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CERTAIN PROPERTIES OF T–EP OPERATORS

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Abstract. An operator A is said to be T-EP if $\mathcal{R}(A)=\mathcal{R}(TA^*)$ and $A=AT^*T$, where T is a partial isometry. In this note, some basic properties of T-EP operators are studied. The invariant characterizations that sum and product of two T-EP operators still keep to be T-EP are obtained. As an extension, we obtain necessary and sufficient conditions for a lower triangular operator matrix to be T-EP.

1. Introduction

EP operators play an important role and have a wide range of applications in operator generalized inverses. Many scholars have paid attention to EP operators and obtained a lot of meaningful results. In this note, we study the basic properties of T-EP operators. Generally, let $\mathcal{H}$ and $\mathcal{K}$ be two complex infinite dimensional separable Hilbert spaces. $\mathcal{H}\oplus\mathcal{K}$ is defined as the Cartesian product $\mathcal{H}\times\mathcal{K}$ where the operations are defined on $\mathcal{H}\times\mathcal{K}$ coordinatewise. If $\mathcal{M}$ and $\mathcal{N}$ are two closed linear subspaces of $\mathcal{H}$, then $\mathcal{M}\ominus\mathcal{N}=\mathcal{M}\cap\mathcal{N}^{\perp}$. The set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ is denoted by $\mathcal{B}(\mathcal{H},\mathcal{K})$. We write $\mathcal{B}(\mathcal{H})$ for $\mathcal{B}(\mathcal{H},\mathcal{H})$. Let $A\in\mathcal{B}(\mathcal{H},\mathcal{K})$. A^*, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ stand for the adjoint, the range and the null space of A, respectively. $\overline{\mathcal{R}(A)}$ is the closure of $\mathcal{R}(A)$ and $P_{\mathcal{M}}$ is the orthogonal projection on a closed subspace $\mathcal{M}$ [15]. We use $\operatorname{asc}(T)$ and $\operatorname{des}(T)$ to denote the ascent and descent of T, i.e., $\operatorname{asc}(T)=\min\{k:\mathcal{N}(T^k)=\mathcal{N}(T^{k+1})\}$ and $\operatorname{des}(T)=\min\{k:\mathcal{R}(T^k)=\mathcal{R}(T^{k+1})\}$ [14]. It is known to us all that $\operatorname{des}(T)=\operatorname{asc}(T)$ when $\operatorname{asc}(T)$ and $\operatorname{des}(T)$ are finite and the index of T is defined by $\operatorname{ind}(T)=\operatorname{asc}(T)=\operatorname{des}(T)$. Note that 0 is not the accumulated point of $\sigma(T)$ when $\operatorname{ind}(T)$ is finite [4]. Let $A\in\mathcal{B}(\mathcal{H})$ and $|A|=(A^*A)^{\frac{1}{2}}$. If $\mathcal{R}(A)=\mathcal{R}(A^*)$, then A is called EP. If $(Ax,x)\geqslant 0$ for all $x\in\mathcal{H}$, we call A is positive.

We remind the reader that $T\in\mathcal{B}(\mathcal{H},\mathcal{K})$ is a partial isometry if $T=TT^*T$. In [6], the concept of relative EP matrix of a rectangular matrix relative to a partial isometry matrix (or, in short, T-EP) and some properties of T-EP are given. The purpose of this paper is to study the basic properties and obtain some new identifying conditions for an operator $A\in\mathcal{B}(\mathcal{H},\mathcal{K})$ on a Hilbert space to be T-EP, where $T\in\mathcal{B}(\mathcal{H},\mathcal{K})$ is a fixed partial isometry. The necessary and sufficient conditions are given for the sum and product of two T-EP operators to be T-EP. As an extension, we then derive necessary and sufficient conditions for a lower triangular operator matrix to be T-EP.

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The paper is organized as follows. In Section 2, we use block operator matrix methods to establish serveral necessary and sufficient conditions and obtain many properties when $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is T-EP. In Section 3, the necessary and sufficient conditions are given for the sum and product of two T-EP operators to be T-EP. And we obtain some interesting conclusions when the fixed partial isometry T has special characteristics.

2. Some basic results

We start with several preliminary results that will be used in our proof. Throughout this paper we will use $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ or $T \in \mathcal{B}(\mathcal{H})$ as a fixed partial isometry. It is well known that, if T is a partial isometry, then $\mathcal{R}(T)$ is closed and there exists an isometry $U \in \mathcal{B}(\mathcal{R}(T^*),\mathcal{R}(T))$ such that

$$T = \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{R}(T^*) \\ \mathcal{N}(T) \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{R}(T) \\ \mathcal{N}(T^*) \end{pmatrix}, \text{ where } U^*U = I_{\mathcal{R}(T^*)},\ UU^* = I_{\mathcal{R}(T)}. \quad (1)$$

The following lemma is a standard result.

LEMMA 2.1. [1, Proposition 2.4], [11, Theorem 4] and [12, Theorem 1.1] *For $A,\ B \in \mathcal{B}(\mathcal{H},\mathcal{K})$, the following conditions are equivalent:*

(i) $\mathcal{R}(A) \subseteq \mathcal{R}(A+B)$.

(ii) $\mathcal{R}(B) \subseteq \mathcal{R}(A+B)$.

(iii) $\mathcal{R}(A)+\mathcal{R}(B) \subseteq \mathcal{R}(A+B)$.

(iv) $\mathcal{R}(A-B) \subseteq \mathcal{R}(A+B)$.

The following ranges results are given by Douglas [8] and Fillmore-Williams [9].

LEMMA 2.2. [8, Theorem 1] and [9, Theorem 2.1] *Let $A,\ B$ be bounded linear operators on a Hilbert space $\mathcal{H}$. Then the following statements are equivalent:*

(i) $\mathcal{R}(A) \subseteq \mathcal{R}(B)$.

(ii) $AA^* \leqslant k^2 BB^*$ *for some* $k > 0$.

(iii) *There exists a bounded linear operator C such that $A = BC$.*

In Lemma 2.2, the operator C can be denoted by $C = B^\dagger A + (I - B^\dagger B)D$ for some bounded linear operator D, where $B^\dagger$ (possibly unbounded) is the Moore-Penrose inverse of B. And $B^\dagger$ exactly bounded when $\mathcal{R}(B)$ is closed [2, 3].

Note that $A \in \mathcal{B}(\mathcal{H})$ is EP if $\mathcal{R}(A) = \mathcal{R}(A^*)$. It is obvious that A is EP $\Longleftrightarrow$ A^* is EP. If $\mathcal{R}(A)$ is closed, A is EP $\Longleftrightarrow$ $A^\dagger$ is EP $\Longleftrightarrow$ $AA^\dagger = A^\dagger A$. The following Definition 2.1 extends the class of EP operators to a class of T-EP operators.

