-p\varphi(w)} dv(w) \\
&\lesssim \sum_{k=1}^\infty \int_{B(a_k,2r)} \|f(z)\|_{\mathcal{H}}^p e^{-p\varphi(z)} dv(z) \\
&\lesssim \int_{\mathbb{C}^n} \|f(z)\|_{\mathcal{H}}^p e^{-p\varphi(z)} dv(z).
\end{aligned}$$

This shows that condition (c) implies (a).

$(a) \Rightarrow (b)$. For any $r > 0$, by (2.1) and (2.2), we get

$$|k_{p,z}(w)|^p e^{-p\varphi(w)} \simeq 1, \ \ w \in B(z,r).$$

If condition (a) holds, since G is positive, by Lemma 2.6, we have

$$\begin{aligned}
&\int_{B(z,r)} \|G(w)\|_{\mathcal{L}(\mathcal{H})} dv(w) \\
&= \int_{B(z,r)} \sup_{\|e\|_{\mathcal{H}}=1} \|G^{\frac{1}{p}}(w)e\|_{\mathcal{H}}^p dv(w) \\
&\simeq \int_{B(z,r)} \sup_{\|e\|_{\mathcal{H}}=1} \|G^{\frac{1}{p}}(w)k_{p,z}(w)e\|_{\mathcal{H}}^p e^{-p\varphi(w)} dv(w) \\
&\lesssim \int_{B(z,r)} \sup_{\|e\|_{\mathcal{H}}=1} \|k_{p,z}(w)e\|_{\mathcal{H}}^p e^{-p\varphi(w)} dv(w) \\
&\lesssim \int_{\mathbb{C}^n} |k_{p,z}(w)|^p e^{-p\varphi(w)} dv(w) = 1.
\end{aligned}$$

It is trivial that condition (b) implies (c). The proof is completed. □

THEOREM 2.8. *Suppose that $G \in \mathcal{T}(\mathcal{L}(\mathcal{H}))$, $\{a_k\}_{k=1}^{\infty}$ is a lattice in $\mathbb{C}^n$ and $1 < p < \infty$. The following conditions are equivalent:*

(a) G satisfies the vanishing p-Fock Carleson condition;

(b) For all $r > 0$, G satisfies

$$\lim_{z\to\infty} \int_{B(z,r)} \|G(w)\|_{\mathcal{L}(\mathcal{H})} dv(w) = 0;$$

(c) For all $r > 0$, G satisfies

$$\lim_{k\to\infty} \int_{B(a_k,r)} \|G(w)\|_{\mathcal{L}(\mathcal{H})} dv(w) = 0.$$

The proof can easily be deduced from that of Theorem 2.7 and the details are omitted here.

Let $L^p_\varphi(\mathcal{H}, Gdv)$ be a positive operator-valued weight measurable function space with the following norm

$$\|f\|_{\varphi,G}^p = \int_{\mathbb{C}^n} \|G^{\frac{1}{p}}(z)f(z)\|_{\mathcal{H}}^p e^{-p\varphi(z)} dv(z) < \infty.$$

REMARK 2.9. It is clear that G satisfies the p-Fock Carleson condition if and only if $F^p_\varphi(\mathcal{H}) \subset L^p_\varphi(\mathcal{H}, Gdv)$ and the inclusion map $i_p : F^p_\varphi(\mathcal{H}) \to L^p_\varphi(\mathcal{H}, Gdv)$ is bounded; G satisfies the vanishing p-Fock Carleson condition if and only if the inclusion map $i_p : F^p_\varphi(\mathcal{H}) \to L^p_\varphi(\mathcal{H}, Gdv)$ is compact.

3. Boundedness and compactness of Toeplitz operators

In this section, we will characterize the boundedness and compactness of the Toeplitz operators with positive operator-valued symbols and give several other equivalent descriptions for a positive operator-valued function to satisfy the p-Fock Carleson condition.

We are now ready to prove the main results.

