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SCALING POSITIVE DEFINITE MATRICES TO ACHIEVE PRESCRIBED EIGENPAIRS

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Abstract. We investigate the problem of scaling a given positive definite matrix A to achieve a prescribed eigenpair (λ, v), by way of a diagonal scaling D^*AD. We consider the case where D is required to be positive, as well as the case where D is allowed to be complex. We generalize a few classical results, and then provide a partial answer to a question of Pereira and Boneng regarding the number of complex scalings of a given 3×3 positive definite matrix A.

1. Introduction

The study of (diagonal) matrix scalings began in earnest in 1964, when Richard Sinkhorn introduced a method for scaling a given entry-wise positive matrix A into a matrix with unit row and column sums (i.e. scaling A into a *doubly stochastic* matrix). In the decades that followed, the problem of scaling a given matrix to obtain desired row sums became a widely studied problem (see [8], [1], [6] or the excellent survey paper [5]).

In this paper, we seek to generalize the above work, investigating the problem of scaling a given positive definite matrix to have desired *eigenpairs*. This can indeed be seen as a generalization of the usual (doubly) stochastic scaling, as a matrix has unit row sums if and only if it preserves the all-ones vector.

We will begin by restricting ourselves to real entries only. In Section 3.1, we will prove that for any given *real* $n\times n$ positive definite matrix A and pair (λ, v), where $\lambda > 0$ and $v \in \mathbb{R}^n$ with no zero components, we can find a unique positive diagonal matrix D such that DAD has eigenpair (λ, v). While the proof of this result is a relatively straightforward consequence of a classical scaling result of Marshall and Olkin, it does not seem to appear anywhere in the literature.

After establishing the above result, we then consider the case where v is permitted to be a *complex* eigenvector in Section 3.2. This is easily seen to be equivalent to the problem of finding positive D such that DAD has a repeated eigenvalue, which is the primary focus of Section 3.2. In particular, we solve the problem in low dimensions

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($n \leqslant 3$), providing precise conditions for when a given 3×3 positive definite real matrix can be scaled to have repeated eigenvalues (and, as a consquence, certain complex eigenvectors).

In Section 4.1, we relax our assumptions on A and D, and consider a slightly different type of scaling, formally introduced by Pereira and Boneng in [11] (though considered earlier in such papers as [10]). In particular, we prove that for any given positive definite *complex* matrix A and pair (λ, v), where $\lambda > 0$ and $v \in \mathbb{C}^n$ with no zero components, we can find a *complex* diagonal matrix D such that D^*AD has eigenpair (λ, v). It is worth noting that (unlike in the case where D is positive, above) there is no guarantee that the desired D^*AD scaling is unique. Indeed, motivated by a question posed in [11], the remainder of the paper is dedicated to the question of *how many* of these "complex scalings" exist for a given A.

In Section 4.2, we provide proof that for 3×3 *real* matrices A, there are at most 6 complex scalings with the desired eigenpair. Although this is a generalization of a result in [3], our proof is shorter and more concise.

Lastly, the purpose of Section 4.3 is to provide a bound on the number of complex scalings when A is allowed to be a *complex* 3×3 matrix. This is an open problem, considered in [3] and [11], to which we provide a partial answer. Our method of proof also gives a method for finding all scalings of a given matrix, as we represent the set of scalings as intersection points of two particular plane curves. In cases where these curves have no common component, an application of Bezout's Theorem then provides the desired upper bound. As in the real case, this bound is found to be 6.

2. Definitions and notation

We take this opportunity to establish some notation and conventions that will we use.

Given a vector $v \in \mathbb{C}^n$, we denote the $j-$th component of v as v_j. We will make frequent use of the following notation:

$$\mathbb{R}^n_+ = \{v \in \mathbb{R}^n | v_j \geqslant 0 \text{ for all } j \in \{1, 2, \ldots, n\}\}$$

$$\mathbb{R}^n_{++} = \{v \in \mathbb{R}^n | v_j > 0 \text{ for all } j \in \{1, 2, \ldots, n\}\}.$$

When the dimension is obvious from context, we will denote the all-ones vector as e (that is, $e = (1, 1, 1, \ldots, 1)^T$). We will use capital roman letters (A, B, C, etc.) to represent matrices, with their entries denoted by their lower-case counterparts (a_{ij}, b_{ij}, c_{ij}, etc.). Absolute value bars around a matrix ($|A|$) will be used to indicate the entry-wise absolute value function (i.e. If $C = |A|$, then $c_{ij} = |a_{ij}|$ for all i, j).

When we are discussing eigenvectors v, it will make our arguments significantly more concise to make the following assumption:

Let v be an eigenvector of a matrix A, where it is known that v has no zero entries. Then we will always assume that $v_1 = 1$.

This will be a safe assumption to make, as we take care to make no arguments that concern the norm of v, nor any other property that might be affected by multiplying v by a scalar.

Lastly, for any positive definite (i.e. Hermitian with positive eigenvalues) matrix $A \in \mathbb{C}^{n\times n}$, we use A_{jj} to denote the (positive definite) $(n-1)\times(n-1)$ principal submatrix obtained by removing the jth row and jth column from A.

3. Positive (DAD) scalings

3.1. Real matrices, real eigenvectors

In 1968, Marshall and Olkin proved the following:

PROPOSITION 3.1. ([7], Corollary 2) *Let A be an $n\times n$ positive definite real matrix, and let $r \in \mathbb{R}^n_{++}$. Then there exists a unique positive diagonal matrix D such that DAD has i-th row (and column) sum equal to r_j for all $j \in \{1,2,\dots,n\}$.*

We will use this result to show that any positive definite real matrix can be scaled to a matrix with a desired (real) eigenpair (λ, v), provided $\lambda > 0$ and v has no zero components. While this is a relatively straightforward consequence of Proposition 3.1, it does not seem to appear anywhere in the literature:

THEOREM 3.2. *Let A be an $n\times n$ positive definite, real matrix. Then for any $\lambda > 0$, and any $v \in \mathbb{R}^n$ with no zero components, there exists a unique, positive definite diagonal matrix D such that $DADv = \lambda v$.*

Proof. Existence: Let us define $p \in \mathbb{R}^n_{++}$ as the vector satisfying $p_k = |v_k|$ for all $k \in \{1,2,\dots,n\}$. Further, let us define U to be the diagonal orthogonal matrix satisfying $U_{kk} = \frac{v_k}{|v_k|}$ for all $k \in \{1,2,\dots,n\}$. A moment's thought should convince the reader that $Up = v$.

Now we define $B = UAU$. As B is positive definite, Proposition 3.1 guarantees the existence of a unique positive definite diagonal matrix F such that FBF has row sums $r_k = (p_k)^2$. Defining $P = diag(p)$, we can express this as $FBFe = P^2 e$, whence we obtain the following:

$$
\begin{aligned}
FBFe &= P^2 e \\
P^{-1}FBFe &= Pe \\
P^{-1}FBFP^{-1}Pe &= Pe
\end{aligned}
$$

Recalling that $B = UAU$, and then exploiting the fact that diagonal matrices commute, this yields:

$$
\begin{aligned}
FP^{-1}UAUFP^{-1}Pe &= Pe \\
UFP^{-1}AFP^{-1}UPe &= Pe
\end{aligned}
$$

As U is orthogonal and diagonal, $U^{-1} = U$. Thus:

$$FP^{-1}AFP^{-1}UPe = UPe$$
$$FP^{-1}AFP^{-1}v = v$$

where the last equality uses the fact that $UPe = v$. Defining $D := \sqrt{\lambda}FP^{-1}$, we see that $DADv = \lambda v$, as desired.

Uniqueness: Suppose now that there is another $n \times n$ positive definite diagonal matrix E satisfying $EAEv = \lambda v$. Then:

$$\begin{aligned} EAEv &= \lambda v \\ EAEPUe &= \lambda PUe \\ UEAEPUe &= \lambda Pe \end{aligned}$$

Again, we use the fact that diagonal matrices commute to obtain:

$$\begin{aligned} E(UAU)EPe &= \lambda Pe \\ EBEPe &= \lambda Pe \\ (P^{-1}P)EBEPe &= \lambda Pe \end{aligned}$$

Multiplying by P:

$$\begin{aligned} PEBEPe &= \lambda P^2 e \\ (EP)B(EP)e &= \lambda P^2 e \\ \left(\frac{1}{\sqrt{\lambda}}EP\right) B \left(\frac{1}{\sqrt{\lambda}}EP\right) e &= P^2 e. \end{aligned}$$

As F is the unique matrix satisfying $FBFe = P^2 e$, this means that $EP = \sqrt{\lambda}F$, yielding $E = \sqrt{\lambda}FP^{-1} = D$, as desired. $\square$

Now that we know that we can scale a given real positive definite matrix to achieve any desired *real* eigenvector with no zero components, we turn our attention to the more difficult problem of complex eigenvectors.

3.2. Real matrices, complex eigenvectors

We now discuss the problem of, given a positive definite real matrix A, finding a positive diagonal matrix D such that DAD has a desired *complex* eigenvector v with no zero entries. We immediately see that (unlike when v is real), there will be vectors v that are unachievable (e.g. It is easy to see that one can never scale a non-diagonal 2×2 positive definite real matrix to obtain the eigenvector $(1, i)^T$). Thus, we wish to discover which complex eigenvectors *are* achievable via our DAD scaling.

Recall that we always assume that eigenvectors with no zero entries satisfy $v_1 = 1$. This ensures that when we discuss eigenvectors in $\mathbb{C}^n \setminus \mathbb{R}^n$, we do not risk discussing scalar multiples of real vectors.

A moment's thought yields the following lemma:

LEMMA 3.3. *Let B be a positive definite real matrix, and suppose that $\lambda > 0$ and $v \in \mathbb{C}^n \setminus \mathbb{R}^n$. If $Bv = \lambda v$, then λ is a repeated eigenvalue for B.*

Proof. Since B is a positive definite real matrix, we may choose the eigenvectors of B to be real. As v lies in the eigenspace of λ, but is not a scalar multiple of a real vector (as $v_1 = 1$ and $v \notin \mathbb{R}^n$), it must be the case that the eigenspace of λ has dimension at least 2. $\square$

Thus, if we wish to scale a given real matrix A to have a specified complex eigenvector, we must find a diagonal matrix D such that DAD has a repeated eigenvalue. This allows us to answer the question for certain sets of matrices, starting with the easy 2×2 matrices. We use the following result, which will also be useful later:

PROPOSITION 3.4. ([9], Lemma 7.7.1) *Let T be a symmetric tridiagonal matrix, such that $t_{ij} \neq 0$ whenever $|i-j| = 1$. Then all eigenvalues of T are simple.*

This yields the following result. (It is fairly obvious, but we include it here for completeness.)