DEFINITION 2.1. [6, Definition 3] Let $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ be a fixed partial isometry. An operator $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is said to be relative EP to T (in short, T-EP) if

$$\mathcal{R}(A) = \mathcal{R}(TA^*), \quad A = AT^*T.$$

For the fixed partial isometry $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and arbitrary $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, it is of great significance to study properties of T-EP. As we know, for every $A \in \mathcal{B}(\mathcal{H})$, there is a unique partial isometry $T_0 \in \mathcal{B}(\mathcal{H})$ such that A has the polar decomposition $A = T_0|A|$ and $\mathcal{N}(T_0) = \mathcal{N}(A)$ [7, Theorem 4.39]. As for this special T_0,

$$A = T_0|A| \text{ is the polar decomposition of } A \Longrightarrow A \text{ is } T_0\text{-EP and } A^* \text{ is } T_0^*\text{-EP.}$$

In fact, $T_0T_0^* = P_{\overline{\mathcal{R}(A)}}$ and $T_0^*T_0 = P_{\overline{\mathcal{R}(A^*)}}$ imply that $A = AT_0^*T_0$ and $A^* = A^*T_0T_0^*$. From $T_0^*AT_0^* = T_0^*T_0|A|T_0^* = |A|T_0^* = A^*$, we have $\mathcal{R}(A^*) = \mathcal{R}(T_0^*AT_0^*) = \mathcal{R}(T_0^*A)$. So, A^* is T_0^*-EP by Definition 2.1. Note that A^* has the polar decomposition $A^* = T_0^*|A^*|$. Similarly, we have that A is T_0-EP.

As for Definition 2.1, we make the following detailed explanations.

REMARK 2.1. Note that T is a fixed partial isometry. Definition 2.1 has some equal descriptions (see [6] for the finite matrix case).

(i)

$$\mathcal{R}(A) = \mathcal{R}(TA^*) \Longleftrightarrow \mathcal{R}(A) = \mathcal{R}(TA^*T).$$

In fact, by Lemma 2.2, $\mathcal{R}(A) = \mathcal{R}(TA^*)$ (resp. $\mathcal{R}(A) = \mathcal{R}(TA^*T)$) implies that $\overline{\mathcal{R}(A)} \subseteq \mathcal{R}(T)$. So,

$$\mathcal{R}(A^*) = A^*\mathcal{R}(T) = \mathcal{R}(A^*T)$$

and

$$\mathcal{R}(TA^*T) = T\mathcal{R}(A^*T) = T\mathcal{R}(A^*) = \mathcal{R}(TA^*).$$

Hence,

$$\mathcal{R}(A) = \mathcal{R}(TA^*) \Longleftrightarrow \mathcal{R}(A) = \mathcal{R}(TA^*T).$$

(ii) Note that $T^*T = P_{\mathcal{R}(T^*)}$ and $TT^* = P_{\mathcal{R}(T)}$. It follows that

$$\begin{aligned}
&A = AT^*T \\
\Longleftrightarrow\ &\mathcal{R}(A^*) \subseteq \mathcal{R}(T^*) \\
\Longleftrightarrow\ &\mathcal{N}(T) \subseteq \mathcal{N}(A) \\
\Longleftrightarrow\ &A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & 0 \end{pmatrix}, \text{ where } A_{11} \in \mathcal{B}(\mathcal{R}(T^*),\mathcal{R}(T)),\ A_{21} \in \mathcal{B}(\mathcal{R}(T^*),\mathcal{N}(T^*))).
\end{aligned}$$

In addition, if $\mathcal{R}(A)$ is closed,

$$A = AT^*T \Longleftrightarrow A^\dagger = T^*TA^\dagger.$$

(iii) Let S_1 and S_2 be two partial isometries, A be a closed range operator such that $AS_2S_2^* = A = S_1^*S_1A$. Then

$$(S_1AS_2)^\dagger = S_2^*A^\dagger S_1^*.$$

(iv) An operator $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is said to be relative selfadjoint to T (in short, T-selfadjoint) if $A = TA^*T$. A is said to be relative normal to T (in short, T-normal) if $A = TT^*A = AT^*T$ and $AA^*T = TA^*A$. It is obvious that every T-selfadjoint operator is T-normal [6, Definitions 1 and 2]. If A is T-normal, then $A = AT^*T$ and

$$\begin{aligned}\mathcal{R}(A) &= \mathcal{R}((AA^*)^{\frac{1}{2}}) = \mathcal{R}((TT^*AA^*)^{\frac{1}{2}}) = \mathcal{R}((TA^*AT^*)^{\frac{1}{2}})\\ &= \mathcal{R}((TA^*TT^*AT^*)^{\frac{1}{2}}) = \mathcal{R}(TA^*T) = \mathcal{R}(TA^*).\end{aligned}$$

Hence, every T-normal operator is T-EP.

For the fixed partial isometry $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$, the next lemma gives the operator matrix representation of the T-EP operator $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$.

LEMMA 2.3. *Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$. The following results are equivalent:*

(i) *A is T-EP.*

(ii) *A and T, as the operators from*

$$\overline{\mathcal{R}(A^*)} \oplus \left[\mathcal{R}(T^*) \ominus \overline{\mathcal{R}(A^*)}\right] \oplus \mathcal{N}(T) \longrightarrow \overline{\mathcal{R}(A)} \oplus \left[\mathcal{R}(T) \ominus \overline{\mathcal{R}(A)}\right] \oplus \mathcal{N}(T^*),$$

can be denoted as

$$A = A_{11} \oplus 0 \oplus 0, \quad T = U_{11} \oplus U_{22} \oplus 0, \tag{2}$$

respectively, where A_{11} is injective with dense range, U_{11} and U_{22} are two isometries satisfying $\mathcal{R}(A_{11}) = \mathcal{R}(U_{11}A_{11}^*)$.

