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THE ESSENTIAL SPECTRUM EQUALITIES OF 2×2 UNBOUNDED UPPER TRIANGULAR OPERATOR MATRICES

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Abstract. Based on the space decomposition theory, the conditions for the essential spectrum equalities

$$\sigma_*(\mathscr{T})=\sigma_*(A)\cup\sigma_*(D),\ (\sigma_*=\sigma_{\{e1,\ e2,\ e3,\ e4,\ e5,\ e6\}}),$$

for the diagonally dominant unbounded upper triangular block operator matrix $\mathscr{T}=\begin{pmatrix}A & B\\ 0 & D\end{pmatrix}$ are given, where the sets $\sigma_{e1}(\cdot)$ and $\sigma_{e2}(\cdot)$ denote the Gustafson and Weidmann essential spectrums, $\sigma_{e3}(\cdot)$ denotes Wolf essential spectrum, $\sigma_{e4}(\cdot)$ denotes the Schechter essential spectrum, $\sigma_{e5}(\cdot)$ and $\sigma_{e6}(\cdot)$ denote the essential approximation point spectrum and the essential defect spectrum, respectively.

1. Introduction

In the theoretical study of linear operators in Hilbert space, the block operator matrix plays a crucial role, and widely appears in the study of many practical problems, such as the problem of solving partial differential equations, fluid mechanics, elasticity theory and quantum mechanics.

Consider the basic partial differential equations in [7]

$$\begin{cases}\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+f_x=0,\\ \frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{xy}}{\partial x}+f_y=0,\\ (\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})(\sigma_x+\sigma_y)+(1+v)(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y})=0,\end{cases}\tag{1}$$

where Ω be a striped region satisfying $-h\leqslant x\leqslant h$ in the direction of x-axis. Introducing state functions

$$p=\frac{\partial\sigma_x}{\partial y}+\frac{\partial\sigma_y}{\partial y},\ q=\frac{\partial\sigma_x}{\partial x}+\frac{\partial\sigma_y}{\partial x}.$$

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Then we have the equivalent form of (1)

$$\frac{\partial}{\partial y}\begin{pmatrix}\tau_{xy}\\ \sigma_y\\ p\\ q\end{pmatrix}=\begin{pmatrix}0 & \frac{\partial}{\partial x} & 0 & -1\\ -\frac{\partial}{\partial x} & 0 & 0 & 0\\ 0 & 0 & 0 & -\frac{\partial}{\partial x}\\ 0 & 0 & \frac{\partial}{\partial x} & 0\end{pmatrix}\begin{pmatrix}\tau_{xy}\\ \sigma_y\\ p\\ q\end{pmatrix}+\begin{pmatrix}-f_x\\ -f_y\\ (1+v)(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y})\\ 0\end{pmatrix}. \tag{2}$$

Let $X=L^2(-h,h)\times L^2(-h,h)$, $A=\begin{pmatrix}0 & \frac{d}{dx}\\ -\frac{d}{dx} & 0\end{pmatrix}$, $B=\begin{pmatrix}0 & -1\\ 0 & 0\end{pmatrix}$, $D=\begin{pmatrix}0 & -\frac{d}{dx}\\ \frac{d}{dx} & 0\end{pmatrix}$.

Then the corresponding block operator matrix of (2) is

$$\mathscr{T}=\begin{pmatrix}A & B\\ 0 & D\end{pmatrix}: D(A)\times D(D)\subset X\times X\to X\times X,$$

and it is a form of upper triangular operator matrix.

Therefore, it is very important to study the properties of the upper triangular block operator matrix (see [10, 11, 15]). In [17], the sufficient and necessary conditions for the spectral equality of the diagonally dominant upper triangular operator matrix were given:

LEMMA 1. *Let* $\mathscr{T}=\begin{pmatrix}A & B\\ 0 & D\end{pmatrix}: D(A)\times D(D)\to X\times X$ *be a densely defined upper triangular operator matrix, where* A, D *are densely defined closed and* B *is closable. Then* $\sigma(\mathscr{T})=\sigma(A)\cup\sigma(D)$ *if and only if* $(\sigma_{r,1}(A)\cap\sigma_{p,1}(D))\cup(\rho(A)\cap\sigma_{p,1}(D))=\emptyset$ *or* $\lambda\in(\sigma_{r,1}(A)\cap\sigma_{p,1}(D))\cup(\rho(A)\cap\sigma_{p,1}(D))$ *satisfies one of the following:*

(i) $N(B)\cap N(D-\lambda I)\neq\emptyset$;

(ii) $BN(D-\lambda I)\cap R(A-\lambda I)\neq\{0\}$;

(iii) $BN(D-\lambda I)+R(A-\lambda I)\neq X$.

Moreover, as an important part of spectral theory, the essential spectrum of the block operator matrices has also received extensive attention (see [1, 2, 3]).

In this paper, we will continue the investigation of the essential spectrums of the diagonally dominant upper triangular block operator matrix, and several types of the essential spectrum equalities are given.