Proof of Theorem 1.1. We first show that $(b) \Rightarrow (a)$. Let (b) holds, using Fubini's theorem, (2.4), and Theorem 2.7, for all $f \in F_\varphi^p(\mathcal{H})$, we have

$$\begin{aligned}
\|T_G f(z)\|_{\mathcal{H}} &= \left\| \int_{\mathbb{C}^n} G(w) f(w) K_\varphi(z,w) e^{-2\varphi(w)} dv(w) \right\|_{\mathcal{H}} \\
&\leqslant \int_{\mathbb{C}^n} \|G(w)\|_{\mathcal{L}(\mathcal{H})} \|f(w)\|_{\mathcal{H}} |K_\varphi(z,w)| e^{-2\varphi(w)} dv(w) \\
&\lesssim \int_{\mathbb{C}^n} \|G(w)\|_{\mathcal{L}(\mathcal{H})} \int_{B(w,r)} \|f(u)\|_{\mathcal{H}} |K_\varphi(z,u)| e^{-2\varphi(u)} dv(u) dv(w) \\
&= \int_{\mathbb{C}^n} \|G(w)\|_{\mathcal{L}(\mathcal{H})} \int_{\mathbb{C}^n} \chi_{B(w,r)}(u) \|f(u)\|_{\mathcal{H}} |K_\varphi(z,u)| e^{-2\varphi(u)} dv(u) dv(w) \\
&= \int_{\mathbb{C}^n} \|f(u)\|_{\mathcal{H}} |K_\varphi(z,u)| e^{-2\varphi(u)} dv(u) \int_{B(u,r)} \|G(w)\|_{\mathcal{L}(\mathcal{H})} dv(w) \\
&\lesssim \int_{\mathbb{C}^n} \|f(u)\|_{\mathcal{H}} |K_\varphi(z,u)| e^{-\varphi(u)} dv(u).
\end{aligned}$$

By Hölder's inequality, we obtain

$$\begin{aligned}
\|T_G f\|^p_{F^p_\varphi(\mathcal{H})} &= \int_{\mathbb{C}^n} \|T_G f(z)\|^p_{\mathcal{H}} e^{-p\varphi(z)} dv(z) \\
&\lesssim \int_{\mathbb{C}^n} \left(\int_{\mathbb{C}^n} \|f(u)\|_{\mathcal{H}} |K_\varphi(z,u)| e^{-\varphi(u)-\varphi(z)} dv(u) \right)^p dv(z) \\
&\leqslant \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} \|f(u)\|^p_{\mathcal{H}} e^{-p\varphi(u)} |K_\varphi(z,u)| e^{-\varphi(u)-\varphi(z)} dv(u) \\
&\quad \times \left(\int_{\mathbb{C}^n} |K_\varphi(z,u)| e^{-\varphi(u)-\varphi(z)} dv(u) \right)^{\frac{p}{q}} dv(z) \\
&\lesssim \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} \|f(u)\|^p_{\mathcal{H}} e^{-p\varphi(u)} |K_\varphi(z,u)| e^{-\varphi(u)-\varphi(z)} dv(u) dv(z) \\
&\lesssim \int_{\mathbb{C}^n} \|f(u)\|^p_{\mathcal{H}} e^{-p\varphi(u)} dv(u) \\
&= \|f\|^p_{F^p_\varphi(\mathcal{H})}.
\end{aligned}$$

where q is the dual exponent of p.

$(a) \Rightarrow (c)$. By [7, (2.4)], (2.1), and (2.2), we obtain

$$\frac{\|K_\varphi(\cdot,z)\|^2_{2,\varphi}}{\|K_\varphi(\cdot,z)\|_{p,\varphi} \|K_\varphi(\cdot,z)\|_{q,\varphi}} \backsimeq 1. \tag{3.1}$$

For any unit vector e in $\mathcal{H}$, by (3.1), we have

$$\begin{aligned}
&\langle T_G k_{p,z}e, k_{q,z}e\rangle \\
&= \int_{\mathbb{C}^n} \langle G(w)e,e\rangle_{\mathcal{H}} \frac{|K_\varphi(w,z)|^2}{\|K_\varphi(\cdot,z)\|_{p,\varphi}\|K_\varphi(\cdot,z)\|_{q,\varphi}} e^{-2\varphi(w)}dv(w) \\
&= \int_{\mathbb{C}^n} \langle G(w)e,e\rangle_{\mathcal{H}} |k_\varphi(w,z)|^2 \frac{\|K_\varphi(\cdot,z)\|_{2,\varphi}^2}{\|K_\varphi(\cdot,z)\|_{p,\varphi}\|K_\varphi(\cdot,z)\|_{q,\varphi}} e^{-2\varphi(w)}dv(w) \\
&\backsimeq \int_{\mathbb{C}^n} \langle G(w)e,e\rangle_{\mathcal{H}} |k_\varphi(w,z)|^2 e^{-2\varphi(w)}dv(w).
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_{\mathbb{C}^n} \langle G(w)e,e\rangle_{\mathcal{H}} |k_\varphi(w,z)|^2 e^{-2\varphi(w)}dv(w) \\
&\lesssim |\langle T_G k_{p,z}e, k_{q,z}e\rangle| \\
&\leqslant \|T_G\| \|k_{p,z}\|_{p,\varphi}\|k_{q,z}\|_{q,\varphi} < \infty,
\end{aligned}$$

which gives

$$\int_{\mathbb{C}^n} \|G(w)\|_{\mathcal{L}(\mathcal{H})} |k_\varphi(w,z)|^2 e^{-2\varphi(w)}dv(w) \leqslant \|T_G\|.$$