(iii) *There exists an EP operator $E \in \mathcal{B}(\mathcal{K})$ such that $A = ET$ and $E = TT^*E$.*

Proof. (i) $\Longrightarrow$ (ii) Let T have the form (1). Then A has the corresponding form

$$A = \begin{pmatrix} A_1 & A_{12} \\ A_{21} & A_{22} \end{pmatrix} : \begin{pmatrix} \mathcal{R}(T^*) \\ \mathcal{N}(T) \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{R}(T) \\ \mathcal{N}(T^*) \end{pmatrix}. \tag{3}$$

So,

$$\mathcal{R}(A) = \mathcal{R}(TA^*T) \Longleftrightarrow A_{21} = 0,\ A_{22} = 0 \text{ and } \mathcal{R}((A_1, A_{12})) = \mathcal{R}(UA_1^*U)$$

and

$$A = AT^*T \Longleftrightarrow A_{12} = 0.$$

One gets $A = A_1 \oplus 0$ with $\mathcal{R}(A_1) = \mathcal{R}(UA_1^*)$. By Definition 2.1, one has $\overline{\mathcal{R}(A)} \subseteq \mathcal{R}(T)$ and $\overline{\mathcal{R}(A^*)} \subseteq \mathcal{R}(T^*)$. The operator matrices (1) and (3) can be rewritten as

$$A = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \overline{\mathcal{R}(A^*)} \\ \mathcal{R}(T^*) \ominus \overline{\mathcal{R}(A^*)} \\ \mathcal{N}(T) \end{pmatrix} \longrightarrow \begin{pmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(T) \ominus \overline{\mathcal{R}(A)} \\ \mathcal{N}(T^*) \end{pmatrix}$$

and

$$T = \begin{pmatrix} U_{11} & U_{12} & 0 \\ U_{21} & U_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \overline{\mathcal{R}(A^*)} \\ \mathcal{R}(T^*) \ominus \overline{\mathcal{R}(A^*)} \\ \mathcal{N}(T) \end{pmatrix} \longrightarrow \begin{pmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(T) \ominus \overline{\mathcal{R}(A)} \\ \mathcal{N}(T^*) \end{pmatrix},$$

respectively, where A_{11} is injective with dense range and $A_1 = A_{11} \oplus 0$, the isometry $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$ with $U^*U = I_{\mathcal{R}(T^*)}$ and $UU^* = I_{\mathcal{R}(T)}$. From $\mathcal{R}(A_1) = \mathcal{R}(UA_1^*)$ and $UA_1^* = \begin{pmatrix} U_{11}A_{11}^* & 0 \\ U_{21}A_{11}^* & 0 \end{pmatrix}$, one gets $U_{21} = 0$ and $\mathcal{R}(A_{11}) = \mathcal{R}(U_{11}A_{11}^*)$. It follows that $\mathcal{R}(U_{11})$ is dense in $\overline{\mathcal{R}(A)}$. From $U^*U = I_{\mathcal{R}(T^*)}$, one gets

$$U_{11}^*U_{11} = I_{\overline{\mathcal{R}(A^*)}}, \quad U_{11}^*U_{12} = 0, \quad U_{12}^*U_{11} = 0, \quad U_{12}^*U_{12} + U_{22}^*U_{22} = I_{\mathcal{R}(T^*)\ominus\overline{\mathcal{R}(A^*)}}.$$

It follows that U_{11} is injective with closed range. Hence, $U_{11} \in \mathcal{B}(\overline{\mathcal{R}(A^*)}, \overline{\mathcal{R}(A)})$ is an isometry, $U_{12} = 0$ and $U_{22}^*U_{22} = I_{\mathcal{R}(T^*)\ominus\overline{\mathcal{R}(A^*)}}$. Furthermore, from $UU^* = I_{\mathcal{R}(T)}$, one derives that $U_{22} \in \mathcal{B}(\mathcal{R}(T^*) \ominus \overline{\mathcal{R}(A^*)}, \mathcal{R}(T) \ominus \overline{\mathcal{R}(A)})$ is an isometry with $U_{22}U_{22}^* = I_{\mathcal{R}(T)\ominus\overline{\mathcal{R}(A)}}$.

(ii) $\Longrightarrow$ (iii) Let $E = AT^*$. By (2), $E = A_{11}U_{11}^* \oplus 0 \oplus 0$. Since

$$\mathcal{R}(A_{11}U_{11}^*) = \mathcal{R}(A_{11}) = \mathcal{R}(U_{11}A_{11}^*),$$

one has E is an EP operator, $A = ET$ and $E = TT^*E$.

(iii) $\Longrightarrow$ (i) Since $A = ET$, $A^* = T^*E^*$, one gets $\mathcal{R}(A^*) \subseteq \mathcal{R}(T^*)$. It follows that $A = AT^*T$. By $TT^*E = E$ and E is an EP operator, we have $\mathcal{R}(E) = \mathcal{R}(E^*) \subseteq \mathcal{R}(T)$. From

$$\mathcal{R}(E) \supseteq \mathcal{R}(A) = \mathcal{R}(ET) = E\mathcal{R}(T) \supseteq E\mathcal{R}(E^*) = \mathcal{R}(E),$$

one has $\mathcal{R}(A) = \mathcal{R}(E)$. In addition, from

$$\mathcal{R}(E) = \mathcal{R}(E^*) \supseteq \mathcal{R}(TA^*T) = \mathcal{R}(TT^*E^*T) = \mathcal{R}(E^*T) \supseteq \mathcal{R}(E^*) = \mathcal{R}(E),$$

one has

$$\mathcal{R}(TA^*) = \mathcal{R}(TA^*T) = \mathcal{R}(E).$$

As a result, $\mathcal{R}(A) = \mathcal{R}(TA^*)$ and therefore A is T-EP. $\square$

REMARK 2.2. By Lemma 2.3 and its proof, one has the following observations immediately (see [6] for the finite matrix case).

(i) If A is T-EP, then $E = AT^*$ satisfies $\mathcal{R}(E) = \mathcal{R}(A) = \mathcal{R}(E^*) \subseteq \mathcal{R}(T)$,

$$E = ETT^* = TT^*E \text{ and } A = AT^*T = TT^*A.$$

(ii) In Lemma 2.3, if T is unitary, then $\mathcal{R}(T) = \mathcal{R}(T^*)$, $T = U_1 \oplus U_2$ and $A = A_{11} \oplus 0$ with $\mathcal{R}(A_{11}) = \mathcal{R}(U_1A_{11}^*)$. Hence,

$$A \text{ is } T\text{-EP} \iff TA^* \text{ is EP} \iff AT^* \text{ is EP} \iff \mathcal{R}(A) = \mathcal{R}(TA^*).$$

In addition, if $\mathcal{R}(A)$ is closed, then

$$\begin{aligned} A \text{ is } T\text{-EP} &\iff \mathcal{R}(A) = \mathcal{R}(TA^*) \\ &\iff \mathcal{N}(A^*) = \mathcal{N}(AT^*) \\ &\iff TA^\dagger \text{ is EP} \\ &\iff TA^\dagger A = AA^\dagger T. \end{aligned}$$

(iii) In Lemma 2.3,

$$\mathcal{R}(A_{11})=\mathcal{R}(U_{11}A_{11}^*)\Longleftrightarrow AT^* \text{ is EP} \Longleftrightarrow\ TA^* \text{ is EP},$$

which is also equivalent to that $TA^\dagger$ is EP when $\mathcal{R}(A)$ is closed.