COROLLARY 3.5. *Let A be a 2×2 positive definite real matrix. Then A cannot be scaled to have a non-real complex eigenvector unless it is diagonal.*

Proof. If A has non-zero off-diagonal entries, then any scaling of A will also have non-zero off-diagonal entries. By Proposition 3.4, this means that any scaling of A cannot have a repeated eigenvalue, nor (by Lemma 3.3) a complex eigenvector. $\square$

REMARK 1. Of course, If A is diagonal, then it can be scaled to λI_n, achieving any eigenpair we wish.

Now let us consider the case of 3×3 matrices. We first consider the case where A has zero entries.

PROPOSITION 3.6. *Let A be a 3×3 positive definite real matrix with zero entries, and let $\lambda > 0$. Then the following conditions hold:*

1) If A has only two zero entries, then A cannot be scaled to a matrix with repeated eigenvalue λ.

2) If A has four zero entries, then A can be scaled to infinitely many matrices with repeated eigenvalue λ.

3) If A is diagonal, then it can be scaled to four matrices with repeated eigenvalue λ.

Proof. The first case follows from Proposition 3.4 and the observation that A is permutationally equivalent to a tridiagonal matrix.

The third case is trivial, as A is a diagonal matrix, and we simply scale A to a diagonal matrix with (at least) two identical diagonal elements in order to achieve repeated eigenvalues. There are four such matrices.

For the second case, let us assume without loss of generality that $a_{12} = a_{13} = 0$, and let C be the 2×2 principal submatrix $C = A_{11}$.

Now, suppose we want to scale A to have repeated eigenvalue λ. In this case, we may let $D = diag(d_1, d_2, d_3)$, where $d_1 = \frac{\sqrt{\lambda}}{\sqrt{a_{11}}}$, and d_2, d_3 are any values such that $E = diag(d_2, d_3)$ scales C into a matrix with eigenvalue λ. Theorem 3.2 guarantees that we have infinitely many of these scalings - one corresponding to each choice of real eigenvector with no zero components. $\square$

REMARK 2. We note that in case 2 of Proposition 3.6, the achievable complex eigenvectors v (with no zero entries) are those that satisfy the condition that $(v_2, v_3)^T$ is a (complex) scalar multiple of some vector $w \in \mathbb{R}^2$. We simply by choose E so that $ECEw = \lambda w$, which guarantees that v is in the eigenspace of λ, i.e. the space spanned by $\{(1,0,0)^T, (0, w_1, w_2)^T\}$.

The case where A has no zero entries is a bit more difficult, and we will use the following lemma of Olaf Dietrich:

LEMMA 3.7. ([2]) *Let A be a 3×3 real symmetric matrix with characteristic polynomial $p(x) = (x-\lambda_1)^2(x-\lambda_2)$. Then the entries of A satisfy both of the following conditions:*

$$a_{11} = a_{33} + a_{13}\left(\frac{a_{12}}{a_{23}} - \frac{a_{23}}{a_{12}}\right)$$

$$a_{11} = a_{22} + a_{12}\left(\frac{a_{13}}{a_{23}} - \frac{a_{23}}{a_{13}}\right).$$

While Dietrich proved that the above conditions are necessary for an eigenvalue to be repeated, we will prove that they are sufficient as well:

LEMMA 3.8. *Let A be a real 3×3 symmetric matrix satisfying:*

$$a_{11} = a_{33} + a_{13}\left(\frac{a_{12}}{a_{23}} - \frac{a_{23}}{a_{12}}\right)$$

$$a_{11} = a_{22} + a_{12}\left(\frac{a_{13}}{a_{23}} - \frac{a_{23}}{a_{13}}\right).$$

Then the characteristic polynomial of A is given by $(x-\lambda_1)^2(x-\lambda_2)$, where:

$$\lambda_1 = a_{11} - \frac{a_{13}a_{12}}{a_{23}} \qquad \lambda_2 = a_{11} + \frac{a_{13}a_{23}}{a_{12}} + \frac{a_{12}a_{23}}{a_{13}}.$$

Proof. It is well-known that the characteristic polynomial of a 3×3 matrix A can be written as

$$p(x) = x^3 - tr(A)x^2 + \frac{1}{2}\left((tr(A))^2 - tr(A^2)\right)x - det(A).$$

Thus, it suffices to show that $tr(A) = 2\lambda_1 + \lambda_2$, $tr(A^2) = 2\lambda_1^2 + \lambda_2^2$, and $det(A) = \lambda_1^2 \lambda_2$.

We will make frequent use of the following equalities, which are just rearrangements of the conditions on A given in the statement of the lemma:

$$a_{22} = a_{11} - \left(\frac{a_{12}a_{13}}{a_{23}} - \frac{a_{12}a_{23}}{a_{13}} \right)$$

$$a_{33} = a_{11} - \left(\frac{a_{13}a_{12}}{a_{23}} - \frac{a_{13}a_{23}}{a_{12}} \right)$$

$\mathbf{tr(A) = 2\lambda_1 + \lambda_2}$: Using the above expressions for a_{22} and a_{33}, we obtain:

$$\begin{aligned} tr(A) &= a_{11} + a_{22} + a_{33} \\ &= a_{11} + \left(a_{11} - \frac{a_{12}a_{13}}{a_{23}} + \frac{a_{12}a_{23}}{a_{13}} \right) + \left(a_{11} - \frac{a_{13}a_{12}}{a_{23}} + \frac{a_{13}a_{23}}{a_{12}} \right) \\ &= 3a_{11} - \frac{a_{12}a_{13}}{a_{23}} + \frac{a_{12}a_{23}}{a_{13}} - \frac{a_{13}a_{12}}{a_{23}} + \frac{a_{13}a_{23}}{a_{12}} \\ &= 2\left(a_{11} - \frac{a_{13}a_{12}}{a_{23}} \right) + \left(a_{11} + \frac{a_{13}a_{23}}{a_{12}} + \frac{a_{12}a_{23}}{a_{13}} \right) \\ &= 2\lambda_1 + \lambda_2. \end{aligned}$$

We can likewise show that $tr(A^2) = 2\lambda_1^2 + \lambda_2^2$, and $det(A) = \lambda_1^2\lambda_2$, but these are a bit more tedious to work through, and for this reason, we relegate the proof of these equalities to Appendix A.1. $\square$

From this result, we obtain the following:

THEOREM 3.9. *Let A be a 3×3 positive definite real matrix with no zero entries. Then (up to multiplication by a scalar) there is at most one positive definite diagonal matrix D such that DAD has a repeated eigenvalue. Further, such a D exists if and only if:*

$$\frac{a_{11} - \frac{a_{13}a_{12}}{a_{23}}}{a_{33} - \frac{a_{13}a_{23}}{a_{12}}} > 0 \qquad \textit{and} \qquad \frac{a_{11} - \frac{a_{12}a_{13}}{a_{23}}}{a_{22} - \frac{a_{12}a_{23}}{a_{13}}} > 0.$$

Proof. Let us denote $D = diag(d_1, d_2, d_3)$, where (multiplying D by an appropriate scalar, if necessary) we may assume that $d_1 = 1$. By Lemma 3.7 and Lemma 3.8, we know that DAD will have a repeated eigenvalue if and only if it satisfies the two conditions in those lemmas. i.e. DAD has a repeated eigenvalue if and only if:

$$a_{11} = d_3^2 a_{33} + d_3 a_{13} \left(\frac{d_2 a_{12}}{d_2 d_3 a_{23}} - \frac{d_2 d_3 a_{23}}{d_2 a_{12}} \right)$$

and

$$a_{11} = d_2^2 a_{22} + d_2 a_{12} \left(\frac{d_3 a_{13}}{d_2 d_3 a_{23}} - \frac{d_2 d_3 a_{23}}{d_3 a_{13}} \right)$$

which simplifies to

$$a_{11} = d_3^2 a_{33} + \frac{a_{13}a_{12}}{a_{23}} - d_3^2 \frac{a_{13}a_{23}}{a_{12}} \qquad \text{and} \qquad a_{11} = d_2^2 a_{22} + \frac{a_{12}a_{13}}{a_{23}} - d_2^2 \frac{a_{12}a_{23}}{a_{13}}$$

or, equivalently:

$$d_3 = \sqrt{\frac{a_{11} - \frac{a_{13}a_{12}}{a_{23}}}{a_{33} - \frac{a_{13}a_{23}}{a_{12}}}} \qquad \text{and} \qquad d_2 = \sqrt{\frac{a_{11} - \frac{a_{12}a_{13}}{a_{23}}}{a_{22} - \frac{a_{12}a_{23}}{a_{13}}}}$$

This completely determines the matrix D, and hence our result is proven. □

REMARK 3. We originally introduced the question of repeated eigenvalues in order to see which complex eigenvectors were attainable. We now have a way of answering that question for 3×3 real matrices. If the conditions in Theorem 3.9 hold, then we simply scale A by the unique (up to scalar multiplication) D that yields a repeated eigenvalue, given in the above proof. The only attainable complex eigenvectors are those in the associated (two-dimensional) eigenspace.

In higher dimensions, the problem of scaling a real matrix to have a repeated eigenvalue seems quite difficult. We pose the following:

OPEN PROBLEM. Find an upper bound $k_n \in \mathbb{N}$ such that for any $n \times n$ positive definite real matrix A, there are at most k_n positive definite diagonal matrices D (up to multiplication by a scalar) such that DAD has a repeated eigenvalue (or prove that no such bound exists).