Let X, Y be the infinite dimensional complex *Hilbert* spaces (see [5]). $\mathscr{C}(X,Y)$ is denoted as the set of all densely defined closed linear operators from X to Y, where $\mathscr{C}(X,X)$ is written as $\mathscr{C}(X)$. $\mathscr{B}(X,Y)$ and $\mathscr{K}(X,Y)$ denote the subset of all bounded linear operators from X to Y and the subset of all compact operators from X to Y (see [9]), respectively. When $T\in\mathscr{C}(X,Y)$, the symbols $D(T)$, $N(T)$, $R(T)$, T^*, $\rho(T)$, $\sigma(T)$, $n(T)$ and $d(T)$ denote the domain, the null space, the range, the adjoint operator of T, the resolvent set, the spectra, the dimension of $N(T)$, the dimension of the orthogonal complement of $R(T)$, respectively.

The set of all upper *semi-Fredholm* operators (see [6]) are defined as

$$\Phi_+(X):=\{T\in\mathscr{C}(X):\alpha(T)<\infty, R(T)\text{ is closed in X}\}.$$

The set of all lower *semi-Fredholm* operators (see [6]) are defined as

$$\Phi_-(X) := \{T \in \mathscr{C}(X) : \beta(T) < \infty, R(T) \text{ is closed in X}\}.$$

The sets of all *Fredholm* operators on X are defined as

$$\Phi(X) := \Phi_+(X) \cap \Phi_-(X).$$

$i(T) := \alpha(T) - \beta(T)$ denotes the index of linear operator. T is called *Weyl* operator (see [12]), if T is a *Fredholm* operator with $i(T) = 0$.

2. Preliminaries

DEFINITION 1. For $T \in \mathscr{C}(X)$, the following types spectrums can be defined:

$$\begin{aligned}
\sigma_{e1}(T) &= \{\lambda \in C : \lambda - T \notin \Phi_+(X)\}, \\
\sigma_{e2}(T) &= \{\lambda \in C : \lambda - T \notin \Phi_-(X)\}, \\
\sigma_{e3}(T) &= \{\lambda \in C : \lambda - T \notin \Phi(X)\}, \\
\sigma_{e4}(T) &= \{\lambda \in C : \lambda - T \text{ is not a Weyl operator}\}, \\
\sigma_{e5}(T) &= \bigcap_{K \in \mathscr{K}(X)} \sigma_{ap}(T+K), \\
\sigma_{e6}(T) &= \bigcap_{K \in \mathscr{K}(X)} \sigma_{\delta}(T+K).
\end{aligned}$$

where

$$\sigma_{ap}(T) = \{\lambda \in C : \inf_{x \in D(T), \|x\|=1} \|(\lambda - T)x\| = 0\},$$

$$\sigma_{\delta}(T) = \{\lambda \in C : \lambda - T \text{ is not surjective}\}.$$

LEMMA 2. (see [17]) *Let* $\mathscr{T} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : D(A) \times D(D) \to X \times X$ *be diagonally dominant and* B *be closable. Then* $\mathscr{T}$ *is closable (closed, respectively) if and only if* A, D *are closable (closed, respectively).*

LEMMA 3. (see [2,4]) *If* $A \in \Phi(X,Y)$, *then there exist* $A_0 \in \mathscr{B}(X,Y)$, $F_1 \in \mathscr{B}(X)$ *and* $F_2 \in \mathscr{B}(Y)$ *such that*

$$A_0 A = I - F_1 \quad on \quad D(A);$$

$$A A_0 = I - F_2 \quad on \quad Y.$$

LEMMA 4. (see [1,3]) *Let* $A \in \mathscr{C}(X,Y)$. *Suppose there exist operators* A_1, $A_2 \in \mathscr{B}(X,Y)$, $K_1 \in \mathscr{K}(X)$, $K_2 \in \mathscr{K}(Y)$ *such that*

$$A_1 A = I - K_1 \quad on \quad D(A);$$

$$A A_2 = I - K_2 \quad on \quad Y.$$

Then $A \in \Phi(X,Y)$.

LEMMA 5. (see [8]) *Let* $T \in \mathscr{C}(X,\ Y)$ *and let* $J:\ X \to Y$ *be a linear operator.*
(i) If $T \in \Phi(X,\ Y)$ *and* $J \in \mathscr{K}(X,\ Y)$*, then* $T+J \in \Phi(X,\ Y)$ *and* $i(T+J) = i(T)$*;*
(ii) If $T \in \Phi_+(X,\ Y)$ *and* $J \in \mathscr{K}(X,\ Y)$*, then* $T+J \in \Phi_+(X,\ Y)$ *and* $i(T+J) = i(T)$;
(iii) If $T \in \Phi_-(X,\ Y)$ *and* $J \in \mathscr{K}(X,\ Y)$*, then* $T+J \in \Phi_-(X,\ Y)$ *and* $i(T+J) = i(T)$.