(iv) If $\mathcal{R}(A)$ is closed, then A is T-EP if and only if $TA^\dagger A=AA^\dagger T$ and $A=AT^*T=TT^*A$. In this case, A_{11} is invertible by (2) and the following relations are obvious:

$$(T^*A)^\dagger=A^\dagger T;\quad (AT^*)^\dagger=TA^\dagger;\quad (TA^\dagger)^\dagger=AT^*;\quad (A^\dagger T)^\dagger=T^*A;$$
$$(TA^\dagger T)^\dagger=T^*AT^*;\quad (AT^*A)^\dagger=A^\dagger TA^\dagger;\quad A^\dagger=T^*E^\dagger=T^*(AT^*)^\dagger.$$

(v) In general,

$$A \text{ is } T\text{-EP}\Longleftrightarrow A^* \text{ is } T^*\text{-EP}.$$

If $\mathcal{R}(A)$ is closed, then

$$A \text{ is } T\text{-EP}\Longleftrightarrow A^\dagger \text{ is } T^*\text{-EP}.$$

In fact, if A is T-EP, by Lemma 2.3, $\mathcal{R}(A_{11})=\mathcal{R}(U_{11}A_{11}^*)$ implies that

$$\mathcal{R}(U_{11}^*A_{11})=\mathcal{R}(U_{11}^*U_{11}A_{11}^*)=\mathcal{R}(P_{\overline{\mathcal{R}(A^*)}}A_{11}^*)=\mathcal{R}(A_{11}^*).$$

So, A^* is T^*-EP. Conversely, if A^* is T^*-EP, similarly to the above proof, $(A^*)^*$ is $(T^*)^*$-EP, i.e., A is T-EP. In case that $\mathcal{R}(A)$ is closed and A^* is T^*-EP, then

$$\mathcal{R}(A^\dagger)=\mathcal{R}(A^*)=\mathcal{R}(T^*A)=\mathcal{R}(T^*(A^\dagger)^*)$$

and $\mathcal{R}((A^\dagger)^*)=\mathcal{R}(A)\subseteq\mathcal{R}(T)$. One has A^* is T^*-EP $\Longleftrightarrow A^\dagger$ is T^*-EP.

(vi)

$$A \text{ is } T\text{-EP}\Longleftrightarrow\ AT^* \text{ is EP and } A=AT^*T.$$

In fact, it is clear that A is T-EP $\Longrightarrow\ AT^*$ is EP and $A=AT^*T$. On the other hand, if AT^* is EP and $A=AT^*T$, then

$$\mathcal{R}(A)=A(\overline{\mathcal{R}(A^*)})\subseteq A(\mathcal{R}(T^*))=\mathcal{R}(TA^*).$$

Hence, $\mathcal{R}(A)\subseteq\mathcal{R}(T)$ and $\mathcal{R}(A^*)\subseteq\mathcal{R}(T^*)$. Therefore, if T is denoted by (1), then $A=A_1\oplus 0$. It follows that $\mathcal{R}(A_1)=\mathcal{R}(A_1U^*)=\mathcal{R}(UA_1^*)$ and $\mathcal{R}(A)=\mathcal{R}(TA^*)$.

(vii)

$$AT^* \text{ is EP}\Longleftrightarrow\ TA^* \text{ is EP } (\Longleftrightarrow TA^\dagger \text{ is EP when } \mathcal{R}(A) \text{ is closed}).$$

(viii)

$$A \text{ is EP and } |A^*| \text{ is } T\text{-EP}\Longrightarrow A^* \text{ is } T\text{-EP}.$$

In fact,

$$\mathcal{R}(A^*)=\mathcal{R}(A)=\mathcal{R}(|A^*|)=\mathcal{R}(T|A^*|)=T\mathcal{R}(A)=\mathcal{R}(TA).$$

By polar decomposition and $|A^*|=|A^*|T^*T$ one has $A^*=A^*T^*T$, i.e., A^* is T-EP.

3. Some properties of T-EP operators

In this section, we study characterizations of T-EP operators. Firstly, we consider a specific case when $T \in \mathcal{B}(\mathcal{H})$ be a fixed selfadjoint partial isometry.

THEOREM 3.1. *Let $A \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{B}(\mathcal{H})$ be a fixed selfadjoint partial isometry. Then A is T-EP if and only if TA is EP and $A = AT^2 = T^2A$.*

Proof. $\Longrightarrow$ Since A is T-EP, $\mathcal{R}(A) \subseteq \mathcal{R}(T)$ and $\mathcal{R}(A^*) \subseteq \mathcal{R}(T)$ by Definition 2.1. Hence, $T^2A = P_{\mathcal{R}(T)}A = A = AT^2$. From $\mathcal{R}(A) = \mathcal{R}(TA^*)$ one has

$$\mathcal{R}(TA) = \mathcal{R}(T^2A^*) = \mathcal{R}(A^*) = A^*(\overline{\mathcal{R}(A)}) \subseteq A^*(\mathcal{R}(T)) = \mathcal{R}(A^*T^*) \subseteq \mathcal{R}(A^*).$$

Therefore, $\mathcal{R}(TA) = \mathcal{R}(A^*T^*)$.

$\Longleftarrow$ From $A = AT^2 = T^2A$ one has $\mathcal{R}(A) \subseteq \mathcal{R}(T)$ and $\mathcal{R}(A^*) \subseteq \mathcal{R}(T)$. In addition,

$$\mathcal{R}(T^*A) = \mathcal{R}(TA) = \mathcal{R}(A^*T^*) = A^*(\mathcal{R}(T^*)) \supseteq A^*(\overline{\mathcal{R}(A)}) = \mathcal{R}(A^*)$$

and $\mathcal{R}(A^*T^*) \subseteq \mathcal{R}(A^*)$. Hence, A^* is T^*-EP. By Remark 2.2 (iii), we get A is T-EP. $\square$

Theorem 3.1 shows that, if $T \in \mathcal{B}(\mathcal{H})$ is an orthogonal projection, then

$$A \in \mathcal{B}(\mathcal{H}) \text{ is } T\text{-EP} \Longleftrightarrow A \text{ is EP and } A = AT = TA.$$

THEOREM 3.2. *Let $A \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{B}(\mathcal{H})$ be a partial isometry such that $AT^* = T^*A$. Then A is T-EP if and only if A is EP and $A = TT^*A = AT^*T$.*

Proof. $\Longrightarrow$ From A is T-EP and $TA^* = A^*T$ one gets

$$\mathcal{R}(A) = \mathcal{R}(TA^*) = \mathcal{R}(A^*T) \subseteq \mathcal{R}(A^*).$$

Note that

$$\mathcal{R}(A^*) = \mathcal{R}(T^*TA^*) = T^*\mathcal{R}(TA^*) = T^*\mathcal{R}(A) = \mathcal{R}(T^*A) = \mathcal{R}(AT^*) \subseteq \mathcal{R}(A).$$

Hence, $\mathcal{R}(A) = \mathcal{R}(A^*)$, i.e., A is EP. By Remark 2.2 (i) one has $A = AT^*T = TT^*A$.