4. Complex (D^*AD) scalings

4.1. Complex matrices, complex eigenvectors

Given a positive definite real matrix A, we have just seen that there are complex eigenvectors that are unobtainable via our DAD scalings. In order to remedy this, we now consider a generalization of these scalings, where we allow D to be a *complex* diagonal matrix, and consider D^*AD scalings. This type of scaling (which we will henceforth call a "complex scaling") was originally introduced in [11], where the authors demonstrated a relationship between complex scalings and the geometric measure of entanglement of certain symmetric states. Further results on complex scalings were developed in [3] and [4]. We prove the following:

THEOREM 4.1. *Let A be an $n \times n$ positive definite (complex) matrix. Then for any $\lambda > 0$ and $v \in \mathbb{C}^n$ with no zero components, there exists an $n \times n$ complex diagonal matrix D such that $D^*ADv = \lambda v$.*

In order to prove Theorem 4.1, we will need the following result, the proof of which is just a very slight generalization of the argument used to prove the more restrictive Lemma 2.9 in [10]:

PROPOSITION 4.2. *Let A be an $n \times n$ positive definite matrix, and suppose we have a positive vector $a = (a_1, a_2, \dots a_n)^T \in \mathbb{R}^n_{++}$. Then there exists a diagonal matrix D such that $D^*ADe = a$ (i.e. D^*AD has ith row sum a_i.)*

Proof. Define $\Omega = \{v \in \mathbb{C}^n : \prod_{j=1}^n |v_j|^{a_j} = 1\}$. We will show that the vector w, defined as $w := \min_{v \in \Omega} v^*Av$, satisfies $(D_w)^*AD_we = ta$, where $D_w = diag(w)$ and t is some positive constant. In order to do this, we must first show that such a minimimizing vector w exists:

Pick any $v_* \in \Omega$, and let λ_1 denote the smallest eigenvalue of A. We immediately see that $(v_*)^*Av_* \geqslant \lambda_1 \|v_*\|^2$. Rearranging, we obtain $\|v_*\| \leqslant \sqrt{\frac{(v_*)^*Av_*}{\lambda_1}} =: k$. Define the set $\mathscr{M} = \{v \in \Omega : \|v\| \leqslant k\}$. This set is compact (it is closed and bounded), and it contains at least one element (v_*). By the extreme value theorem, the function $f(v) = v^*Av$ has a minimum over $\mathscr{M}$. Further, we claim that this must be a minimum over *all* of Ω, since for any element $v \in \Omega \setminus \mathscr{M}$, we must have

$$f(v) = v^*Av \geqslant \lambda_1 ||v||^2 \geqslant \lambda_1 k^2 = \lambda_1 \frac{(v_*)^*Av_*}{\lambda_1} = f(v_*).$$

Now that we know that w necessarily exists, we are going to show that $B = (D_w)^*AD_w$ satisfies $Be = ta$ for some $t > 0$ (i.e. B is a positive scalar multiple of a matrix with the desired row sums). We begin by showing that B has all *real* row sums:

Indeed, suppose that one of the rows of B has non-real row sum, $r_j \in \mathbb{C} \setminus \mathbb{R}$, and let $r_j - b_{jj} = se^{i\theta}$. Now define $y \in \Omega$ to be the vector identical to w, except with jth component equal to $w_je^{-i(\pi-\theta)}$. Then

$$\begin{aligned} y^*Ay &= w^*Aw - 2(r_j - b_{jj}) + 2(r_j - b_{jj})e^{i(\pi-\theta)} \\ &= w^*Aw - 2(r_j - b_{jj}) - |2(r_j - b_{jj})| \\ &< w^*Aw \end{aligned}$$

contradicting the definition of w. Thus, B cannot have any non-real row sums.

We will now show that for any two row sums of B, r_i and r_j, it must be the case that $\frac{r_i}{a_i} = \frac{r_j}{a_j}$. Without loss of generality, we show that $\frac{r_1}{a_1} = \frac{r_2}{a_2}$:

To this end, let $\varepsilon > 0$, and suppose without loss of generality that $\frac{r_2}{a_2} \leqslant \frac{r_1}{a_1}$. We consider the vector $s = \left((1-\varepsilon)^{\frac{1}{a_1}}, (1-\varepsilon)^{\frac{-1}{a_2}}, 1, 1, \dots, 1\right)^T \in \Omega$. Then:

$$\begin{aligned} f(s) = s^*Bs =& (1-\varepsilon)^{\frac{2}{a_1}} b_{11} + 2(1-\varepsilon)^{\left(\frac{1}{a_1} - \frac{1}{a_2}\right)} \Re(b_{12}) + 2(1-\varepsilon)^{\frac{1}{a_1}} \sum_{j=3}^n \Re(b_{1j}) \\ &+ (1-\varepsilon)^{\frac{-2}{a_2}} b_{22} + 2(1-\varepsilon)^{\frac{-1}{a_2}} \sum_{j=3}^n \Re(b_{2j}) \\ &+ \sum_{j=3}^n \sum_{i=3}^n \Re(b_{ij}) \end{aligned}$$

Making use of the Maclaurin series expansions of $(1-\varepsilon)^{\frac{2}{a_1}}$, $(1-\varepsilon)^{\left(\frac{1}{a_1}-\frac{1}{a_2}\right)}$, $(1-\varepsilon)^{\frac{1}{a_1}}$, $(1-\varepsilon)^{\frac{-2}{a_2}}$, and $(1-\varepsilon)^{\frac{-1}{a_2}}$, the above expression becomes:

$$\begin{aligned} f(s) = &\left(1-\frac{2}{a_1}\varepsilon+\mathscr{O}(\varepsilon^2)\right)b_{11}+2\left(1-\left(\frac{1}{a_1}-\frac{1}{a_2}\right)\varepsilon+\mathscr{O}(\varepsilon^2)\right)\Re(b_{12}) \\ &+2\left(1-\frac{1}{a_1}\varepsilon+\mathscr{O}(\varepsilon^2)\right)\sum_{j=3}^{n}\Re(b_{1j})+\left(1+\frac{2}{a_2}\varepsilon+\mathscr{O}(\varepsilon^2)\right)b_{22} \\ &+2\left(1+\frac{1}{a_2}\varepsilon+\mathscr{O}(\varepsilon^2)\right)\sum_{j=3}^{n}\Re(b_{2j})+\sum_{j=3}^{n}\sum_{i=3}^{n}\Re(b_{ij}) \end{aligned}$$

Rearranging (and dropping the "$\Re$", as we know row sums of B are real), we obtain...

$$\begin{aligned} f(s) &= \sum_{j=1}^{n}\sum_{i=1}^{n}b_{ij} \\ &\quad -\frac{2}{a_1}\varepsilon(b_{11}+b_{12}+b_{13}+b_{14}+\ldots+b_{1n}) \\ &\quad +\frac{2}{a_2}\varepsilon(b_{21}+b_{22}+b_{23}+b_{24}+\ldots+b_{2n}) \\ &= \sum_{j=1}^{n}\sum_{i=1}^{n}b_{ij}+2\varepsilon\left(\frac{r_2}{a_2}-\frac{r_1}{a_1}\right)+\mathscr{O}(\varepsilon^2) \end{aligned}$$

As we know the minimum is obtained when $\varepsilon=0$, we must have $\frac{r_2}{a_2}-\frac{r_1}{a_1}=0$, i.e. $\frac{r_2}{a_2}=\frac{r_1}{a_1}$. Repeating this argument for all other pairs of rows, we find that $\frac{r_i}{a_i}=\frac{r_j}{a_j}$ for all $i\neq j$, as desired.

To finish the proof, we define $t:=\frac{r_j}{a_j}>0$. Then $r_j=ta_j$, and thus $Be=ta$. We finish by noting that the diagonal matrix $D=\frac{D_w}{\sqrt{t}}$ satisfies $D^*ADe=\frac{Be}{t}=a$, as desired. □

We can now prove Theorem 4.1

Proof of Theorem 4.1. The proof is identical to that of the "Existence" proof of Theorem 3.2, replacing all instances of UAU (and DAD) with U^*AU (and D^*AD) where required, and using Proposition 4.2 in place of Proposition 3.1. □

An important part of Theorem 3.2 was the uniqueness of the diagonal matrix D. In Theorem 4.1, however, we had to sacrifice uniqueness in order to guarantee existence. It is therefore worth asking *how many* matrices D (up to multiplication by a scalar) exist that provide the prescribed eigenpair.

4.2. How many complex scalings exist – real A, real eigenvectors

The problem of ennumerating complex scalings is a difficult one, and the case where the desired eigenpair is $(1,e)$ has been considered before. Indeed, the authors of [11] posed the following question (though they framed it as a conjecture which was later falsified in [3]):

QUESTION 4.3. ([11]) *Given an $n \times n$ positive definite matrix A, how many complex scalings $B = D^*AD$ exist that preserve the all-ones vector?*

REMARK 4. The above question is equivalent to searching for complex scalings that have all row and column sums equal to one (a property known as being "doubly quasi-stochastic").

In [3], it was shown that when A is a real, 3×3 matrix, the answer to the above question is 6. Here, we provide a much simpler (and shorter) proof of this fact, while also generalizing the statement to arbitrary real eigenpairs (with no zero entries):

PROPOSITION 4.4. *Let A be a 3×3 positive definite real matrix, and let $\lambda > 0$ and $v \in \mathbb{R}^3$ with no zero entries. Then there are at most six complex scalings $B = D^*AD$ such that $Bv = \lambda v$.*

Proof. We first consider the case where A has zero entries. If A is diagonal, then the only scaling is $D^*AD = \lambda I_3$. If A has four zero entries, let us assume that $a_{12} = a_{13} = 0$. We must have two scalings, where $b_{11} = \lambda$ and the submatrix B_{11} is a real matrix satisfying $B_{11}(v_2, v_3)^T = \lambda(v_2, v_3)^T$, of which there can only be two, by the uniqueness of D in Theorem 3.2 (one for each possible sign pattern of B_{11}). Lastly, if A has two zero entries, let us assume that $a_{13} = 0$. If $Bv = \lambda v$, then (since v is real), we must have that B_{12} is real, which means that B_{23} must also be real, and hence B is real matrix. Again, we see that (by the uniqueness of D in Theorem 3.2), there can only be as many scalings as possible sign patterns of B, i.e. four. As we have now proven the result for matrices with zero entries, hereafter we will assume that A has no zero entries.

Suppose that $D^*ADv = \lambda v$. Multiplying D by a scalar ω from the complex unit circle will not affect the scaling D^*AD, and so we may assume $d_{11} > 0$. Further, let $D = PU$ be the polar decomposition of D (so P is a diagonal matrix with strictly positive entries and U is a diagonal unitary matrix with $u_{11} = 1$). Then $D^*ADv = \lambda v$ is equivalent to $U^*PAPUv = \lambda v$, or $PAP(Uv) = \lambda(Uv)$.

If U is real, then it must be one of $U_1 := diag(1,1,1)$, $U_2 := diag(1,1,-1)$, $U_3 := diag(1,-1,1)$ or $U_4 := diag(1,1,-1)$. Theorem 3.2 states that for each of these U_k, there is exactly one P_k that yields $P_kAP_k(U_kv) = \lambda(U_kv)$. Thus we have (exactly) 4 real scalings, corresponding to $D_k = P_kU_k$ for each $k \in \{1,2,3,4\}$. These scalings will necessarily be distinct, as each $D_k^*AD_k$ has a different sign pattern (by the definitions of U_k).