LEMMA 6. (see [3, 14]) *Let* $\mathscr{T} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : D(A) \times D(D) \subset X \times X \to X \times X$ *be a densely defined upper triangular operator matrix, where* $A,\ D$ *are closed operators, and* B *is a closable operator. Then*
(i) $n(A) \leqslant n(\mathscr{T}) \leqslant n(A) + n(D)$;
(ii) $d(D) \leqslant d(\mathscr{T}) \leqslant d(A) + d(D)$.

LEMMA 7. (see [15]) *Let* $T, S \in \mathscr{C}(X,\ Y)$. *Then* $T^{-1}S$ *is bounded if and only if* $D(T^*) \subset D(S^*)$.

LEMMA 8. (see [8]) *Let* $T \in \mathscr{C}(X)$. *Then*
(i) $\lambda \notin \sigma_{e5}(T)$ *if and only if* $\lambda I - T \in \Phi_+(X)$ *and* $i(\lambda I - T) \leqslant 0$;
(ii) $\lambda \notin \sigma_{e6}(T)$ *if and only if* $\lambda I - T \in \Phi_-(X)$ *and* $i(\lambda I - T) \geqslant 0$.

LEMMA 9. (see [4]) *Let* $\mathscr{T} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : D(A) \times D(D) \subset X \times X \to X \times X$*, where* $A,\ D$ *are closed operators,* B *is a closable operator. Then*
(i) If $A,\ D \in \Phi_+(X)$*, then* $\mathscr{T} \in \Phi_+(X)$;
(ii) If $A,\ D \in \Phi_-(X)$*, then* $\mathscr{T} \in \Phi_-(X)$.

REMARK 1. Obviously, Lemma 9 is not a necessary and sufficient relation. For example, define linear operators $S_r,\ S_l$ in *Hilbert* space $X = l^2[1,\ \infty]$

$$S_r x = (0,\ x_1,\ x_2,\ x_3,\ \cdots),$$
$$S_l x = (x_2,\ x_3,\ x_4,\ \cdots),$$

where $x = (x_1,\ x_2,\ x_3,\ \cdots) \in X$. Let

$$\mathscr{T} = \begin{pmatrix} S_r & I - S_r S_l \\ 0 & S_l \end{pmatrix}.$$

It is easy to prove that $\mathscr{T}$ is a left *semi-Fredholm* operator, but S_l is not a left *semi-Fredholm* operator.

3. The essential spectrum equalities

THEOREM 1. *Let* $\mathscr{T} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : D(A) \times D(D) \subset X \times X \to X \times X$, *where* A, D *are closed operators,* B *is a closable operator,* $D(A^*) \subset D(B^*)$, $P_{R(\lambda I - A)^\perp} B|_{D(D)}$ *is compact, where* $\lambda \notin \sigma_{e1}(A)$ *and* $P_{R(\lambda I - A)^\perp} : X \to R(\lambda I - A)^\perp$ *is a orthogonal projection. Then*

$$\sigma_{e1}(\mathscr{T}) = \sigma_{e1}(A) \cup \sigma_{e1}(D).$$

Proof. By Lemma 9, the proof of $\sigma_{e1}(\mathscr{T}) \subseteq \sigma_{e1}(A) \cup \sigma_{e1}(D)$ is obvious.

Next, we will show that $\sigma_{e1}(\mathscr{T}) \supseteq \sigma_{e1}(A) \cup \sigma_{e1}(D)$. Let $\lambda \notin \sigma_{e1}(\mathscr{T})$. By Lemma 3, there exists $\mathscr{M} \in \mathscr{B}(X \times X)$, $\mathscr{N} \in \mathscr{K}(X \times X)$ such that

$$\mathscr{M}(\lambda I - \mathscr{T}) = \mathscr{I} - \mathscr{N}.$$

Let

$$\mathscr{M} = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}, \quad \mathscr{N} = \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix}.$$

Then, $M_1(\lambda I - A) = I - N_1$ on $D(A)$, where $M_1 \in \mathscr{B}(X)$, and $N_1 \in \mathscr{K}(X)$, so $\lambda \notin \sigma_{e1}(A)$ by Lemma 4. And $\lambda I - \mathscr{T}$ can be decomposed into

$$\lambda I - \mathscr{T} = \begin{pmatrix} (\lambda I - A)_1 & 0 & B_1 \\ 0 & 0 & B_2 \\ 0 & 0 & \lambda I - D \end{pmatrix} : \begin{pmatrix} N(\lambda I - A)^\perp \cap D(\lambda I - A) \\ N(\lambda I - A) \\ D(\lambda I - D) \end{pmatrix} \to \begin{pmatrix} R(\lambda I - A) \\ R(\lambda I - A)^\perp \\ X \end{pmatrix}.$$

Obviously, $(\lambda I - A)_1 = P_{R(\lambda I - A)}(\lambda I - A)|_{N(\lambda I - A)^\perp \cap D(\lambda I - A)}$ is invertible. Let

$$Q_1 = \begin{pmatrix} I & 0 & \overline{-(\lambda I - A)_1^{-1} B_1} \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} : \begin{pmatrix} N(\lambda I - A)^\perp \\ N(\lambda I - A) \\ X \end{pmatrix} \to \begin{pmatrix} N(\lambda I - A)^\perp \\ N(\lambda I - A) \\ X \end{pmatrix}.$$