$(c) \Rightarrow (b)$. For $z \in \mathbb{D}$ and $w \in B(z,r)$, by (3.1), we have

$$\int_{B(z,r)} \|G(w)\|_{\mathcal{L}(\mathcal{H})} dv(w) \lesssim \int_{\mathbb{C}^n} \|G(w)\|_{\mathcal{L}(\mathcal{H})} |k_\varphi(w,z)|^2 e^{-2\varphi(w)}dv(w) < \infty.$$

The proof is finished. □

LEMMA 3.1. *For $1 < p < \infty$, the functions $k_{p,z}(\cdot)e$ converges weakly to 0 in $F^p_\varphi(\mathcal{H})$ as $z \to \infty$, where e is an unit element in $\mathcal{H}$.*

Proof. Let $F_z = k_{p,z}(\cdot)e$. By Proposition 2.3, for any $g \in F^q_\varphi(\mathcal{H})$, where $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\begin{aligned}
\langle F_w, g\rangle &= \int_{\mathbb{C}^n} \langle F_z(w), g(w)\rangle_{\mathcal{H}} e^{-2\varphi(w)}dv(w) \\
&= \int_{\mathbb{C}^n} k_{p,z}(w)\langle e, g(w)\rangle_{\mathcal{H}} e^{-2\varphi(w)}dv(w) \\
&= \int_{\mathbb{C}^n} k_{p,z}(w)\overline{\langle g(w), e\rangle_{\mathcal{H}}} e^{-2\varphi(w)}dv(w).
\end{aligned}$$

Lemma 2.4 tells us that $\langle g(z), e\rangle_{\mathcal{H}} \in F^q_\varphi(\mathbb{C}^n)$. Thus, we obtain

$$\lim_{z\to\infty} \int_{\mathbb{C}^n} k_{p,z}(w)\overline{\langle g(w), e\rangle_{\mathcal{H}}} e^{-2\varphi(w)}dv(w) = 0,$$

since $k_{p,z}(\cdot)$ weakly converges to 0 in $F^p_\varphi(\mathbb{C}^n)$ as $z \to \infty$, see Proposition 4.2 in [14]. Which means that $k_{p,z}(\cdot)e$ weakly converges to 0 in $F^p_\varphi(\mathcal{H})$ as $z \to \infty$. □

We now turn to the proof of Theorem 1.2.

Proof of Theorem 1.2. Assuming (b) holds, let $\{f_j\}$ be a sequence weakly convergent to 0 in $F_\varphi^p(\mathcal{H})$. For any $r>0$, by Theorem 2.8, we have

$$\lim_{z\to\infty}\int_{B(z,r)}\|G(w)\|_{\mathcal{L}(\mathcal{H})}dv(w)=0.$$

Thus, for $\varepsilon>0$, there is a positive number R such that

$$\int_{B(z,r)}\|G(w)\|_{\mathcal{L}(\mathcal{H})}dv(w)<\varepsilon,\quad |z|\geqslant R.$$

Then, by the same process of the proof of $(b)\Rightarrow(a)$ of the Theorem 1.1, we have