If U is non-real, then PAP has a complex eigenvector $Uv = (1, z_1, z_2)^T$ where at least one of z_1, z_2 is non-real. Thus, by Lemma 3.3, PAP must have a repeated eigenvalue λ. By Theorem 3.9, there is at most one P such that PAP has repeated eigenvalue λ. Supposing that such a P exists, we claim that there are at most two U satisfying $PAP(Uv) = \lambda(Uv)$. To this end, let us observe that PAP must also have a simple eigenvalue λ_2 (as this matrix has no zero entries, it is not a multiple of the identity matrix). Let us denote the (real) eigenvector associated with λ_2 as $w \in \mathbb{R}^n$. If

U satisfies $PAP(Uv) = \lambda(Uv)$, then it must be the case that Uv is in the orthogonal complement of w. i.e.

$$\begin{aligned}(Uv, w) &= 0 \\ 1w_1 + u_{22}v_2w_2 + u_{33}v_3w_3 &= 0 \qquad (*)\end{aligned}$$

For the purposes of contradiction, suppose that exactly one of the components of w is zero. (If more than one component satisfies this, then PAP must have a zero entry, contradicting our assumptions on A.) Without loss of generality, we will assume $w_1 = 0$. As the eigenspace associated with λ is 2-dimensional, it must contain a vector $s \in \mathbb{C}^3$ with first component 0 (and other components nonzero, for the same reason given above for why w cannot have two zero entries). However, denoting $C = PAP$, this means that $(Cw)_1 = 0$ and $(Cs)_1 = 0$ i.e.

$$0 = c_{11}(0) + c_{12}w_2 + c_{13}w_3 = c_{11}(0) + c_{12}s_2 + c_{13}s_3,$$

yielding $c_{12} = -\frac{c_{13}w_3}{w_2} = -\frac{c_{13}s_3}{s_2}$, and thus $\frac{w_3}{w_2} = \frac{s_3}{s_2}$. But this means that $w_3 = \frac{w_2 s_3}{s_2}$, whence we obtain:

$$\begin{aligned}\langle w, s\rangle &= 0 + w_2\overline{s_2} + w_3\overline{s_3} \\ &= w_2\overline{s_2} + \frac{w_2 s_3}{s_2}\overline{s_3} \\ &= w_2\left(\overline{s_2} + \frac{|s_3|^2}{s_2}\right) \\ &= w_2\left(\frac{|s_2|^2 + |s_3|^2}{s_2}\right) \neq 0\end{aligned}$$

which cannot be the case, as w and s are orthogonal. Thus, we conclude that w cannot have any zero components.

As none of the w_j are 0, then we may assume that $w_1 = 1$. The equality $(*)$, above, is then equivalent to $\{1, |v_2w_2|, |v_3w_3|\}$ being the side lengths of a triangle. There are, then, at most two solutions for $U = diag(1, u_{22}, u_{33})$ (corresponding to the only possible angles of a triangle with fixed side lengths and first side lying on the horizontal). Further, our two choices of U are complex conjugates of each other.

Thus, we have seen that we have at most six complex scalings, corresponding to the 4 real choices of U and the 2 complex choices of U. Further, the non-real scalings are, in fact, complex conjugates of each other. □

REMARK 5. In the above proof, note that the four real scalings are always guaranteed to exist, while the two complex scalings will exist if and only if two easily-checked conditions are satisfied:

1) There exists a positive diagonal matrix P such that PAP has repeated eigenvalue λ (checked via Theorem 3.9).

2) A triangle exists with side lengths $\{1, |v_2w_2|, |v_3w_3|\}$, where (once P is known) w is defined as in the above proof.

This result is easily extended to complex eigenvectors:

COROLLARY 4.5. *Let A be a 3×3 positive definite real matrix, and let $\lambda > 0$ and $v \in \mathbb{C}^3$ with no zero entries. Then there are at most six complex scalings $B = D^*AD$ such that $Bv = \lambda v$.*

Proof. Let $v = Up$, where U is the diagonal unitary matrix $diag\left(1, \frac{v_2}{|v_2|}, \frac{v_3}{|v_3|}\right)$ and $p = |v|$. Then $D^*ADUp = \lambda Up$ if and only if $(DU)^*A(DU)p = \lambda p$. The result follows from Proposition 4.4. $\square$

As mentioned at the beginning of this section, Proposition 4.4 is a generalization of the following corollary, originally found in [3]. Recall that we say that a positive definite matrix is *doubly quasi-stochastic* if it has all row and column sums equal to 1 (or, equivalently, if it preserves the all-ones vector):

COROLLARY 4.6. *Let A be a 3×3 positive definite real matrix. Then there are at most 6 doubly quasi-stochastic matrices B of the form $B = D^*AD$ for some diagonal (possibly complex) matrix D.*

4.3. How many scalings exist – complex A, complex eigenvectors

In Proposition 4.4, we provided a proof that at most 6 scalings exist when A is a real, 3×3 matrix, and we wish to obtain a *real* eigenpair. We mentioned that the case where the desired eigenpair is $(1,e)$ was already proven in [3], where the author was motivated by Question 4.3. The case where A is allowed to be a *complex* 3×3 matrix, however, remained open. We will once again generalize the question to arbitrary eigenpairs (λ, v) (where v no zero entries), and then provide a partial answer. In particular, we will prove the following:

THEOREM 4.7. *Let A be a 3×3 (possibly complex) positive definite matrix, and let $\lambda > 0$ and $v \in \mathbb{C}^3$ with no zero entries. If there are finitely many complex scalings $B = D^*AD$ such that $Bv = \lambda v$, then there are at most six such scalings.*

Before we prove Theorem 4.7, we note that we may force A to be of a particular form. First note that if A can be scaled to a real matrix (which it can if A has any zero entries), we may appeal to Corollary 4.5. Henceforth, we assume that A has no zero entries, and cannot be scaled to a real matrix. We define $E = \frac{1}{\sqrt{a_{22}}} diag\left(\frac{\sqrt{a_{22}}}{\sqrt{a_{11}}}, 1, \frac{a_{12}}{a_{13}}\right)$. Then

$$C := E^*AE = \begin{pmatrix} 1 & \frac{a_{12}}{\sqrt{a_{11}a_{22}}} & \frac{a_{12}}{\sqrt{a_{11}a_{22}}} \\ \frac{\overline{a_{12}}}{\sqrt{a_{11}a_{22}}} & 1 & * \\ \frac{\overline{a_{12}}}{\sqrt{a_{11}a_{22}}} & * & * \end{pmatrix}$$

where $*$ denotes entries we are not concerned with at the moment.

Now define $F = diag(e^{i\theta}, 1, 1)$, where θ is the argument of c_{12}. Then

$$G := F^*CF = \begin{pmatrix} 1 & |c_{12}| & |c_{12}| \\ |c_{12}| & 1 & * \\ |c_{12}| & * & * \end{pmatrix},$$

whence we see that (by applying the appropriate scaling) we may assume that our matrix A is of the form

$$\begin{pmatrix} 1 & b & b \\ b & 1 & f \\ b & \overline{f} & g \end{pmatrix} \tag{$\dagger$}$$

where $b, g > 0$, and $f \in \mathbb{C} \setminus \mathbb{R}$ (as A cannot be scaled to a real matrix).

For the remainder of the paper, we will assume that our given matrix A is of the form $(\dagger)$. We simplify our problem further with the following lemma. Note that the following argument can be deduced by inspecting the proof of Theorem 4.1, but we include the proof here for completeness.

LEMMA 4.8. *Let A be a $n \times n$ complex matrix, of the form $(\dagger)$ and suppose that $\lambda > 0$ and $v \in \mathbb{C}^n$ with no zero entries. Define the diagonal matrices P and U, where $P_{kk} = diag(|v_k|)$ and $U_{kk} = diag\left(\frac{v_k}{|v_k|}\right)$, and let $C = U^*AU$. Then the set of scalings*

$$\mathscr{L} = \{B \mid B = D^*AD \text{ for some diagonal } D \in \mathbb{C}^{n\times n} \text{ and } Bv = \lambda v\}$$

is in one-to-one correspondence with set of scalings

$$\mathscr{K} = \{B \mid B = E^*CE \text{ for some diagonal } E \in \mathbb{C}^{n\times n} \text{ with } E_{11} = 1 \text{ and } Be = tP^2e \text{ for some } t > 0\}.$$

Proof. We begin by defining a function $f : \mathscr{K} \to \mathscr{L}$, defined as $f(E^*CE) = D^*AD$, where $D = \frac{\sqrt{\lambda}}{\sqrt{t}}EP^{-1}$. We will first show that f does indeed have range $\mathscr{L}$, and then we will show that it is a bijection.

Let $E^*CE \in \mathscr{K}$. By the definition of the set, we know that $E^*CE = tP^2e$ for some $t > 0$, and that $E_{11} = 1$. Then

$$\begin{aligned} E^*CEe &= tP^2e \\ \frac{1}{t}E^*(U^*AU)Ee &= P^2e \\ \frac{1}{t}P^{-1}E^*U^*AUEP^{-1}Pe &= Pe \\ U^*\frac{1}{\sqrt{t}}P^{-1}E^*AE\frac{1}{\sqrt{t}}P^{-1}UPe &= Pe \end{aligned}$$

where the last line follows from the fact that $P, E,$ and U are all diagonal, and therefore commute. Left multiplying both sides by U and by λ (which we break up into $\sqrt{\lambda}\sqrt{\lambda}$ on the left side of the equality):

$$\frac{\sqrt{\lambda}}{\sqrt{t}}P^{-1}E^*AE\frac{\sqrt{\lambda}}{\sqrt{t}}P^{-1}UPe = \lambda UPe.$$

As $UPe = v$, we see $D^*ADv = \lambda v$, where $D = \frac{\sqrt{\lambda}}{\sqrt{t}} EP^{-1}$. As $D^*AD = f(E^*CE)$, this demonstrates that $f(E^*CE) \in \mathscr{L}$. We now show that this function is injective and surjective.

Injective: Suppose that $f(E_1^*CE_1) = f(E_2^*CE_2)$. This means that $D_1^*AD_1 = D_2^*AD_2$, where $D_1 = \frac{\sqrt{\lambda}}{\sqrt{t_1}} E_1 P^{-1}$ and $D_2 = \frac{\sqrt{\lambda}}{\sqrt{t_2}} E_2 P^{-1}$. It is easy to show (see, for example, Proposition 2.1 in [11]) that this can only be the case if $D_1 = \omega D_2$ for some ω from the complex unit circle. This yields $\frac{\sqrt{\lambda}}{\sqrt{t_1}} E_1 P^{-1} = \omega \frac{\sqrt{\lambda}}{\sqrt{t_2}} E_2 P^{-1}$. We note that the $(1,1)$ entry of both $\frac{\sqrt{\lambda}}{\sqrt{t_1}} E_1 P^{-1}$ and $\frac{\sqrt{\lambda}}{\sqrt{t_2}} E_2 P^{-1}$ are necessarily positive real (by the definition of $\mathscr{K}$), demonstrating that $\omega = 1$. Rearranging, and multiplying both sides by $\frac{\sqrt{t_2}}{\sqrt{\lambda}} P$, we obtain:

$$\frac{\sqrt{t_2}}{\sqrt{t_1}} E_1 = E_2$$

This shows that E_1 is a positive scalar multiple of E_2, and since both matrices have $(1,1)$ component equal to 1, we see that they must be equal.