Then,

$$\begin{aligned} (\lambda I - \mathscr{T})Q_1 &= \begin{pmatrix} (\lambda I - A)_1 & 0 & 0 \\ 0 & 0 & B_2 \\ 0 & 0 & \lambda I - D \end{pmatrix} \\ &= \begin{pmatrix} (\lambda I - A)_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda I - D \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & B_2 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{3}$$

By given condition, $B_2 = P_{R(\lambda I - A)^\perp} B|_{D(D)}$ is compact. So $\lambda \notin \sigma_{e1}(D)$, i.e. $\lambda \notin \sigma_{e1}(A) \cup \sigma_{e1}(D)$. □

However, considering whether $R(\lambda I - A)$ is dense when $\lambda \notin \sigma_{e1}(A)$, we have the following remark:

REMARK 2. Let $\mathscr{T}=\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}: D(A)\times D(D)\subset X\times X\to X\times X$, where A, D is closed, B is a closable operator, $D(A^*)\subset D(B^*)$ and $\sigma_{p3}(A)=\sigma_{r1}(A)=\emptyset$. Then

$$\sigma_{e1}(\mathscr{T})=\sigma_{e1}(A)\cup\sigma_{e1}(D),$$

where $\sigma_{p3}(T)=\{\lambda\in C:\lambda-T \text{ is not injective}, \overline{R(\lambda I-T)}\neq X,\ R(\lambda I-T) \text{ is not closed in } X\}$, $\sigma_{r1}(T)=\{\lambda\in C:\lambda-T \text{ is injective}, \overline{R(\lambda I-T)}\neq X,\ R(\lambda I-T) \text{ is closed in } X\}$.

Proof. For $\lambda\notin\sigma_{e1}(A)$ and considering the condition in Theorem 1, we have $R(\lambda I-A)$ is closed. i.e.

$$\lambda\in\rho(A)\cup\sigma_{p1}(A)\cup\sigma_{p3}(A)\cup\sigma_{r1}(A).$$

By given condition $\sigma_{p3}(A)=\sigma_{r1}(A)=\emptyset$, we have

$$R(\lambda I-A)^{\perp}=\{0\}, \text{ i.e. } P_{R(\lambda I-A)^{\perp}}B|_{D(D)}=0.$$

So $\sigma_{e1}(\mathscr{T})=\sigma_{e1}(A)\cup\sigma_{e1}(D)$. □

THEOREM 2. *Let* $\mathscr{T}=\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}: D(A)\times D(D)\subset X\times X\to X\times X$, *where* A, D *are closed operators,* B *is a closable operator,* $D(A^*)\subset D(B^*)$ *and* $B|_{N(\lambda I-D)}$ *is compact operator on* X, *where* $\lambda\notin\sigma_{e2}(D)$. *Then*

$$\sigma_{e2}(\mathscr{T})=\sigma_{e2}(A)\cup\sigma_{e2}(D).$$

Proof. By Lemma 9, the proof of $\sigma_{e2}(\mathscr{T})\subseteq\sigma_{e2}(A)\cup\sigma_{e2}(D)$ is obvious.

It suffices to show that $\sigma_{e2}(\mathscr{T})\supseteq\sigma_{e2}(A)\cup\sigma_{e2}(D)$. Let $\lambda\notin\sigma_{e2}(\mathscr{T})$. Then $\lambda\notin\sigma_{e2}(D)$ by Lemma 3 and Lemma 4. We can obtain

$$\lambda I-\mathscr{T}=\begin{pmatrix} \lambda I-A & B_1 & B_2 \\ 0 & (\lambda I-D)_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}:\begin{pmatrix} D(\lambda I-A) \\ N(\lambda I-D)^{\perp}\cap D(\lambda I-D) \\ N(\lambda I-D) \end{pmatrix}\to\begin{pmatrix} X \\ R(\lambda I-D) \\ R(\lambda I-D)^{\perp} \end{pmatrix}.$$

Obviously, $(\lambda I-D)_1=P_{R(\lambda I-D)}(\lambda I-D)|_{N(\lambda I-D)^{\perp}\cap D(\lambda I-D)}$ are invertible. There exists Q_2,

$$\begin{aligned} Q_2(\lambda I-\mathscr{T}) &= \begin{pmatrix} \lambda I-A & 0 & B_2 \\ 0 & (\lambda I-D)_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \lambda I-A & 0 & 0 \\ 0 & (\lambda I-D)_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 & B_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{4}$$

By given condition, B_2 is compact. Similar to the proof process of Theorem 1, $\lambda\notin\sigma_{e2}(A)$, i.e. $\lambda\notin\sigma_{e2}(A)\cup\sigma_{e2}(D)$. □

Same as remark 2, considering whether $\lambda I-D$ is injective when $\lambda\notin\sigma_{e2}(D)$, we have

REMARK 3. Let $\mathscr{T}=\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}: D(A)\times D(D)\subset X\times X\to X\times X$, where A, D is closed, B is a closable operator, $D(A^*)\subset D(B^*)$, $\sigma_{p1}(D)=\sigma_{p3}(D)=\emptyset$. Then

$$\sigma_{e2}(\mathscr{T})=\sigma_{e2}(A)\cup\sigma_{e2}(D),$$

where $\sigma_{p1}(T)=\{\lambda\in C:\lambda-T$ is not injective and $R(\lambda I-T)=X\}$.