$\Longleftarrow$ Since $\mathcal{R}(A) = \mathcal{R}(A^*)$, A can be denoted as $A = A_1 \oplus 0$, where $A_1 \in \mathcal{B}(\overline{\mathcal{R}(\mathcal{A})})$ is injective with dense range. From $AT^* = T^*A$ one can write T as the corresponding form $T = U_{11} \oplus U_{22}$, where U_{11} and U_{22} are partial isometries on $\overline{\mathcal{R}(A)}$ and $\mathcal{N}(A)$, respectively. Since $A = AT^*T = TT^*A$, one derives that U_{11} is an isometry. Hence $\mathcal{R}(A) = \mathcal{R}(A^*) = \mathcal{R}(A^*T) = \mathcal{R}(TA^*)$. A is T-EP. $\square$

Next, we consider the sum and the product of two T-EP operators and obtain the sufficient and necessary conditions which ensure $A + B$ and AB are T-EP.

THEOREM 3.3. *Let A and $B \in \mathcal{B}(\mathcal{H},\mathcal{K})$ be T-EP. Then $A+B$ is T-EP if and only if*

$$\begin{aligned}&\mathcal{R}(P_{\overline{\mathcal{R}(A)}}(A+B)P_{\overline{\mathcal{R}(A^*)}})+\mathcal{R}(P_{\overline{\mathcal{R}(A)}}BP_{\mathcal{R}(T^*)\ominus\overline{\mathcal{R}(A^*)}})\\&=\mathcal{R}(P_{\overline{\mathcal{R}(A)}}T(A+B)^*P_{\overline{\mathcal{R}(A)}})+\mathcal{R}(P_{\overline{\mathcal{R}(A)}}TB^*P_{\mathcal{R}(T)\ominus\overline{\mathcal{R}(A)}}).\end{aligned}\tag{4}$$

Proof. Let A and T have the matrix forms (2). If A and B are T-EP, then $A=AT^*T$, $B=BT^*T$ and $(A+B)=(A+B)T^*T$. By Lemma 2.3, T-EP operator B can be represented as

$$B=\begin{pmatrix}B_{11}&B_{12}&0\\B_{21}&B_{22}&0\\0&0&0\end{pmatrix}:\begin{pmatrix}\overline{\mathcal{R}(A^*)}\\\mathcal{R}(T^*)\ominus\overline{\mathcal{R}(A^*)}\\\mathcal{N}(T)\end{pmatrix}\longrightarrow\begin{pmatrix}\overline{\mathcal{R}(A)}\\\mathcal{R}(T)\ominus\overline{\mathcal{R}(A)}\\\mathcal{N}(T^*)\end{pmatrix}.\tag{5}$$

By (2) and $\mathcal{R}(B)=\mathcal{R}(TB^*)$ one has

$$\mathcal{R}\left(\begin{pmatrix}B_{11}&B_{12}\\B_{21}&B_{22}\end{pmatrix}\right)=\mathcal{R}\left(\begin{pmatrix}U_{11}B_{11}^*&U_{11}B_{21}^*\\U_{22}B_{12}^*&U_{22}B_{22}^*\end{pmatrix}\right).$$

One gets $\mathcal{R}(\begin{pmatrix}B_{11}&B_{12}\end{pmatrix})=\mathcal{R}(\begin{pmatrix}U_{11}B_{11}^*&U_{11}B_{21}^*\end{pmatrix})$, i.e.,

$$\begin{aligned}&\mathcal{R}(P_{\overline{\mathcal{R}(A)}}BP_{\overline{\mathcal{R}(A^*)}})+\mathcal{R}(P_{\overline{\mathcal{R}(A)}}BP_{\mathcal{R}(T^*)\ominus\overline{\mathcal{R}(A^*)}})\\&=\mathcal{R}(P_{\overline{\mathcal{R}(A)}}TB^*P_{\overline{\mathcal{R}(A)}})+\mathcal{R}(P_{\overline{\mathcal{R}(A)}}TB^*P_{\mathcal{R}(T)\ominus\overline{\mathcal{R}(A)}})\end{aligned}\tag{6}$$

and $\mathcal{R}(\begin{pmatrix}B_{21}&B_{22}\end{pmatrix})=\mathcal{R}(\begin{pmatrix}U_{22}B_{12}^*&U_{22}B_{22}^*\end{pmatrix})$. Therefore,

$$\begin{aligned}&\mathcal{R}(A+B)=\mathcal{R}(T(A+B)^*)\\\Longleftrightarrow\ &\mathcal{R}\left(\begin{pmatrix}A_{11}+B_{11}&B_{12}\\B_{21}&B_{22}\end{pmatrix}\right)=\mathcal{R}\left(\begin{pmatrix}U_{11}(A_{11}+B_{11})^*&U_{11}B_{21}^*\\U_{22}B_{12}^*&U_{22}B_{22}^*\end{pmatrix}\right)\\\Longleftrightarrow\ &\mathcal{R}((A_{11}+B_{11}\ B_{12}))=\mathcal{R}((U_{11}(A_{11}+B_{11})^*\ U_{11}B_{21}^*))\\\Longleftrightarrow\ &\mathcal{R}(A_{11}+B_{11})+\mathcal{R}(B_{12})=\mathcal{R}(U_{11}(A_{11}+B_{11})^*)+\mathcal{R}(U_{11}B_{21}^*)\\\Longleftrightarrow\ &\mathcal{R}(P_{\overline{\mathcal{R}(A)}}(A+B)P_{\overline{\mathcal{R}(A^*)}})+\mathcal{R}(P_{\overline{\mathcal{R}(A)}}BP_{\mathcal{R}(T^*)\ominus\overline{\mathcal{R}(A^*)}})\\&=\mathcal{R}(P_{\overline{\mathcal{R}(A)}}T(A+B)^*P_{\overline{\mathcal{R}(A)}})+\mathcal{R}(P_{\overline{\mathcal{R}(A)}}TB^*P_{\mathcal{R}(T)\ominus\overline{\mathcal{R}(A)}}).\quad\square\end{aligned}$$

Theorem 3.3 contains some special cases.