Surjective: Suppose that $D^*AD \in \mathscr{L}$, so that $D^*ADv = \lambda v$. We show that there exists a scaling $E^*CE \in \mathscr{K}$ such that $f(E^*CE) = D^*AD$. Indeed, defining $t = \frac{\lambda}{d_{11}^2}$, then $E = \frac{\sqrt{t}}{\sqrt{\lambda}} PD$ satisfies $f(E^*CE) = D^*AD$ (this is easy to see, by the definition of f). It remains to show that $E^*CE \in \mathscr{K}$ i.e. that $E_{11} = 1$ and $E^*CEe = tP^2e$. Indeed, recalling that $p_{11} = |v_1| = 1$, we see:

$$E_{11} = \frac{\sqrt{t}}{\sqrt{\lambda}} p_{11} d_{11} = \frac{\sqrt{\lambda}}{d_{11}\sqrt{\lambda}} d_{11} = 1$$

and, recalling that $v = UPe$, we see:

$$\begin{aligned}
D^*ADv &= \lambda v \\
D^*ADUPe &= \lambda UPe \\
U^*D^*ADUPe &= \lambda Pe \\
\left(\frac{1}{d_{11}}\right) D^*(U^*AU) \left(\frac{1}{d_{11}}\right) DPe &= \frac{\lambda}{d_{11}^2} Pe \\
P\left(\frac{1}{d_{11}}\right) D^*(U^*AU) \left(\frac{1}{d_{11}}\right) DPe &= \frac{\lambda}{d_{11}^2} P^2 e \\
E^*CEe &= tP^2 e
\end{aligned}$$

as desired. □

By the above lemma, we conclude that the following is equivalent to Theorem 4.7:

THEOREM 4.7B. *Let A be a 3×3 positive definite matrix of the form $(\dagger)$, and let $p \in \mathbb{R}^3_{++}$. If there are finitely many complex scalings $B = D^*AD$ such that $d_{11} = 1$ and $Bv = tp$ for some $t > 0$, then there are at most six such scalings.*

This is the form of Theorem 4.7 that we will work towards proving. Combining the assumed form of A given by † with our assumption that $d_{11} = 1$, we see that our scalings take the following form, where $D = diag(1, d_2, d_3)$:

$$D^*AD = \begin{pmatrix} 1 & d_2 b & d_3 b \\ \overline{d_2} b & |d_2|^2 & \overline{d_2} d_3 f \\ \overline{d_3} b & \overline{d_3} d_2 \overline{f} & |d_3|^2 g \end{pmatrix}.$$

where $b, g > 0$, and $f \in \mathbb{C} \setminus \mathbb{R}$. We invite the reader to make note of the above, as it will be a useful reference going forward. For the remainder of the section, we will use the above notation.

We now work towards our proof of Theorem 4.7. We begin by introducing a special case that we will need to consider separately. Recall that we use $\Im(z)$ and $\Re(z)$ to denote the imaginary and real part of a complex number z, respectively.

LEMMA 4.9. *Suppose that $D^*ADe = tp$ for some $t > 0$ and $p \in \mathbb{R}^3_{++}$. If D satsfies $d_3 = \frac{-b}{f}$, then it must be the case that*

$$g = \frac{\left(2\Re\left(\frac{b^2}{f}\right) + 2kb + 1 + t\left(p_3 - p_1\right)\right)}{\left|\frac{b}{f}\right|^2},$$

where k is one of:

$$k = \frac{-b\left(2\Re\left(\frac{1}{f}\right) - 1\right) \pm \sqrt{\left(b\left(2\Re\left(\frac{1}{f}\right) - 1\right)\right)^2 - 4\left(\left|\frac{b}{f}\right|^2 - 1 + t\left(p_1 - p_2\right)\right)}}{2}.$$

Proof. Suppose that $d_3 = -\frac{b}{f}$, then our second row sum becomes

$$tp_2 = \overline{d_2} b + |d_2|^2 + \overline{d_2}\left(-\frac{b}{f} f\right) = \overline{d_2}(b - b) + |d_2|^2 = |d_2|^2.$$

As we know that our first row sum must be real, we see that $(d_2 + d_3)b \in \mathbb{R}$, yielding $d_2 = -d_3 + k$ for some $k \in \mathbb{R}$. (We will show further down that this k is indeed given by the expression in statement of the lemma.) We may then conclude that the sum of the elements in the first row is

$$tp_1 = 1 + d_2 b + d_3 b = 1 - bd_3 + kb + bd_3 = 1 + kb.$$

Subtracting the first row sum from the second, we obtain:

$$t\left(p_2 - p_1\right) = |d_2|^2 - (1 + kb) = |k - d_3|^2 - (1 + kb).$$

We let $d_3 = r + is$, so that $r = \Re\left(\frac{-b}{f}\right)$ and $s = \Im\left(\frac{-b}{f}\right)$, and obtain

$$\begin{aligned} t(p_2 - p_1) &= (k - r)^2 + s^2 - (1 + kb) \\ t\left(p_2 - p_1\right) &= k^2 - 2kr + r^2 + s^2 - 1 - kb \end{aligned}$$

$$0 = k^2 + k(-2r - b) + |d_3|^2 - 1 + t(p_1 - p_2)$$
$$0 = k^2 + k\left(2\Re\left(\frac{b}{f}\right) - b\right) + \left|\frac{b}{f}\right|^2 - 1 + t(p_1 - p_2)$$
$$0 = k^2 + b\left(2\Re\left(\frac{1}{f}\right) - 1\right)k + \left(\left|\frac{b}{f}\right|^2 - 1 + t(p_1 - p_2)\right)$$

whence we obtain

$$k = \frac{-b\left(2\Re\left(\frac{1}{f}\right) - 1\right) \pm \sqrt{\left(b\left(2\Re\left(\frac{1}{f}\right) - 1\right)\right)^2 - 4\left(\left|\frac{b}{f}\right|^2 - 1 + t(p_1 - p_2)\right)}}{2}.$$

(Note: This may fail to be real, but if this is the case, then we may conclude that $d_3 \neq -\frac{b}{f}$, contradicting the assumption of the lemma.)

Lastly, we subtract row 1 from row 3, obtaining:

$$t(p_3 - p_1) = \overline{d_3}b + d_2\overline{d_3 f} + |d_3|^2 g - (1 + kb)$$
$$t(p_3 - p_1) = -\frac{b^2}{\overline{f}} - d_2 b + \left|\frac{b}{f}\right|^2 g - 1 - kb$$
$$t(p_3 - p_1) = -\frac{b^2}{\overline{f}} - (-d_3 + k)b + \left|\frac{b}{f}\right|^2 g - 1 - kb$$
$$t(p_3 - p_1) = -\frac{b^2}{\overline{f}} - \frac{b^2}{f} - kb + \left|\frac{b}{f}\right|^2 g - 1 - kb$$
$$g = \frac{\left(2\Re\left(\frac{b^2}{f}\right) + 2kb + 1 + t(p_3 - p_1)\right)}{\left|\frac{b}{f}\right|^2}$$

as desired. □

The condition in the lemma above is, of course, very specific. We will consider the matrices that do *not* satisfy this condition first. We proceed via a series of lemmas.

LEMMA 4.10. *Suppose that* $D^*ADe = tp$ *for some* $t > 0$ *and* $p \in \mathbb{R}^n_{++}$, *and suppose that* $d_3 \neq -\frac{b}{f}$. *Then* $d_2 = lb + ld_3 f$, *where* $l = -\frac{\Im(d_3)}{\Im(d_3 f)}$

Proof. As in the proof of Lemma 4.9, we may use the fact that row 1 is real to see that $d_2 = -d_3 + k$ for some $k \in \mathbb{R}$. Substituting this into the second row sum and recognizing that the off-diagonal elements must sum to a real number, we see that:

$$\overline{d_2}b + \overline{d_2}(d_3 f) = \overline{d_2}(b + d_3 f) = (\overline{-d_3 + k})(b + d_3 f) \in \mathbb{R} \setminus \{0\}$$

(we know this cannot be zero, since $d_3 \neq -\frac{b}{f}$, and of course $d_2 \neq 0$).

Thus, we may conclude that $-d_3+k=l(b+d_3f)$, for some $l\in\mathbb{R}\setminus\{0\}$. Rearranging, we obtain $k=lb+ld_3f+d_3$, yielding $d_2=lb+ld_3f$, as desired.

To obtain the value of l, we recall that $k\in\mathbb{R}$ and inspect the imaginary parts of $k=lb+ld_3f+d_3$ to obtain $0=l\Im(d_3f)+\Im(d_3)$, or $l=-\frac{\Im(d_3)}{\Im(d_3f)}$. We note that $\Im(d_3f)$ cannot be zero, as this would make $l(b+d_3f)$ real, and (since $-d_3+k=l(b+d_3f)$) this would mean that $d_2=-d_3+k$ is also real, meaning that d_3 and (by extension) f must be real, contradicting our assumptions on f from $(\dagger)$. $\square$

LEMMA 4.11. *Suppose that $D^*ADe=tp$, for some $t>0$, $p\in\mathbb{R}^3_{++}$ and suppose that $d_3\neq-\frac{b}{f}$. If $d_3=r+is$ and $f=m+in$, then it must be the case that:*

$$\begin{aligned}0=&\,s^3\left(mg-\left(m^2+n^2\right)\right)+s^2\left(rng\right)+s\left(mgr^2-r^2m^2-r^2n^2+b^2-m+Rm\right)\\&+\left(r^3ng-rn+Rrn\right)\end{aligned}$$

where $R=t(p_1-p_3)$.

Proof. We subtract row 3 from row 1 to obtain

$$1+d_2b+d_3b-\left(\overline{d_3}b+d_2\overline{d_3f}+|d_3|^2g\right)=t(p_1-p_3).$$

We set $R=t(p_1-p_3)$, substitute $d_2=lb+ld_3f$ from Lemma 4.10, and inspect the real parts of the expression (losing no information, as the right side of the equation is real). We obtain:

$$1+lb^2+lb\Re(d_3f)+b\Re(d_3)-b\Re(d_3)-lb\Re(d_3f)-l|d_3f|^2-|d_3|^2g=R,$$

which simplifies to

$$1+lb^2-l|d_3f|^2-|d_3|^2g=R.$$

Substituting $d_3=r+is$, $f=m+in$, and $l=-\frac{\Im(d_3)}{\Im(d_3f)}=\frac{-s}{rn+ms}$, this becomes

$$1-\frac{sb^2}{rn+ms}+\frac{s(r^2+s^2)(m^2+n^2)}{rn+ms}-(r^2+s^2)g=R.$$

Multiplying by $rn+ms$, we have

$$rn+ms-sb^2+s\left(r^2+s^2\right)\left(m^2+n^2\right)-\left(r^2+s^2\right)g\left(rn+ms\right)=Rrn+Rms.$$

Expanding and moving all terms to the right side of the equation, we obtain our desired expression. $\square$

LEMMA 4.12. *Suppose that $D^*ADe=tp$, for some $t>0$, $p\in\mathbb{R}^3_{++}$ and suppose that $d_3\neq-\frac{b}{f}$. Using the notation of Lemma 4.11, it must be the case that:*

$$\begin{aligned}0=&\,s^4\left(m^2+n^2-m^2g\right)+s^3\left(mnb-2bn-2gnmr\right)\\&+s^2\left(rn^2b-mb^2-2rbm^2+2rbm+b^2+r^2m^2+r^2n^2-r^2n^2g-r^2m^2g-R_2m^2\right)\\&+s\left(-rnb^2-3r^2bnm-2r^3nmg-2R_2rnm\right)+\left(-r^3n^2b-r^4gn^2-R_2r^2n^2\right)\end{aligned}$$

where $R_2=t(p_2-p_3)$.