THEOREM 3. *Let* $\mathscr{T}=\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}: D(A)\times D(D)\subset X\times X\to X\times X$*, where* A, D *are closed operators,* B *is a closable operator,* $D(A^*)\subset D(B^*)$ *and* $P_{R(\lambda I-A)^{\perp}}B|_{N(\lambda I-D)}$ *is compact, where* $\lambda\notin\sigma_{e1}(A)\cup\sigma_{e2}(D)$. *Then*

$$\sigma_{e3}(\mathscr{T})=\sigma_{e3}(A)\cup\sigma_{e3}(D).$$

Proof. By Lemma 9, the proof of $\sigma_{e3}(\mathscr{T})\subseteq\sigma_{e3}(A)\cup\sigma_{e3}(D)$ is obvious.

It suffices to show that $\sigma_{e3}(\mathscr{T})\supseteq\sigma_{e3}(A)\cup\sigma_{e3}(D)$.

Let $\lambda\notin\sigma_{e3}(\mathscr{T})$. Then $\lambda\notin\sigma_{e1}(A)\cup\sigma_{e2}(D)$ by Lemma 3 and Lemma 4. And $\lambda I-\mathscr{T}$ can be decomposed into

$$\lambda I-\mathscr{T}=\begin{pmatrix} D_\lambda & 0 & B_1 & B_2 \\ 0 & 0 & B_3 & B_4 \\ 0 & 0 & D_\lambda & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}:\begin{pmatrix} N(\lambda I-A)^{\perp}\cap D(\lambda I-A) \\ N(\lambda I-A) \\ N(\lambda I-D)^{\perp}\cap D(\lambda I-D) \\ N(\lambda I-D) \end{pmatrix}$$

$$\to\begin{pmatrix} R(\lambda I-A) \\ R(\lambda I-A)^{\perp} \\ R(\lambda I-D) \\ R(\lambda I-D)^{\perp} \end{pmatrix}.$$

Obviously, $A_\lambda=P_{R(\lambda I-A)}(\lambda I-A)|_{N(\lambda I-A)^{\perp}\cap D(\lambda I-A)}$ and $D_\lambda=P_{R(\lambda I-D)}(\lambda I-D)|_{N(\lambda I-D)^{\perp}\cap D(\lambda I-D)}$ are invertible. Let

$$Q_3=\begin{pmatrix} I & 0 & \overline{A_\lambda B_1} & \overline{-A_\lambda^{-1}B_2} \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}:\begin{pmatrix} N(\lambda I-A)^{\perp} \\ N(\lambda I-A) \\ N(\lambda I-D)^{\perp} \\ N(\lambda I-D) \end{pmatrix}\to\begin{pmatrix} N(\lambda I-A)^{\perp} \\ N(\lambda I-A) \\ N(\lambda I-D)^{\perp} \\ N(\lambda I-D) \end{pmatrix}.$$

$$Q_4=\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & -B_3D_\lambda^{-1} & 0 \\ 0 & 0 & 0 & I \end{pmatrix}:\begin{pmatrix} R(\lambda I-A) \\ R(\lambda I-A)^{\perp} \\ R(\lambda I-D) \\ R(\lambda I-D)^{\perp} \end{pmatrix}\to\begin{pmatrix} R(\lambda I-A) \\ R(\lambda I-A)^{\perp} \\ R(\lambda I-D) \\ R(\lambda I-D)^{\perp} \end{pmatrix}.$$

Then

$$Q_4(\lambda I-\mathscr{T})Q_3=\begin{pmatrix} A_\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & B_4 \\ 0 & 0 & D_\lambda & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}=\begin{pmatrix} A_\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & D_\lambda & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_4 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

By given condition, $B_4 = P_{R(\lambda I-A)^{\perp}}B|_{N(\lambda I-D)}$ is compact. So $\lambda \notin \sigma_{e2}(A)\cup\sigma_{e1}(D)$, i.e. $\lambda \notin \sigma_{e3}(A)\cup\sigma_{e3}(D)$. □

REMARK 4. Let $\mathscr{T} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : D(A)\times D(D) \subset X\times X \to X\times X$, where A, D is closed, B is a closable operator and $D(A^*)\subset D(B^*)$. If one of the following conditions is satisfied:

(i) $\sigma_{p3}(A) = \sigma_{r1}(A) = \emptyset$;

(ii) $\sigma_{p1}(D) = \sigma_{p3}(D) = \emptyset$.

Then,

$$\sigma_{e3}(\mathscr{T}) = \sigma_{e3}(A)\cup\sigma_{e3}(D).$$

REMARK 5. Let $\mathscr{T} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : D(A)\times D(D) \subset X\times X \to X\times X$, where A, D are closed operators, B is a closable operator, $D(A^*)\subset D(B^*)$.