COROLLARY 3.1. *Let A, $B\in\mathcal{B}(\mathcal{H},\mathcal{K})$ be T-EP. If $P_{\overline{\mathcal{R}(A)}}BP_{\overline{\mathcal{R}(A^*)}}=0$, then $A+B$ is T-EP.*

Proof. By Lemma 2.3, if A is T-EP, then $P_{\overline{\mathcal{R}(A)}}T=TP_{\overline{\mathcal{R}(A^*)}}$ and

$$\mathcal{R}(A)=\mathcal{R}(P_{\overline{\mathcal{R}(A)}}AP_{\overline{\mathcal{R}(A^*)}})=\mathcal{R}(P_{\overline{\mathcal{R}(A)}}TA^*P_{\overline{\mathcal{R}(A)}}).\tag{7}$$

If $P_{\overline{\mathcal{R}(A)}}BP_{\overline{\mathcal{R}(A^*)}}=0$, then

$$\mathcal{R}(P_{\overline{\mathcal{R}(A)}}BP_{\mathcal{R}(T^*)\ominus\overline{\mathcal{R}(A^*)}})=\mathcal{R}(P_{\overline{\mathcal{R}(A)}}TB^*P_{\mathcal{R}(T)\ominus\overline{\mathcal{R}(A)}})$$

by (6),

$$\mathcal{R}(P_{\overline{\mathcal{R}(A)}}(A+B)P_{\overline{\mathcal{R}(A^*)}})=\mathcal{R}(P_{\overline{\mathcal{R}(A)}}AP_{\overline{\mathcal{R}(A^*)}})$$

and

$$\mathcal{R}(P_{\overline{\mathcal{R}(A)}}T(A+B)^*P_{\overline{\mathcal{R}(A)}})=\mathcal{R}(P_{\overline{\mathcal{R}(A)}}TA^*P_{\overline{\mathcal{R}(A)}}).$$

Hence, the result in (4) holds and $A+B$ is T-EP. □

REMARK 3.1. Let A, $B\in\mathcal{B}(\mathcal{H},\mathcal{K})$ be T-EP. We can derive the following results.
(i) In Theorem 3.3, if $\mathcal{R}(A)$ is closed, then $A+B$ is T-EP if and only if

$$\begin{aligned}&\mathcal{R}(A+AA^\dagger BA^\dagger A)+\mathcal{R}(AA^\dagger B(T^*T-A^\dagger A))\\ &=\mathcal{R}(AA^\dagger T(A+B)^*AA^\dagger)+\mathcal{R}(AA^\dagger TB^*(TT^*-AA^\dagger)).\end{aligned}\tag{8}$$

(ii) If $BA^*=0$, then

$$B=\begin{pmatrix}0&B_{12}&0\\0&B_{22}&0\\0&0&0\end{pmatrix},\quad \mathcal{R}(B_{12})=\mathcal{R}(U_{11}B_{12}^*),\quad \mathcal{R}(B_{22})=\mathcal{R}(U_{22}B_{22}^*).$$

By Theorem 3.3, $A+B$ is T-EP. Similarly, if $A^*B=0\Rightarrow B_{11}=0$ and $B_{12}=0\Rightarrow A+B$ is T-EP.
(iii) If one of $\overline{\mathcal{R}(A)}\cap\mathcal{R}(B)=\{0\}$ and $\overline{\mathcal{R}(A^*)}\cap\mathcal{R}(B^*)=\{0\}$ holds, then $P_{\overline{\mathcal{R}(A)}}BP_{\overline{\mathcal{R}(A^*)}}=0$ and $A+B$ is T-EP by Corollary 3.1. These are the main results in [6, Corollaries 5 and 6, Theorems 23, 25 and 26].

COROLLARY 3.2. *Let A, $B\in\mathcal{B}(\mathcal{H},\mathcal{K})$ be T-EP. If*

$$\mathcal{R}(A)\subseteq\mathcal{R}(P_{\overline{\mathcal{R}(A)}}(A+B)P_{\overline{\mathcal{R}(A^*)}})$$

and

$$\mathcal{R}(P_{\overline{\mathcal{R}(A)}}TA^*)\subseteq\mathcal{R}(P_{\overline{\mathcal{R}(A)}}T(A+B)^*P_{\overline{\mathcal{R}(A)}}),$$

then $A+B$ is T-EP.

Proof. If

$$\mathcal{R}(A)=\mathcal{R}(P_{\overline{\mathcal{R}(A)}}AP_{\overline{\mathcal{R}(A^*)}})\subseteq\mathcal{R}(P_{\overline{\mathcal{R}(A)}}(A+B)P_{\overline{\mathcal{R}(A^*)}})$$

and

$$\mathcal{R}(P_{\overline{\mathcal{R}(A)}}TA^*)\subseteq\mathcal{R}(P_{\overline{\mathcal{R}(A)}}T(A+B)^*P_{\overline{\mathcal{R}(A)}}),$$

then

$$\mathcal{R}(P_{\overline{\mathcal{R}(A)}}AP_{\overline{\mathcal{R}(A^*)}})+\mathcal{R}(P_{\overline{\mathcal{R}(A)}}BP_{\overline{\mathcal{R}(A^*)}})=\mathcal{R}(P_{\overline{\mathcal{R}(A)}}(A+B)P_{\overline{\mathcal{R}(A^*)}})$$

and

$$\mathcal{R}(P_{\overline{\mathcal{R}(A)}}TA^*P_{\overline{\mathcal{R}(A)}})+\mathcal{R}(P_{\overline{\mathcal{R}(A)}}TB^*P_{\overline{\mathcal{R}(A)}})=\mathcal{R}(P_{\overline{\mathcal{R}(A)}}T(A+B)^*P_{\overline{\mathcal{R}(A)}})$$

by Lemma 2.1. Since A and B are T-EP, by (6) and (7),

$$\mathcal{R}(P_{\overline{\mathcal{R}(A)}}(A+B)P_{\overline{\mathcal{R}(A^*)}})+\mathcal{R}(P_{\overline{\mathcal{R}(A)}}BP_{\mathcal{R}(T^*)\ominus\overline{\mathcal{R}(A^*)}})$$

$$=\mathcal{R}(P_{\overline{\mathcal{R}(A)}}T(A+B)^*P_{\overline{\mathcal{R}(A)}})+\mathcal{R}(P_{\overline{\mathcal{R}(A)}}TB^*P_{\mathcal{R}(T)\ominus\overline{\mathcal{R}(A)}}).$$

Therefore, $A+B$ is T-EP. □

As for the product of two T-EP operators, we have the following results.

THEOREM 3.4. *Let $A,\ B\in\mathcal{B}(\mathcal{H},\mathcal{K})$ be T-EP. Then*

(i) *A^*B is T^*T-EP $\Longleftrightarrow A^*BP_{\overline{\mathcal{R}(A^*)}}$ is EP and $P_{\overline{\mathcal{R}(A)}}BP_{\mathcal{R}(T^*)\ominus\overline{\mathcal{R}(A^*)}}=0$.*

(ii) *AB^* is TT^*-EP $\Longleftrightarrow AB^*P_{\overline{\mathcal{R}(A)}}$ is EP and $P_{\mathcal{R}(T)\ominus\overline{\mathcal{R}(A)}}BP_{\overline{\mathcal{R}(A^*)}}=0$.*

Proof. (i) It is obvious that $A^*B=A^*BT^*T$ since B is T-EP. By Lemma 2.3, A and T have the forms as in (2). Let B have the corresponding form as in (5). One has $T^*T=I\oplus I\oplus 0$,

$$A^*B=\begin{pmatrix}A_{11}^*B_{11} & A_{11}^*B_{12} & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad T^*TB^*A=\begin{pmatrix}B_{11}^*A_{11} & 0 & 0\\ B_{12}^*A_{11} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Hence, A^*B is T^*T-EP if and only if $\mathcal{R}(A^*B)=\mathcal{R}(T^*TB^*A)\Longleftrightarrow A_{11}^*B_{11}$ is EP and $B_{12}=0\Longleftrightarrow P_{\overline{\mathcal{R}(A)}}BP_{\mathcal{R}(T^*)\ominus\overline{\mathcal{R}(A^*)}}=0$ and $A^*BP_{\overline{\mathcal{R}(A^*)}}$ is EP.