Proof. We subtract row 3 from row 2 to obtain

$$t(p_2-p_3)=\overline{d_2}b+|d_2|^2+\overline{d_2}d_3f-(\overline{d_3}b+d_2\overline{d_3f}+|d_3|^2g)$$

Defining $R_2:=t(p_2-p_3)$, and inspecting the real parts of the expression:

$$R_2=\Re(d_2)b+|d_2|^2+\Re(\overline{d_2}d_3f)-\Re(d_3)b-\Re(d_2\overline{d_3f})-|d_3|^2g$$
$$R_2=b(\Re(d_2)-\Re(d_3))+|d_2|^2-|d_3|^2g.$$

As $d_2=lb+ld_3f$, we obtain

$$\begin{aligned}R_2&=b(\Re(lb+l(d_3f))-\Re(d_3))+|lb+ld_3f|^2-|d_3|^2g\\&=lb^2+lb\Re(d_3f)-b\Re(d_3)+(lb+l\Re(d_3f))^2+(l\Im(d_3f))^2-|d_3|^2g\\&=lb^2+lb(rm-sn)-br+(lb+l(rm-sn))^2+(l(rn+sm))^2-(r^2+s^2)g.\end{aligned}$$

Substituting $l=-\frac{s}{rn+ms}$, this becomes:

$$\begin{aligned}R_2=&\left(\frac{-s}{rn+ms}\right)b^2+\left(\frac{-s}{rn+ms}\right)b(rm-sn)-br+\left(\frac{-s}{rn+ms}\right)^2(b+(rm-sn))^2\\&+\left(\frac{-s}{rn+ms}\right)^2(rn+sm)^2-(r^2+s^2)g.\end{aligned}$$

Multiplying by $(rn+ms)^2$, we obtain:

$$\begin{aligned}R_2(rn+ms)^2=&-sb^2(rn+ms)-sb(rm-sn)(rn+ms)-br(rn+ms)^2\\&+s^2(b+(rm-sn))^2+s^2(rn+sm)^2-(r^2g+s^2g)(rn+ms)^2.\end{aligned}$$

Expanding:

$$\begin{aligned}0=&-sb^2rn-s^2b^2m-sbr^2mn+s^2brn^2-s^2brm^2+s^3bnm-br^3n^2-2br^2nms\\&-brm^2s^2+s^2b^2+2brms^2-2bs^3n+r^2m^2s^2-2rms^3n+s^4n^2+s^2r^2n^2\\&+2rns^3m+s^4m^2-r^4gn^2-2r^3nmsg-r^2gm^2s^2-s^2gr^2n^2-2rnms^3g\\&-s^4m^2g-R_2r^2n^2-2R_2rnms-R_2m^2s^2.\end{aligned}$$

Rearranging (and cancelling the $-2rms^3n+2rns^3m$ terms), we obtain the expression given in the statement of the lemma. ☐

Lemma 4.11 and Lemma 4.12 give us the following characterization of our scalings:

PROPOSITION 4.13. *Suppose that $D^*ADe=tp$, for some $t>0$, $p\in\mathbb{R}^3_{++}$ and suppose that $d_3\neq-\frac{b}{f}$. If we denote $d_3=r+is$ and $f=m+in$, then (r,s) must be an intersection point of the plane curves:*

$$0=s^3(mg-(m^2+n^2))+s^2(rng)+s(mgr^2-r^2m^2-r^2n^2+b^2-m+Rm)+(r^3ng-rn+Rrn)$$

and

$$\begin{aligned}0 &= s^4(m^2+n^2-m^2g)+s^3(mnb-2bn-2gnmr)\\&\quad +s^2(rn^2b-mb^2-2rbm^2+2rbm+b^2+r^2m^2+r^2n^2-r^2n^2g-r^2m^2g-R_2m^2)\\&\quad +s(-rnb^2-3r^2bnm-2r^3nmg-2R_2rnm)+(-r^3n^2b-r^4gn^2-R_2r^2n^2).\end{aligned}$$

Let us denote the first curve by C_1 and the second as C_2. If C_1 and C_2 have no common component, Bezout's Theorem tells us that there are 12 intersection points, counted with multiplicities and including points at infinity. We will show, however, that some of these intersection points do not yield scalings. Our next three lemmas demonstrate the existence of such intersection points.

LEMMA 4.14. *C_1 and C_2 have exactly two intersection points at infinity.*

Proof. Setting $z=0$ in the homogenized equations of C_1 and C_2, we obtain:

$$\begin{aligned}H_1: 0 &= s^3\left(mg-(m^2+n^2)\right)+s^2(rng)+s\left(mgr^2-r^2m^2-r^2n^2\right)+r^3ng\\&= s^2\left(smg-sm^2-sn^2+rng\right)+r^2\left(smg-sm^2-sn^2+rng\right)\\&= \left(s^2+r^2\right)\left(smg-sm^2-sn^2+rng\right)\end{aligned}$$

and

$$\begin{aligned}H_2: 0 &= s^4\left(m^2+n^2-m^2g\right)+s^3(-2gnmr)+s^2\left(r^2m^2+r^2n^2-r^2n^2g-r^2m^2g\right)\\&\quad +s\left(-2r^3mng\right)-gr^4n^2\\&= s^2\left(s^2m^2+s^2n^2-s^2m^2g-2sgnmr-gr^2n^2\right)\\&\quad +r^2\left(s^2m^2+s^2n^2-s^2m^2g-2sgnmr-gr^2n^2\right)\\&= \left(s^2+r^2\right)\left(s^2m^2+s^2n^2-s^2m^2g-2sgnmr-gr^2n^2\right)\end{aligned}$$

By inspection, we see that $(r,s)=(i,1)$ and $(r,s)=(-i,1)$ are solutions to both of these curves.

It is easy to verify that these are the only common solutions at infinity. Indeed, the third solution to H_1, $(r,s)=\left(\frac{m^2+n^2-mg}{ng},1\right)$ does not solve H_2, as plugging this into the second factor of H_2 yields

$$\begin{aligned}&m^2+n^2-m^2g-2m(m^2+n^2-mg)-\frac{(m^2+n^2-mg)^2}{g}\\=&|f|^2-m^2g-2m|f|^2+2m^2g-\frac{(|f|^2-mg)^2}{g}\\=&|f|^2-2m|f|^2+m^2g-\frac{|f|^4}{g}+2|f|^2m-m^2g\\=&|f|^2-\frac{|f|^4}{g}\\=&|f|^2\left(1-\frac{|f|^2}{g}\right)\end{aligned}$$

which cannot be 0, as this would require $|f|^2 = g$, contradicting the fact that A is positive definite. $\square$

LEMMA 4.15. *C_1 and C_2 intersect at $(r,s) = (0,0)$, with intersection multiplicity at least 2.*

Proof. It is easy to see that $(0,0)$ is an intersection point of both curves. We see that the multiplicity is (at least) 2, as the partial derivatives $\frac{\partial C_2}{\partial s}$ and $\frac{\partial C_2}{\partial r}$ vanish at $(0,0)$ (see Appendix A.2 for details). $\square$

LEMMA 4.16. *C_1 and C_2 intersect at $(r,s) = \left(\frac{-mb}{m^2+n^2}, \frac{nb}{m^2+n^2}\right)$, with intersection multiplicity at least 2.*

Proof. It can be verified that this point falls on both curves. Again, we see that the partial derivatives $\frac{\partial C_2}{\partial s}$ and $\frac{\partial C_2}{\partial r}$ vanish at this point (see Appendix A.2 for details), demonstrating that the multiplicity of this point is at least 2. $\square$

We can now prove Theorem 4.7B for matrices that do not satisfy the special case in Lemma 4.9:

PROPOSITION 4.17. *Theorem 4.7B holds for matrices that do not satisfy the condition in Lemma 4.9.*

Proof. From Proposition 4.13, we know that for every D satisfying $D^*ADe = tp$, $(r,s) = (\Re(d_3), \Im(d_3))$ is an intersection point of C_1 and C_2. If C_1 and C_2 have a common component, then there are infinitely many intersection points. Let us suppose that this is not the case for the remainder of the proof.

From Bezout's Theorem, we know that there are twelve such intersection points counted with multiplicity. Lemma 4.14, Lemma 4.15, and Lemma 4.16 give 6 of these intersection points, but (as we show below) it is easy to see that none of these can correspond to $(\Re(d_3), \Im(d_3))$ for a scaling matrix D:

Lemma 4.14 gives points at infinity with imaginary components. Of course, points at infinity do not give a meaningful intersections of our curve (and $(\Re(d_3), \Im(d_3))$ must be real). Thus these two points do not correspond to a valid d_3.

Lemma 4.15 gives $(0,0)$ as a solution, but this of course cannot correspond to a valid d_3, as D^*AD would have 3rd row consisting of all zeroes (which cannot be the case, as $tp_3 > 0$).

Lemma 4.16 gives $\left(\frac{-mb}{m^2+n^2}, \frac{nb}{m^2+n^2}\right)$ as a solution. However, this corresponds to $d_3 = \frac{-b(m-in)}{|f|^2} = \frac{-b\overline{f}}{|f|^2} = \frac{-b}{f}$, which (by our assumption) does not scale A (as A does not satisfy the condition in Lemma 4.9).

Thus, we have a maximum of 6 intersection points that may correspond to d_3 for a scaling matrix D. Once d_3 is found, D is fully determined, as $d_2 = lb + ld_3 f$ (by Lemma 4.10), and $d_1 = 1$, as in the statement of Theorem 4.7B. Hence, each choice of d_3 corresponds to exactly one scaling D^*AD. Thus there are at most 6 scalings satisfying $D^*ADe = tp$. $\square$

We now investigate the matrices that *do* satisfy the condition in Lemma 4.9.