(i) If $P_{R(A)^{\perp}}B|_{D(D)}$ is compact, then $\mathscr{T}\in\Phi_+(X)$ if and only if $A\in\Phi_+(X)$ and $D\in\Phi_+(X)$.

(ii) If $B|_{N(D)}$ is compact on X, then $\mathscr{T}\in\Phi_-(X)$ if and only if $A\in\Phi_-(X)$ and $D\in\Phi_-(X)$.

(iii) If $P_{R(A)^{\perp}}B|_{N(D)}$ is compact, then $\mathscr{T}\in\Phi(X)$ if and only if $A\in\Phi(X)$ and $D\in\Phi(X)$.

THEOREM 4. *Let* $\mathscr{T} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : D(A)\times D(D) \subset X\times X \to X\times X$, *where* A, D *are closed operators,* B *is a closable operator,* $D(A^*)\subset D(B^*)$, $P_{R(\lambda I-A)^{\perp}}B|_{N(\lambda I-D)}$ *is compact and* $i(\lambda I-A)i(\lambda I-D)\geqslant 0$, *where* $\lambda\notin\sigma_{e1}(A)\cup\sigma_{e2}(D)$. *Then*

$$\sigma_{e4}(\mathscr{T}) = \sigma_{e4}(A)\cup\sigma_{e4}(D).$$

Proof. Let $\lambda\notin\sigma_{e4}(A)\cup\sigma_{e4}(D)$. Then by Lemma 5 and Theorem 3, we have $\lambda I-\mathscr{T}\in\Phi(X)$ and

$$i(\lambda I-\mathscr{T}) = i(\lambda I-A)+i(\lambda I-D) = 0.$$

Therefore, $\lambda\notin\sigma_{e4}(\mathscr{T})$.

On the other hand, let $\lambda\notin\sigma_{e4}(\mathscr{T})$, by formula (5) of Theorem 3, $\lambda I-A$ and $\lambda I-D$ are *Fredholm* operators, and

$$i(\lambda I-A)+i(\lambda I-D) = i(\lambda I-\mathscr{T}) = 0.$$

By given condition $i(\lambda I-A)i(\lambda I-D)\geqslant 0$, it is easy to prove

$$i(\lambda I-A) = i(\lambda I-D) = 0.$$

i.e. $\lambda\notin\sigma_{e4}(A)\cup\sigma_{e4}(D)$. □

REMARK 6. Let $\mathscr{T} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : D(A) \times D(D) \subset X \times X \to X \times X$, where D is closed, B is a closable operator, $D(A^*) \subset D(B^*)$. If one of the following conditions is satisfied:

(i) $\sigma_{p1}(A) = \sigma_{p3}(A) = \sigma_{r1}(A) = \emptyset$;
(ii) $\sigma_{p1}(D) = \sigma_{p3}(D) = \sigma_{r1}(D) = \emptyset$.
Then,

$$\sigma_{e4}(\mathscr{T}) = \sigma_{e4}(A) \cup \sigma_{e4}(D).$$

Proof. We only prove the case of condition (i). Let $\lambda \notin \sigma_{e1}(A) \cup \sigma_{e2}(D)$, considering the condition in Theorem 4, $R(\lambda I - A)$, $R(\lambda I - D)$ are closed. i.e.

$$\lambda \in \rho(A) \cup \sigma_{p1}(A) \cup \sigma_{p3}(A) \cup \sigma_{r1}(A) \cup \rho(D) \cup \sigma_{p1}(D) \cup \sigma_{p3}(D) \cup \sigma_{r1}(D).$$

Then, we have

$$R(\lambda I - A)^{\perp} = \{0\},\ N(\lambda I - A) = \{0\},$$

i.e.

$$P_{R(\lambda I - A)^{\perp}} B|_{D(D)} = 0,\ i(\lambda I - A) = 0.$$

So $\sigma_{e4}(\mathscr{T}) = \sigma_{e4}(A) \cup \sigma_{e4}(D)$. $\square$

THEOREM 5. *Let* $\mathscr{T} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : D(A) \times D(D) \subset X \times X \to X \times X$, *where* A, D *are closed operators,* B *is a closable operator,* $D(A^*) \subset D(B^*)$, $P_{R(\lambda I - A)^{\perp}} B|_{D(D)}$ *is compact and* $i(\lambda I - A) i(\lambda I - D) \geqslant 0$, *where* $\lambda \notin \sigma_{e1}(A)$. *Then*

$$\sigma_{e5}(\mathscr{T}) = \sigma_{e5}(A) \cup \sigma_{e5}(D).$$

Proof. Let $\lambda \notin \sigma_{e5}(A) \cup \sigma_{e5}(D)$. Then by lemma 8, we have

$$\lambda I - A, \lambda I - D \in \Phi_{+}(X),\ i(\lambda I - A) \leqslant 0,\ i(\lambda I - D) \leqslant 0.$$

And by Lemma 5 and Lemma 9, we can obtain

$$\lambda I - \mathscr{T} \in \Phi_{+}(X),\ i(\lambda I - \mathscr{T}) = i(\lambda I - A) + i(\lambda I - D) \leqslant 0,$$

i.e. $\lambda \notin \sigma_{e5}(\mathscr{T})$.

