

## NUMERICAL RADII OF WEIGHTED SHIFT MATRICES WITH PALINDROMIC WEIGHTS USING DETERMINANTAL POLYNOMIALS

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*Abstract.* In this paper, we formulate the determinantal polynomials of weighted shift matrices with palindromic weights

$$\begin{aligned} &(a, br, ar^2, \dots, br^{2n-3}, ar^{2n-2}, c, ar^{2n-2}, br^{2n-3}, \dots, ar^2, br, a), \\ &(a, br, ar^2, \dots, ar^{2n-2}, br^{2n-1}, c, br^{2n-1}, ar^{2n-2}, \dots, ar^2, br, a), \\ &(a, br, ar^2, \dots, br^{2n-3}, ar^{2n-2}, c, c, ar^{2n-2}, br^{2n-3}, \dots, ar^2, br, a) \text{ and} \\ &(a, br, ar^2, \dots, ar^{2n-2}, br^{2n-1}, c, c, br^{2n-1}, ar^{2n-2}, \dots, ar^2, br, a). \end{aligned}$$

Also, we obtain an explicit expression of the numerical radius for each of the weighted shift matrices using these determinantal polynomials. The purpose of this paper is to generalize the results given in [12] and [4].

### 1. Introduction

Let  $M_n(\mathbb{C})$  denote the set of all  $n \times n$  matrices with complex entries. The *numerical range* of  $T \in M_n(\mathbb{C})$  is the subset of the complex plane  $\mathbb{C}$  defined as

$$W(T) = \{ \langle Tx, x \rangle : x \in \mathbb{C}^n, \|x\| = 1 \},$$

and the *numerical radius* of  $T$  is defined as

$$w(T) = \sup\{ |z| : z \in W(T) \}.$$

It is well known that  $W(T)$  is a nonempty, compact and convex subset of  $\mathbb{C}$  (see [8]). Let  $T$  be a weighted shift matrix with weights  $(w_1, \dots, w_{n-1})$  denoted as follows,

$$T(w_1, \dots, w_{n-1}) = \begin{pmatrix} 0 & w_1 & & & \\ & 0 & w_2 & & \\ & & 0 & \ddots & \\ & & & \ddots & w_{n-1} \\ & & & & 0 \end{pmatrix}.$$

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From [5], we know that the numerical range of a weighted shift matrix  $T(w_1, \dots, w_{n-1})$  is a closed circular disc centered at origin and the numerical radius is given by

$$w(T(w_1, \dots, w_{n-1})) = \max \{z \in \mathbb{R} : \det(zI_n - \text{Re}(T(w_1, \dots, w_{n-1}))) = 0\}.$$

The polynomial  $p_n(t) = \det(tI_n - \text{Re}(T(w_1, w_2, \dots, w_{n-1})))$  is known as determinantal polynomial of  $T(w_1, w_2, \dots, w_{n-1})$  and it has the recurrence relation

$$p_n(t) = tp_{n-1}(t) - \frac{1}{4}w_{n-1}^2 p_{n-2}(t).$$

From Lemma 1 of [10],  $p_n(t)$  can be represented as

$$p_n(t) = t^n + \sum_{1 \leq k \leq \lfloor n/2 \rfloor} \left(\frac{-1}{4}\right)^k S_k(w_1, \dots, w_{n-1}) t^{n-2k},$$

where the circularly symmetric function is

$$S_k(w_1, w_2, \dots, w_{n-1}) = \sum w_{i_1}^2 w_{i_2}^2 \dots w_{i_k}^2,$$

the sum being taken over

$$1 \leq i_1 < i_2 < \dots < i_k \leq n-1, \quad i_2 - i_1 \geq 2, i_3 - i_2 \geq 2, \dots, i_k - i_{k-1} \geq 2.$$

But for the sake of simplicity of calculation, we assume

$$\det(tI_n - 2\text{Re}(T(w_1, w_2, \dots, w_{n-1})))$$

as the determinantal polynomial of  $T(w_1, w_2, \dots, w_{n-1})$ .

The numerical range  $W(T)$  of an operator  $T$  defined on an infinite dimensional complex, separable Hilbert space is studied by various mathematicians. For the comprehensive study in this case, one can see [8, 7]. Since a weighted shift matrix or operator  $T$  is unitarily equivalent to its entry-wise modulus operator  $|T|$ , therefore one can consider the weights of the weighted shift matrix or operator to be nonnegative. In this article, we assume that all the weights of the weighted shift operators (or matrices) are positive.