PROPOSITION 4.18. *Theorem 4.7B holds for matrices that satisfy the condition in Lemma 4.9.*

Proof. Firstly, note that any scaling that does *not* satisfy $d_3 = \frac{-b}{f}$ must still be an intersection of C_1 and C_2. Thus, all of the reasoning found in the proof of Proposition 4.17 still holds, except that we can no longer assume that the intersection point $(p,q) = \left(\frac{-mb}{m^2+n^2}, \frac{nb}{m^2+n^2}\right)$ does not correspond to a scaling of A. At first, this would appear to provide us with a maximum of 7 scalings (the six from Proposition 4.17 and the additional scaling corresponding to (p,q)). However, we will show that in this case, the intersection multiplicity at (p,q) is at least *three* (where it was only guaranteed to be at least *two* in the proof of Proposition 4.17). Given that we still have the solutions from Lemma 4.14 and Lemma 4.15 (each of multiplicity two) which do not correspond to scalings, Bezout's Theorem tells us that there are at most 5 other solutions, giving (again) a maximum of 6 scalings.

Thus, it suffices to show that the multiplicity of (p,q) is at least three. To that end, we note that the argument from the proof of Lemma 4.16 still holds (i.e. $\frac{\partial C_1}{\partial s}$ and $\frac{\partial C_2}{\partial r}$ vanish at (p,q)), but we will now also show that C_1 and C_2 must share a tangent at this point. This guarantees that we have a multiplicity of at least three.

We leave the details to Appendix A.3, but it can be shown that the tangent to C_1 at (p,q) in variables (r',s') is given by:

$$s' = -\left(\frac{p^2ng + 2pmgq - 2pqm^2 - 2pqn^2 + 3q^2ng - n + Rn}{3p^2mg - 3p^2m^2 - 3p^2n^2 + 2pqng + mgq^2 - q^2m^2 - q^2n^2 + b^2 - m + Rm}\right) r' \tag{1}$$

while the tangents to C_2 at (p,q) are the linear solutions to

$$\begin{aligned}
0 = {} & 6(s')^2p^2\left(m^2+n^2-m^2g\right) + 3(s')^2p\left(mnb - 2bn - 2gnmq\right) + 3s'p^2\left(-2gnmr'\right) \\
& + (s')^2\left(qn^2b - mb^2 - 2qbm^2 + 2qbm + b^2 + q^2m^2 + q^2n^2 - q^2n^2g - q^2m^2g - R_2m^2\right) \\
& + 2s'p\left(r'n^2b - 2r'bm^2 + 2r'bm + 2r'qm^2 + 2r'qn^2 - 2r'qn^2g - 2r'qm^2g\right) \\
& + p^2\left((r')^2m^2 + (r')^2n^2 - (r')^2n^2g - (r')^2m^2g\right) \\
& + s'\left(-r'nb^2 - 6r'qbnm - 6r'q^2nmg - 2R_2r'nm\right) \\
& + p\left(-3(r')^2bnm - 6(r')^2qnmg\right) + \left(-3(r')^2qn^2b - 6(r')^2q^2gn^2 - R_2(r')^2n^2\right)
\end{aligned} \tag{2}$$

It is easy to verify (with computer assistance) that (1) is indeed a linear solution to (2), provided g satisfies the condition given in Lemma 4.9. Thus, C_1 and C_2 share a tangent at (p,q), and we may conclude that the intersection multiplicity at this point is at least three, as desired.

We complete the proof by noting that once we know that $d_3 = \frac{-b}{f}$, D is fully determined (and so there is only one possible scaling satisfying $\frac{-b}{f}$). Indeed, as seen in the proof of Lemma 4.9, it must be the case $d_2 = -d_3 + k$, where k is well-defined from the (fixed) value of g (and $d_1 = 1$). $\square$

We finally arrive at our proof of Theorem 4.7, which we restate here:

THEOREM 4.7. *Let A be a 3×3 positive definite matrix, and let $\lambda > 0$ and $v \in \mathbb{C}^3$ with no zero entries. If there are finitely many complex scalings $B = D^*AD$ such that $Bv = \lambda v$, then there are at most six such scalings.*

Proof. Proposition 4.17 and Proposition 4.18 prove Theorem 4.7B, which is equivalent to Theorem 4.7 (by Lemma 4.8). $\square$

REMARK 6. We note that in our proof of Theorem 4.7, we have provided a way to actually find the scalings of a given matrix A. One can simply use a solver to find the intersections points of C_1 and C_2 to find all possible d_3, and then use $d_2 = lb + ld_3 f$. (In the case where the matrix satisfies the condition of Lemma 4.9, there may also be a scaling that satisfies $d_3 = -\frac{b}{f}$, with d_2 defined accordingly). We may then use the function from Lemma 4.8 to obtain the desired scalings.

Lastly, we apply the above result to Question 4.3 of Pereira and Boneng, by letting $\lambda = 1$ and $v = e$ (the all-ones vector):

COROLLARY 4.19. *Let A be a 3×3 positive definite matrix. If there are finitely many complex scalings satisfying satisfying $D^*ADe = e$, then there are six such scalings.*

We close by noting that we have not been able to construct a 3×3 matrix with infinitely many scalings (i.e. A 3×3 positive definite matrix where the corresponding curves C_1 and C_2 have a common component). We make the following conjecture:

CONJECTURE 4.20. *Let A be a 3×3 positive definite matrix, and suppose that $\lambda > 0$ and $v \in \mathbb{C}^3$ with no zero entries. Then there are finitely many complex scalings $B = D^*AD$ such that $Bv = \lambda v$.*

It would be nice to prove this conjecture, as this would fully answer the question of ennumerating scalings when $n = 3$. In higher dimensions ($n \geqslant 4$), the problem is more delicate, as it has been shown that there are certain matrices and eigenpairs with infinitely many scalings (see [3]), and we must thus place more restrictions on A or v if we are to obtain interesting results.

A. Appendix

Here, you will find details that were left out of certain proofs, for the sake of brevity.

A.1. Proofs of the equalities from Lemma 3.8

$\mathbf{tr}(\mathbf{A^2}) = \mathbf{2\lambda_1^2 + \lambda_2^2}$: Examining the diagonal elements of A^2 and exploiting symmetry, we obtain:

$$\begin{aligned}
tr(A^2) &= a_{11}^2 + 2a_{12}^2 + 2a_{13}^2 + 2a_{23}^2 + a_{22}^2 + a_{33}^2 \\
&= a_{11}^2 + 2a_{12}^2 + 2a_{13}^2 + 2a_{23}^2 + \left(a_{11} - \left(\frac{a_{12}a_{13}}{a_{23}} - \frac{a_{12}a_{23}}{a_{13}}\right)\right)^2 \\
&\quad + \left(a_{11} - \left(\frac{a_{13}a_{12}}{a_{23}} - \frac{a_{13}a_{23}}{a_{12}}\right)\right)^2 \\
&= a_{11}^2 + 2a_{12}^2 + 2a_{13}^2 + 2a_{23}^2 \\
&\quad + \left(a_{11}^2 - \frac{2a_{11}a_{12}a_{13}}{a_{23}} + \frac{2a_{11}a_{12}a_{23}}{a_{13}} + \frac{a_{12}^2a_{13}^2}{a_{23}^2} + \frac{a_{12}^2a_{23}^2}{a_{13}^2} - 2a_{12}^2\right) \\
&\quad + \left(a_{11}^2 - \frac{2a_{11}a_{13}a_{12}}{a_{23}} + \frac{2a_{11}a_{13}a_{23}}{a_{12}} + \frac{a_{13}^2a_{12}^2}{a_{23}^2} + \frac{a_{13}^2a_{23}^2}{a_{12}^2} - 2a_{13}^2\right)
\end{aligned}$$

We see that the $2a_{12}^2$ and $2a_{13}^2$ cancel out, and we collect like terms to obtain:

$$\begin{aligned}
tr(A^2) &= 3a_{11}^2 + 2a_{23}^2 - \frac{4a_{11}a_{12}a_{13}}{a_{23}} + \frac{2a_{11}a_{12}a_{23}}{a_{13}} + \frac{2a_{12}^2a_{13}^2}{a_{23}^2} + \frac{a_{12}^2a_{23}^2}{a_{13}^2} \\
&\quad + \frac{2a_{11}a_{13}a_{23}}{a_{12}} + \frac{a_{13}^2a_{23}^2}{a_{12}^2}.
\end{aligned}$$

Rearranging, we arrive at our desired equality:

$$\begin{aligned}
tr(A^2) &= 2a_{11}^2 - 4\frac{a_{11}a_{12}a_{13}}{a_{23}} + \frac{2a_{12}^2a_{13}^2}{a_{23}^2} + a_{11}^2 + 2a_{23}^2 + \frac{2a_{11}a_{12}a_{23}}{a_{13}} + \frac{a_{12}^2a_{23}^2}{a_{13}^2} \\
&\quad + \frac{2a_{11}a_{13}a_{23}}{a_{12}} + \frac{a_{13}^2a_{23}^2}{a_{12}^2} \\
&= 2\left(a_{11}^2 - \frac{2a_{11}a_{12}a_{13}}{a_{23}} + \frac{a_{12}^2a_{13}^2}{a_{23}^2}\right) \\
&\quad + \left(a_{11}^2 + 2a_{23}^2 + \frac{2a_{11}a_{12}a_{23}}{a_{13}} + \frac{a_{12}^2a_{23}^2}{a_{13}^2} + \frac{2a_{11}a_{13}a_{23}}{a_{12}} + \frac{a_{13}^2a_{23}^2}{a_{12}^2}\right) \\
&= 2\left(a_{11} - \frac{a_{13}a_{12}}{a_{23}}\right)^2 + \left(a_{11} + \frac{a_{13}a_{23}}{a_{12}} + \frac{a_{12}a_{23}}{a_{13}}\right)^2 \\
&= 2\lambda_1^2 + \lambda_2^2
\end{aligned}$$