(ii) Similar to (i). □

THEOREM 3.5. *Let A and $B\in\mathcal{B}(\mathcal{H})$ be T-EP.*

(i) *If $\mathcal{R}(AB)=\mathcal{R}(A)\cap\mathcal{R}(B)$ and $\mathcal{R}(B^*A^*)=\mathcal{R}(B^*)\cap\mathcal{R}(A^*)$, then AB is T-EP.*

(ii) *If AB is T-EP, then $\mathcal{R}(AB)\subseteq\mathcal{R}(A)\cap\mathcal{R}(B)$ and $\mathcal{R}(B^*A^*)\subseteq\mathcal{R}(B^*)\cap\mathcal{R}(A^*)$.*

(iii) *If $\operatorname{ind}(A)\leqslant 1$ and $\operatorname{ind}(B)\leqslant 1$, then*

$$AB \text{ is } T\text{-EP} \Longleftrightarrow \mathcal{R}(AB)=\mathcal{R}(A)\cap\mathcal{R}(B) \text{ and } \mathcal{R}(B^*A^*)=\mathcal{R}(B^*)\cap\mathcal{R}(A^*).$$

Proof. (i) If $\mathcal{R}(AB)=\mathcal{R}(A)\cap\mathcal{R}(B)$ and $\mathcal{R}(B^*A^*)=\mathcal{R}(B^*)\cap\mathcal{R}(A^*)$, then

$$\mathcal{R}(AB)=\mathcal{R}(A)\cap\mathcal{R}(B)=\mathcal{R}(TB^*)\cap\mathcal{R}(TA^*)$$

and

$$\mathcal{R}(TB^*A^*) = T(\mathcal{R}(B^*A^*)) = T(\mathcal{R}(B^*)\cap\mathcal{R}(A^*)) = \mathcal{R}(TB^*)\cap\mathcal{R}(TA^*).$$

And it is clear that $AB = ABT^*T$. Hence, AB is T-EP.

(ii) Obviously, $\mathcal{R}(AB)\subseteq\mathcal{R}(A)$ by Lemma 2.2. Since AB is T-EP,

$$\mathcal{R}(AB) = \mathcal{R}(TB^*A^*) = TB^*(\mathcal{R}(A^*)) \subseteq \mathcal{R}(TB^*) = \mathcal{R}(B).$$

Hence, $\mathcal{R}(AB)\subseteq\mathcal{R}(A)\cap\mathcal{R}(B)$. Since AB is T-EP, B^*A^* is T^*-EP and

$$\mathcal{R}(B^*A^*) = \mathcal{R}(T^*AB) = T^*A(\mathcal{R}(B)) \subseteq \mathcal{R}(T^*A) = \mathcal{R}(A^*).$$

Obviously, $\mathcal{R}(B^*A^*)\subseteq\mathcal{R}(B^*)$. Hence, $\mathcal{R}(B^*A^*)\subseteq\mathcal{R}(B^*)\cap\mathcal{R}(A^*)$.

(iii) By items (i) and (ii), we only need to verify that, if AB is T-EP,

$$\mathcal{R}(AB)\supseteq\mathcal{R}(A)\cap\mathcal{R}(B), \quad \mathcal{R}(B^*A^*)\supseteq\mathcal{R}(B^*)\cap\mathcal{R}(A^*).$$

Since $\operatorname{ind}(B)\leqslant 1$, $\mathcal{R}(B)$ is closed and B can be denoted as

$$B = \begin{pmatrix} B_{11} & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{pmatrix},$$

where B_{11} is invertible. Denote A by

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} : \begin{pmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{pmatrix}.$$

Then $AB = \begin{pmatrix} A_{11}B_{11} & 0 \\ A_{21}B_{11} & 0 \end{pmatrix}$. Since AB is T-EP, by item (ii), $\mathcal{R}(AB)\subseteq\mathcal{R}(B)$. One gets $A_{21}=0$ since B_{11} is invertible. It follows that

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad A^2 = \begin{pmatrix} A_{11}^2 & A_{11}A_{12}+A_{12}A_{22} \\ 0 & A_{22}^2 \end{pmatrix},$$

where $\mathcal{N}(A_{22}) = \mathcal{N}(A_{22}^2)$ since $\max\{\operatorname{ind}(A_{11}),\operatorname{ind}(A_{22})\}\leqslant\operatorname{ind}(A)\leqslant 1$. If $x\in\mathcal{R}(A)\cap\mathcal{R}(B) = \mathcal{R}(A^2)\cap\mathcal{R}(B)$, then there exists x_i and y_i, $i=1,2$ such that

$$x = \begin{pmatrix} A_{11}^2 & A_{11}A_{12}+A_{12}A_{22} \\ 0 & A_{22}^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} B_{11} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

So, $A_{11}^2x_1 + (A_{11}A_{12}+A_{12}A_{22})x_2 = B_{11}y_1$ and $A_{22}x_2 = 0$. One has

$$x = A_{11}^2x_1 + A_{11}A_{12}x_2 = A_{11}(A_{11}x_1 + A_{12}x_2) = B_{11}y_1.$$

Note that $A_{11}x_1 + A_{12}x_2 \in \mathcal{R}(B)$. It follows that $x = B_{11}y_1 \in \mathcal{R}(AB)$, i.e., $\mathcal{R}(AB)\supseteq\mathcal{R}(A)\cap\mathcal{R}(B)$. Similarly, $\mathcal{R}(B^*A^*)\supseteq\mathcal{R}(B^*)\cap\mathcal{R}(A^*)$. $\square$

If A and B are EP with closed range, then $\operatorname{ind}(A)\leqslant 1$ and $\operatorname{ind}(B)\leqslant 1$. It follows that

$$AB \text{ is EP} \Longleftrightarrow \mathcal{R}(AB) = \mathcal{R}(A) \cap \mathcal{R}(B) \text{ and } \mathcal{R}(B^*A^*) = \mathcal{R}(B^*) \cap \mathcal{R}(A^*).$$

It is the result in [5, Theorem 1] and [13].