Conversely, let $\lambda \notin \sigma_{e5}(\mathscr{T})$. By formula (3) of Theorem 1, $\lambda I - A$ and $\lambda I - D$ are upper *semi-Fredholm* operators, and

$$i(\lambda I - A) + i(\lambda I - D) = i(\lambda I - \mathscr{T}) \leqslant 0.$$

By given condition $i(\lambda I - A) i(\lambda I - D) \geqslant 0$, it is easy to prove that

$$i(\lambda I - A) \leqslant 0,\ i(\lambda I - D) \leqslant 0.$$

Therefore, $\lambda \notin \sigma_{e5}(A) \cup \sigma_{e5}(D)$. $\square$

REMARK 7. Let $\mathscr{T}=\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}: D(A)\times D(D)\subset X\times X\to X\times X$, where D is closed, B is a closable operator, $D(A)\subset D(B^*)$ and $\sigma_{p1}(A)=\sigma_{p3}(A)=\sigma_{r1}(A)=\emptyset$. Then

$$\sigma_{e5}(\mathscr{T})=\sigma_{e5}(A)\cup\sigma_{e5}(D),$$

THEOREM 6. *Let* $\mathscr{T}=\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}: D(A)\times D(D)\subset X\times X\to X\times X$, *where* A, D *are closed operators,* B *is a closable operator,* $D(A^*)\subset D(B^*)$, $B|_{N((\lambda I-D)}$ *is compact operator on* X *and* $i(\lambda I-A)i(\lambda I-D)\geqslant 0$, *where* $\lambda\notin\sigma_{e2}(D)$. *Then*

$$\sigma_{e6}(\mathscr{T})=\sigma_{e6}(A)\cup\sigma_{e6}(D).$$

Proof. The proof process of Theorem 6 is similar to Theorem 5. □

REMARK 8. Let $\mathscr{T}=\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}: D(A)\times D(D)\subset X\times X\to X\times X$, where A is closed, B is a closable operator, $D(A^*)\subset D(B^*)$ and $\sigma_{p1}(D)=\sigma_{p3}(D)=\sigma_{r1}(D)=\emptyset$. Then

$$\sigma_{e6}(\mathscr{T})=\sigma_{e6}(A)\cup\sigma_{e6}(D).$$

4. Application

As applications of the main results, we shall characterize various essential spectrums of the upper-triangular infinite-dimensional Hamiltonian operators.

COROLLARY 1. *Let* $\mathscr{H}=\begin{pmatrix} A & B \\ 0 & -A^* \end{pmatrix}: D(A)\times D(A^*)\subset X\times X\to X\times X$ *be an upper triangular infinite dimensional Hamiltonian operator,* $P_{R(A)^\perp}B|_{D(A^*)}$ *is compact. Then* $\mathscr{H}$ *is a Fredholm operator if and only if* A *is a Fredholm operator.*

COROLLARY 2. *Let* $\mathscr{H}=\begin{pmatrix} A & B \\ 0 & -A^* \end{pmatrix}: D(A)\times D(A^*)\subset X\times X\to X\times X$ *be an upper triangular infinite dimensional Hamiltonian operator,* $P_{R(A)^\perp}B|_{D(A^*)}$ *is compact and* $i(A)i(A^*)\geqslant 0$. *Then* $\mathscr{H}$ *is a Weyl operator if and only if* A *is a Weyl operator.*

Now we will give an example to illustrate the validity of the main results.

EXAMPLE 1. [18] Consider the boundary value problem of the plate bending equation:

$$\begin{cases} D(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})^2 w=0, & 0<x<h,\ 0<y<1; \\ w(x,0)=u(x,1)=0, & 0\leqslant x\leqslant h; \\ \frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=0, & y=0 \text{ or } y=1. \end{cases}$$

Let

$$\theta=\frac{\partial w}{\partial x},\ q=D\left(\frac{\partial^3 w}{\partial x^3}+\frac{\partial^3 w}{\partial xy^2}\right),\ m=-D\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right).$$

Here q is the Lagrangian parameter function, and m is the bending moment. Using the multivariate polynomial with remainder division of the matrix, the Hamilton canonical equation can be obtained

$$\frac{\partial}{\partial x}\begin{pmatrix} w\\ \theta\\ q\\ m\end{pmatrix}=\begin{pmatrix} 0 & I & 0 & 0\\ -\frac{\partial^2}{\partial y^2} & 0 & 0 & -\frac{I}{D}\\ 0 & 0 & 0 & \frac{\partial^2}{\partial y^2}\\ 0 & 0 & -I & 0\end{pmatrix}\begin{pmatrix} w\\ \theta\\ q\\ m\end{pmatrix}.$$