$\mathbf{det(A) = \lambda_1^2\lambda_2}$: Exploiting symmetry, we obtain:

$$\begin{aligned}
det(A) &= a_{11}a_{22}a_{33} + 2a_{12}a_{23}a_{13} - a_{11}a_{23}^2 - a_{22}a_{13}^2 - a_{33}a_{12}^2 \\
&= a_{11}\left(a_{11} - \left(\frac{a_{12}a_{13}}{a_{23}} - \frac{a_{12}a_{23}}{a_{13}}\right)\right)\left(a_{11} - \left(\frac{a_{13}a_{12}}{a_{23}} - \frac{a_{13}a_{23}}{a_{12}}\right)\right) + 2a_{12}a_{23}a_{13} \\
&\quad - a_{11}a_{23}^2 - \left(a_{11} - \left(\frac{a_{12}a_{13}}{a_{23}} - \frac{a_{12}a_{23}}{a_{13}}\right)\right)a_{13}^2 - \left(a_{11} - \left(\frac{a_{13}a_{12}}{a_{23}} - \frac{a_{13}a_{23}}{a_{12}}\right)\right)a_{12}^2 \\
&= a_{11}^3 - a_{11}^2\left(\frac{a_{13}a_{12}}{a_{23}} - \frac{a_{13}a_{23}}{a_{12}}\right) - a_{11}^2\left(\frac{a_{12}a_{13}}{a_{23}} - \frac{a_{12}a_{23}}{a_{13}}\right) \\
&\quad + a_{11}\left(\frac{a_{12}a_{13}}{a_{23}} - \frac{a_{12}a_{23}}{a_{13}}\right)\left(\frac{a_{13}a_{12}}{a_{23}} - \frac{a_{13}a_{23}}{a_{12}}\right) + 2a_{12}a_{23}a_{13} - a_{11}a_{23}^2 - a_{11}a_{13}^2 \\
&\quad + \frac{a_{12}a_{13}^3}{a_{23}} - a_{13}a_{12}a_{23} - a_{11}a_{12}^2 + \frac{a_{13}a_{12}^3}{a_{23}} - a_{12}a_{13}a_{23} \\
&= a_{11}^3 + \frac{a_{11}^2a_{13}a_{23}}{a_{12}} + \frac{a_{11}^2a_{12}a_{23}}{a_{13}} - \frac{2a_{11}^2a_{13}a_{12}}{a_{23}} + \frac{a_{11}a_{12}^2a_{13}^2}{a_{23}^2} - a_{11}a_{12}^2 - a_{11}a_{13}^2 \\
&\quad + a_{11}a_{23}^2 + 2a_{12}a_{23}a_{13} - a_{11}a_{23}^2 - a_{11}a_{13}^2 + \frac{a_{12}a_{13}^3}{a_{23}} - 2a_{13}a_{12}a_{23} - a_{11}a_{12}^2 + \frac{a_{13}a_{12}^3}{a_{23}}
\end{aligned}$$

Collecting terms and cancelling, we obtain:

$$\begin{aligned}
det(A) = a_{11}^3 &+ \frac{a_{11}^2a_{13}a_{23}}{a_{12}} + \frac{a_{11}^2a_{12}a_{23}}{a_{13}} - \frac{2a_{11}^2a_{13}a_{12}}{a_{23}} + \frac{a_{11}a_{12}^2a_{13}^2}{a_{23}^2} - 2a_{11}a_{12}^2 \\
&- 2a_{11}a_{13}^2 + \frac{a_{13}^3a_{12}}{a_{23}} + \frac{a_{13}a_{12}^3}{a_{23}}
\end{aligned}$$

We now show that $\lambda_1^2\lambda_2$ is also given by the above expression. Indeed:

$$\begin{aligned}
\lambda_1^2\lambda_2 &= \left(a_{11} - \frac{a_{13}a_{12}}{a_{23}}\right)^2\left(a_{11} + \frac{a_{13}a_{23}}{a_{12}} + \frac{a_{12}a_{23}}{a_{13}}\right) \\
&= \left(a_{11}^2 - \frac{2a_{11}a_{13}a_{12}}{a_{23}} + \frac{a_{13}^2a_{12}^2}{a_{23}^2}\right)\left(a_{11} + \frac{a_{13}a_{23}}{a_{12}} + \frac{a_{12}a_{23}}{a_{13}}\right) \\
&= a_{11}^3 + \frac{a_{11}^2a_{13}a_{23}}{a_{12}} + \frac{a_{11}^2a_{12}a_{23}}{a_{13}} - \frac{2a_{11}^2a_{13}a_{12}}{a_{23}} - 2a_{11}a_{13}^2 - 2a_{11}a_{12}^2 \\
&\quad + \frac{a_{11}a_{13}^2a_{12}^2}{a_{23}^2} + \frac{a_{13}^3a_{12}}{a_{23}} + \frac{a_{13}a_{12}^3}{a_{23}},
\end{aligned}$$

which is equivalent to our above expansion of $det(A)$. Hence $det(A) = \lambda_1^2\lambda_2$, as desired.

A.2. Proof that partial derivatives of C_2 vanish in Lemma 4.15 and Lemma 4.16

We begin by rewriting our expression for C_2, for reference:

$$\begin{aligned}0 &= s^4\left(m^2+n^2-m^2g\right)+s^3\left(mnb-2bn-2gnmr\right)\\&\quad+s^2\left(rn^2b-mb^2-2rbm^2+2rbm+b^2+r^2m^2+r^2n^2-r^2n^2g-r^2m^2g-R_2m^2\right)\\&\quad+s\left(-rnb^2-3r^2bnm-2r^3nmg-2R_2rnm\right)+\left(-r^3n^2b-r^4gn^2-R_2r^2n^2\right)\end{aligned}$$

We take the partial derivative with respect to r:

$$\begin{aligned}0 &= -2s^3gnm+s^2\left(n^2b-2bm^2+2bm+2rm^2+2rn^2-2rn^2g-2rm^2g\right)\\&\quad+s\left(-nb^2-6rbnm-6r^2nmg-2R_2nm\right)-3r^2n^2b-4r^3gn^2-2R_2rn^2,\end{aligned}$$

and the partial derivative with respect to s:

$$\begin{aligned}0 &= 4s^3\left(m^2+n^2-m^2g\right)+3s^2\left(mnb-2bn-2gnmr\right)\\&\quad+2s\left(rn^2b-mb^2-2rbm^2+2rbm+b^2+r^2m^2+r^2n^2-r^2n^2g-r^2m^2g-R_2m^2\right)\\&\quad+\left(-rnb^2-3r^2bnm-2r^3nmg-2R_2rnm\right),\end{aligned}$$

whence one can verify that both of these partial derivatives vanish at $(r,s)=(0,0)$ (as claimed in the proof of Lemma 4.15) and at $\left(\frac{-mb}{m^2+n^2},\frac{nb}{m^2+n^2}\right)$ (as claimed in the proof of Lemma 4.16).

A.3. Proof of the tangents given in Proposition 4.18

Recall that in order to find the tangents to C_1 and C_2, at the point $(s,r)=(p,q)$, we first shift the coordinates of our curve to $s'=s-p$ and $r'=r-q$, after which the tangents are given by the linear factors of the homogeneous part C_m, where m is the smallest natural number such that C_m is non-zero. (These definitions are standard, but we encourage the reader to refer to [12] for an excellent introductory treatment of this material. The pertinent definitions for our purposes are Definitions 2.20 and 2.18 therein).

Tangent to C_1 at (p,q): We make the coordinate change $s\mapsto s'+p$ and $r\mapsto r'+q$. We will call this new curve F, so

$$\begin{aligned}F(s',r') &= (s'+p)^3\left(mg-(m^2+n^2)\right)+(s'+p)^2\left((r'+q)ng\right)\\&\quad+(s'+p)\left(mg(r'+q)^2-(r'+q)^2m^2-(r'+q)^2n^2+b^2-m+Rm\right)\\&\quad+((r'+q)^3ng-(r'+q)n+R(r'+q)n).\end{aligned}$$

Of course, $F(0,0)=0$ (since $C_1(p,q)=0$), so the multiplicity of this point is at least 1. We look to the homogeneous part F_1 (i.e. the degree 1 terms):

$$\begin{aligned}F_1(s',r') &= 3s'p^2mg-3s'p^2m^2-3s'p^2n^2+2s'pqng+p^2r'ng+s'mgq^2-s'q^2m^2\\&\quad-s'q^2n^2+s'b^2-s'm+s'Rm+2pmgr'q-2pr'qm^2-2pr'qn^2+3r'q^2ng\\&\quad-r'n+Rr'n.\end{aligned}$$

Factoring out the s' and the r', we obtain:

$$\begin{aligned} F_1(s',r') = s'&\left(3p^2mg-3p^2m^2-3p^2n^2+2pqng+mgq^2-q^2m^2-q^2n^2+b^2-m+Rm\right)\\ &+r'\left(p^2ng+2pmgq-2pqm^2-2pqn^2+3q^2ng-n+Rn\right). \end{aligned}$$

As the tangent at (p,q) is defined to be the linear factors of this curve, we see that the tangent is given by:

$$s' = -\left(\frac{p^2ng+2pmgq-2pqm^2-2pqn^2+3q^2ng-n+Rn}{3p^2mg-3p^2m^2-3p^2n^2+2pqng+mgq^2-q^2m^2-q^2n^2+b^2-m+Rm}\right)r'$$

as given in the proof of Proposition 4.18.

Tangent to C_2 at (p,q): As above, we make the coordinate change $s\mapsto s'+p$ and $r\mapsto r'+q$, and call this new curve G:

$$\begin{aligned} G(s',r') =&(s'+p)^4(m^2+n^2-m^2g)+(s'+p)^3(mnb-2bn-2gnm(r'+q))\\ &+(s'+p)^2\big((r'+q)n^2b-mb^2-2(r'+q)bm^2+2(r'+q)bm+b^2\\ &+(r'+q)^2m^2+(r'+q)^2n^2-(r'+q)^2n^2g-(r'+q)^2m^2g-R_2m^2\big)\\ &+(s'+p)\left(-(r'+q)nb^2-3(r'+q)^2bnm-2(r'+q)^3nmg-2R_2(r'+q)nm\right)\\ &+\left(-(r'+q)^3n^2b-(r'+q)^4gn^2-R_2(r'+q)^2n^2\right). \end{aligned}$$

By Lemma 4.16, we know that $(0,0)$ is a zero of $G(s',r')$ of multiplicity two (this corresponds to $C_2(p,q)$ in our original coordinates). Thus, we look to the homogeneous part G_2 (i.e. the terms of order 2), and see that the tangents are given by the linear solutions to:

$$\begin{aligned} 0 =\ & 6(s')^2p^2\left(m^2+n^2-m^2g\right)+3(s')^2p\,(mnb-2bn-2gnmq)+3s'p^2\left(-2gnmr'\right)\\ &+(s')^2\left(qn^2b-mb^2-2qbm^2+2qbm+b^2+q^2m^2+q^2n^2-q^2n^2g-q^2m^2g-R_2m^2\right)\\ &+2s'p\left(r'n^2b-2r'bm^2+2r'bm+2r'qm^2+2r'qn^2-2r'qn^2g-2r'qm^2g\right)\\ &+p^2\left((r')^2m^2+(r')^2n^2-(r')^2n^2g-(r')^2m^2g\right)\\ &+s'\left(-r'nb^2-6r'qbnm-6r'q^2nmg-2R_2r'nm\right)\\ &+p\left(-3(r')^2bnm-6(r')^2qnmg\right)+\left(-3(r')^2qn^2b-6(r')^2q^2gn^2-R_2(r')^2n^2\right) \end{aligned}$$

as given in the proof of Proposition 4.18.

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