$$\begin{aligned}
&\|T_Gf_j(z)\|^p_{F^p_\varphi(\mathcal{H})}\\
&=\int_{\mathbb{C}^n}\|T_Gf_j(z)\|^p_{\mathcal{H}}e^{-p\varphi(z)}dv(z)\\
&\lesssim\int_{\mathbb{C}^n}\left(\int_{\mathbb{C}^n}\|f_j(u)\|_{\mathcal{H}}|K_\varphi(z,u)|e^{-2\varphi(u)}\int_{B(u,r)}\|G(w)\|_{\mathcal{L}(\mathcal{H})}dv(w)dv(u)\right)^p e^{-p\varphi(z)}dv(z)\\
&\lesssim\int_{\mathbb{C}^n}\left(\int_{B(0,R)}\|f_j(u)\|_{\mathcal{H}}|K_\varphi(z,u)|e^{-2\varphi(u)}\int_{B(u,r)}\|G(w)\|_{\mathcal{L}(\mathcal{H})}dv(w)dv(u)\right)^p\\
&\quad\times e^{-p\varphi(z)}dv(z)\\
&\quad+\int_{\mathbb{C}^n}\left(\int_{\mathbb{C}^n\setminus B(0,R)}\|f_j(u)\|_{\mathcal{H}}|K_\varphi(z,u)|e^{-2\varphi(u)}\int_{B(u,r)}\|G(w)\|_{\mathcal{L}(\mathcal{H})}dv(w)dv(u)\right)^p\\
&\quad\times e^{-p\varphi(z)}dv(z)\\
&\lesssim\int_{\mathbb{C}^n}\left(\int_{B(0,R)}\|f_j(u)\|_{\mathcal{H}}|K_\varphi(z,u)|e^{-2\varphi(u)}dv(u)\right)^p e^{-p\varphi(z)}dv(z)+\varepsilon\|f_j\|^p_{p,\varphi}\\
&\lesssim\int_{B(0,R)}\|f_j(u)\|^p_{\mathcal{H}}e^{-p\varphi(u)}dv(u)+\varepsilon\|f_j\|^p_{p,\varphi}\\
&\lesssim\varepsilon.
\end{aligned}$$

This shows (b) can deduce (a).

$(a)\Rightarrow(c)$. If T_G is compact, by Lemma 3.1, we get

$$\lim_{z\to\infty}\langle T_Gk_{p,z}e,k_{q,z}e\rangle=0.$$

By the proof process of $(a)\Rightarrow(c)$ of the Theorem 1.1, one can see

$$\int_{\mathbb{C}^n}\langle G(w)e,e\rangle_{\mathcal{H}}|k_\varphi(w,z)|^2e^{-2\varphi(w)}dv(w)\lesssim|\langle T_Gk_{p,z}e,k_{q,z}e\rangle|\to 0,$$

as $z\to\infty$.

The proof of implication $(c)\Rightarrow(b)$ is similar to the same part of Theorem 1.1. □

REMARK 3.2. In [7], Hu and Lv defined (p,q)-Fock Carleson measure on the scalar-valued space F^p_φ. We define the (vanishing) (p,q)-Fock Carleson condition on the vector-valued space $F^p_\varphi(\mathcal{H})$ as follows:

Let $1<p,q<\infty$ and $G(z)$ be an $\mathcal{L}(\mathcal{H})$-valued positive operator on $\mathbb{C}^n$. We say that G satisfies the (p,q)-Fock Carleson condition for $F^p_\varphi(\mathcal{H})$ if the inclusion map $i_p: F^p_\varphi(\mathcal{H}) \to L^q_\varphi(\mathcal{H}, Gdv)$ is bounded. And that G satisfies the vanishing (p,q)-Fock Carleson condition if the inclusion map $i_p: F^p_\varphi(\mathcal{H}) \to L^q_\varphi(\mathcal{H}, Gdv)$ is compact.

Then, in view of [7, 10], we raise the following two conjectures:

CONJECTURE 1. Suppose that $G \in \mathcal{T}(\mathcal{L}(\mathcal{H}))$, $\{a_k\}_{k=1}^\infty$ is a lattice in $\mathbb{C}^n$ and $1<p\leqslant q<\infty$. The following conditions are equivalent:

(a) G satisfies the (p,q)-Fock Carleson condition;

(b) For all $r>0$ and $k>0$, G satisfies

$$\int_{B(a_k,r)} \|G(w)\|_{\mathcal{L}(\mathcal{H})} dv(w) < \infty.$$

CONJECTURE 2. Suppose that $G \in \mathcal{T}(\mathcal{L}(\mathcal{H}))$, $\{a_k\}_{k=1}^\infty$ is a lattice in $\mathbb{C}^n$ and $1<q<p<\infty$. The following conditions are equivalent:

(a) G satisfies the vanishing (p,q)-Fock Carleson condition;

(b) G satisfies the (p,q)-Fock Carleson condition;

(c) For all $r>0$, G satisfies

$$\sum_{k=1}^\infty \int_{B(a_k,r)} \|G(w)\|_{\mathcal{L}(\mathcal{H})} dv(w) < \infty.$$

Noticed that the weight function φ in the present paper satisfies the doubling measure hypothesis [8]. It is natural to consider the properties of Toeplitz operators on vector-valued doubling Fock spaces. So, we raise the following open problem:

OPEN PROBLEM. *How to characterize the boundedness and compactness of Toeplitz operators on vector-valued doubling Fock spaces?*

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