Let $Y=L^2[0,1]\times L^2[0,1]$. Then the corresponding infinite-dimensional Hamiltonian operator is

$$\mathscr{H}=\begin{pmatrix} A & B\\ 0 & -A^*\end{pmatrix}: D(A)\times D(A^*)\subset Y\times Y,$$

where

$$A=\begin{pmatrix} 0 & I\\ -\frac{d^2}{dy^2} & 0\end{pmatrix},\ B=\begin{pmatrix} 0 & 0\\ 0 & -\frac{I}{D}\end{pmatrix},$$

$$D(A)=\left\{\begin{pmatrix} w\\ \theta\end{pmatrix}\in X: w' \text{ absolutely continuous, } w(0)=w(1)=0,\ w',\ w''\in X\right\}.$$

Obviously, $A=\begin{pmatrix} 0 & I\\ -\frac{d^2}{dy^2} & 0\end{pmatrix}$ is invertible, so $P_{R(A)^\perp}B|_{N(A^*)}=0$, i.e. it is compact and $i(A)=i(-A^*)=0$.

By Theorem $1-6$, we have

$$\sigma_{ei}(\mathscr{H})=\sigma_{ei}(A)\cup\sigma_{ei}(-A^*)=\emptyset,\ (i=1,\cdots,6).$$

On the other hand,

$$\sigma(A)=\sigma(-A^*)=\sigma_p(A)=\{k\pi:\ k=\pm 1,\ \pm 2,\ \cdots\}.$$

Thus, $\sigma_{r,1}(A)=\emptyset$, and so

$$\sigma(\mathscr{H})=\{k\pi:\ k=\pm 1,\ \pm 2,\ \cdots\}.$$

Then, $\sigma_{ei}(A)=\sigma_{ei}(-A^*)=\sigma_{ei}(\mathscr{H})=\emptyset,(i=1,\cdots,6)$, and

$$\sigma_{ei}(\mathscr{H})=\sigma_{ei}(A)\cup\sigma_{ei}(-A^*)=\emptyset,\ (i=1,\cdots,6).$$

REFERENCES

[1] ALTANTANG, Q. M. BAI AND D. Y. WU, *Spectra of* 2×2 *Upper Triangular Operator Matrices with Unbounded entries*, Sci Sin Math. **46**, (2016), 157–168.
[2] Q. M. BAI AND ALTANTANG, *Essential, Weyl and Browder spectra of unbounded upper triangular operator matrices*, Linear and Multilinear Algebra. **64**, (2016), 1583–1594.
[3] Q. M. BAI, J. J. HUANG AND ALATANCANG, *Essential Spectrum and Weyl Spectrum Properties of Upper Triangular Closed Operator Matrix*, Journal of Mathematics. **64**, (2021), 761–772.
[4] Q. M. BAI, J. J. HUANG AND ALATANCANG, *Weyl type theorems of* 2×2 *upper triangular operator matrices*, Journal of Mathematical Analysis and Applications. **434**, (2015).
[5] I. GOHBERG, *Classes of Linear Operator Theory*, Advances and Applications, 1990.
[6] K. GUSTAFSON AND J. WEIDMANN, *On essential spectrum*, Journal of Mathematical Analysis and Application. (1969), 121–127.
[7] J. J. HUANG, ALATANCANG AND H. WANG, *An intrinsic expansion method for two-dimensional elastic problems based on stress forms*, Applied Mathematics and Mechanics. **31**, (2010), 992–1000.
[8] A. JERIBI, *Linear Operators and Their Essential Pseudospectra*, Apple Academic Press, 2018.
[9] T. KATO, *Perturbation theory for linear operators*, Springer Science & Business Media, 2013.
[10] R. F. LI, D. Y. WU AND ALATANCANG, *Some general quadratic numerical radius inequalities for the off-diagonal parts of* 2×2 *block operator matrices*, Operators and Matrices. **15**, (2021), 1427–1444.
[11] Y. Y. QIN, Z. P. XIONG AND W. N. ZHOU, *On the Drazin inverse and M-P inverse for sum of matrices*, Operators and Matrices. **15**, (2021), 209–223.
[12] M. SCHECHTER, *On the essential spectrum of an arbitrary operator. I*, Journal of Mathematical Analysis and Applications. **13**, (1966), 205–215.
[13] A. E. TAYLOR, *Theorems on ascent, descent, nullity and defect of linear operators*, Mathematische Annalen. **163**, (1996), 18–49.
[14] A. E. TAYLOR AND D. C. LAY, *Introduction to functional analysis*, second ed., Wiley, 1980.
[15] C. TRETTER, *Spectral theory of block operators matrices and applications*, Imperial College Press, 2008.
[16] M. VLADIMIR, *Spectral Theory of Linear Operators and Spectral Systems in Banach Algebras*, Springer Science & Business Media, 2007.
[17] D. Y. WU, ALATANCANG AND T. Y. TAM, *The Spectral Equality for Upper Triangular Operator Matrices with Unbounded Entries*, Operators and Matrices. **11**, (2017), 505–517.
[18] W. X. ZHONG, *Separation Variable Method and Hamiltonian System*, Computational Structural Mechanics and Its Applications. (1991), 229–240.

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