In Theorem 3.5 (iii), the conditions that $\operatorname{ind}(A) \leqslant 1$ and $\operatorname{ind}(B) \leqslant 1$ are necessary. The next example shows that, if A and B are T-EP with closed range and AB is T-EP, then $\mathcal{R}(AB) = \mathcal{R}(A) \cap \mathcal{R}(B)$ and $\mathcal{R}(B^*A^*) = \mathcal{R}(B^*) \cap \mathcal{R}(A^*)$ do not always hold.

EXAMPLE. Let $A = B = T = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{pmatrix} \in B(\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H})$. Then A, B and AB and T-EP. However,

$$\mathcal{R}(AB) \neq \mathcal{R}(A) \cap \mathcal{R}(B), \quad \mathcal{R}(B^*A^*) \neq \mathcal{R}(B^*) \cap \mathcal{R}(A^*).$$

Note that

$$TA^* = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad AB = T(AB)^* = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to show that T is partial isometry and $\mathcal{R}(A) = \mathcal{R}(B) = \mathcal{R}(TA^*)$. Hence, A and B are T-EP. The relations $\mathcal{R}(AB) = \mathcal{R}(T(AB)^*)$ and $\mathcal{R}((AB)^*) \subseteq \mathcal{R}(T^*)$ imply that AB is T-EP. But it is obvious that $\mathcal{R}(A) \cap \mathcal{R}(B) \neq \mathcal{R}(AB)$ and $\mathcal{R}(B^*A^*) \neq \mathcal{R}(B^*) \cap \mathcal{R}(A^*)$.

Denoted by

$$T := \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad M := \begin{pmatrix} A & 0 \\ C & B \end{pmatrix}, \tag{9}$$

where A and $T_1 \in \mathcal{B}(\mathcal{H}_1, \mathcal{K}_1)$, B and $T_2 \in \mathcal{B}(\mathcal{H}_2, \mathcal{K}_2)$ and $C \in \mathcal{B}(\mathcal{H}_1, \mathcal{K}_2)$ satisfy that $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed, A is T_1-EP and B is T_2-EP, T_1 and T_2 are partial isometries. Then we have the following result.

THEOREM 3.6. *Let M and T be defined by (9). The 2×2 lower triangular operator matrix M is T-EP if and only if $P_{\mathcal{N}(B^*)}CA^* = 0$ and $CP_{\mathcal{N}(A)} = 0$ if and only if $C = BXA$ for some $X \in \mathcal{B}(\mathcal{K}_1, \mathcal{H}_2)$.*

Proof. By Lemma 2.3, M and T, as operators from $\mathcal{R}(A^*) \oplus [\mathcal{R}(T_1^*) \ominus \mathcal{R}(A^*)] \oplus \mathcal{N}(T_1) \oplus \mathcal{R}(B^*) \oplus [\mathcal{R}(T_2^*) \ominus \mathcal{R}(B^*)] \oplus \mathcal{N}(T_2)$ into $\mathcal{R}(A) \oplus [\mathcal{R}(T_1) \ominus \mathcal{R}(A)] \oplus \mathcal{N}(T_1^*) \oplus \mathcal{R}(B) \oplus [\mathcal{R}(T_2) \ominus \mathcal{R}(B)] \oplus \mathcal{N}(T_2^*)$, can be denoted as

$$M = \begin{pmatrix} A_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ C_{11} & C_{12} & C_{13} & B_1 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} U_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & U_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & U_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & U_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where A_1, B_1 is invertible, U_i $(i=1,2,4,5)$ is isometry. One derives that

$$\begin{aligned}\mathcal{R}(M) &= \mathcal{R}(A_1)\oplus\mathcal{R}(B_1)\oplus\mathcal{R}\left(\begin{pmatrix} C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33}\end{pmatrix}\right)\\ &= \mathcal{R}(A)\oplus\mathcal{R}(B)\oplus\mathcal{R}((I-BB^{\dagger})C).\end{aligned}$$

$M = MT^*T$ if and only if

$$\begin{pmatrix} A_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ C_{11} & C_{12} & C_{13} & B_1 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} A_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ C_{11} & C_{12} & 0 & B_1 & 0 & 0 \\ C_{21} & C_{22} & 0 & 0 & 0 & 0 \\ C_{31} & C_{32} & 0 & 0 & 0 & 0 \end{pmatrix} \tag{10}$$

if and only if

$$\begin{pmatrix} C_{13} \\ C_{23} \\ C_{33} \end{pmatrix} = 0. \tag{11}$$

Note that

$$TM^* = \begin{pmatrix} U_1A_1^* & 0 & 0 & U_1C_{11}^* & U_1C_{21}^* & U_1C_{31}^* \\ 0 & 0 & 0 & U_2C_{12}^* & U_2C_{22}^* & U_2C_{32}^* \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & U_4B_1^* & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since A_1 and B_2 are invertible, and U_i $(i=1,2,4,5)$ is isometry, one has

$$\begin{aligned}\mathcal{R}(TM^*) &= \mathcal{R}(U_1A_1^*)\oplus\mathcal{R}(\begin{pmatrix} U_2C_{12}^* & U_2C_{22}^* & U_2C_{32}^* \end{pmatrix})\oplus\mathcal{R}(U_4B_1^*)\\ &= \mathcal{R}(A)\oplus\mathcal{R}(\begin{pmatrix} U_2C_{12}^* & U_2C_{22}^* & U_2C_{32}^* \end{pmatrix})\oplus\mathcal{R}(B).\end{aligned}$$

Therefore,

$$\mathcal{R}(M)=\mathcal{R}(TM^*) \Longleftrightarrow \begin{pmatrix} 0 & C_{12} & 0 \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} = 0. \tag{12}$$

Hence,

$$M \text{ is } T\text{-EP} \Longleftrightarrow (11) \text{ and } (12) \text{ hold} \Longleftrightarrow \begin{pmatrix} 0 & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} = 0.$$

Applying the representations of A, B and C in (10), one has $C_{i,j}=0$, $1\leqslant i,j\leqslant 3$ and $(i,j)\neq(1,1) \Longleftrightarrow P_{\mathcal{N}(B^*)}CA^*=0$ and $CP_{\mathcal{N}(A)}=0 \Longleftrightarrow C=BXA$ for some $X\in\mathcal{B}(\mathcal{K}_1,\mathcal{H}_2)$. □

In [10, Lemma 1], R. E. Hartwig and I. J. Katz show that $\begin{pmatrix} A & 0 \\ C & B \end{pmatrix}$ is EP if and only if A and B are EP with closed range and $\mathcal{R}(CA^*)\subseteq\mathcal{R}(B)$ and $CP_{\mathcal{N}(A)}=0$, i.e., $C=BXA$ for some X. Theorem 3.6 generalizes this result to the T-EP case.

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