Since the numerical range of a weighted shift operator  $T(w_1, w_2, w_3, \dots)$  defined on  $\ell^2(\mathbb{N})$  is either open or closed circular disc centered at origin (see [15]), therefore calculating the radius of this circular disc gives the numerical radius of a weighted shift operator. In 1976, Ridge [9] has obtained an expression of numerical radius of a weighted shift operator with periodic weights. He has shown that the numerical radius of weighted shift operator with weights  $(a, b, a, b, \dots)$  is  $\frac{a+b}{2}$ . Further, Berger & Stampfli [1], Chien & Sheu [6] and Vandanjav & Undrakh [14] have obtained the numerical radius of weighted shift operators with weights  $(h, 1, 1, \dots)$ ,  $(1, h, 1, 1, \dots)$  and  $(h, k, 1, 1, \dots)$ , respectively. Recently, Chakraborty et al. [2] have shown that the numerical radius of the weighted shift operator  $T$  with weights  $(h, k, a, b, a, b, \dots)$  is given by

$$w(T)^2 = \frac{1}{4} \left[ ab \left( q + \frac{1}{q} \right) + a^2 + b^2 \right],$$

where  $q = \frac{h^2+k^2-a^2-b^2+\sqrt{(a^2+b^2-h^2-k^2)^2-4a^2(b^2-k^2)}}{2ab}$ , provided  $bh^2 + (a + b)k^2 > (a + b)^2b$ .

Undrakh et al. [11] have expressed the numerical radius of a weighted shift operator with weights  $(w_1, w_2, \dots, w_n, 1, \dots)$  by the help of determinantal polynomial of the weighted shift matrix with weights  $(w_1, w_2, \dots, w_n)$ . This is further generalized by Chakraborty et al. [3] for the weighted shift operators with weights

$$(w_1, w_2, \dots, w_{2n-1}, b, a, b, a, b, \dots) \text{ and } (w_1, w_2, \dots, w_{2n}, a, b, a, b, \dots).$$

In 2012, Vandanjav and Undrakh [13] have calculated the determinantal polynomials of weighted shift matrices with weights  $(1, r, r^2, \dots, r^{n-2})$  and  $(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n-1})$ , where  $r > 0$ . Later on, Undrakh [12] has derived the determinantal polynomials of weighted shift matrices with palindromic geometric weights

$$(1, r, r^2, \dots, r^{n-2}, r^{n-1}, r^{n-2}, r^{n-3}, \dots, r^2, r, 1) \text{ and } (1, r, r^2, \dots, r^{n-2}, r^{n-1}, r^{n-1}, r^{n-2}, \dots, r^2, r, 1) \text{ with } r > 0.$$

Recently, Chakraborty et al. [4] have explicitly obtained the determinantal polynomials of weighted shift matrices with palindromic positive weights

$$(a, b, a, b, \dots, a, b, c, b, a, b, a, \dots, b, a), (a, b, a, b, \dots, a, c, a, b, a, \dots, b, a) \\ (a, b, a, b, \dots, a, c, c, a, b, a, \dots, b, a) \text{ and } (a, b, \dots, a, b, c, c, b, a, \dots, b, a).$$

In this paper, we focus on the weighted shift matrices with positive weights (for  $n \geq 1$ ) of the following types given in Table 1.

Table 1: *Weighted shift matrices of Type I and Type II with palindromic weights*

Type-I	(i) $T(a, br, ar^2, \dots, br^{2n-3}, ar^{2n-2}, c, ar^{2n-2}, br^{2n-3}, \dots, ar^2, br, a)$ (ii) $T(a, br, ar^2, \dots, ar^{2n-2}, br^{2n-1}, c, br^{2n-1}, ar^{2n-2}, \dots, ar^2, br, a)$
Type-II	(i) $T(a, br, ar^2, \dots, br^{2n-3}, ar^{2n-2}, c, c, ar^{2n-2}, br^{2n-3}, \dots, ar^2, br, a)$ (ii) $T(a, br, ar^2, \dots, ar^{2n-2}, br^{2n-1}, c, c, br^{2n-1}, ar^{2n-2}, \dots, ar^2, br, a)$

In Section 2, we compute the determinantal polynomials of weighted shift matrices with weights

$$(a, br, ar^2, br^3, \dots, br^{2n-3}, ar^{2n-2}) \text{ and } (a, br, ar^2, br^3, \dots, ar^{2n-2}, br^{2n-1}),$$

where  $a, b, r > 0$ . In Section 3 and 4, we calculate the determinantal polynomials of the above Type-I and Type-II matrices, which is a generalization of the results given in [12,4]. Also using these polynomials, we determine the numerical radii of the matrices.

### 2. Some classes of weighted shift matrices

First, we consider the weighted shift matrices

$$A_{2n}(a, b, r) = T(a, br, ar^2, br^3, \dots, br^{2n-3}, ar^{2n-2}) \in M_{2n}(\mathbb{C}) \tag{1}$$

and

$$B_{2n+1}(a, b, r) = T(a, br, ar^2, br^3, \dots, ar^{2n-2}, br^{2n-1}) \in M_{2n+1}(\mathbb{C}), \tag{2}$$

where  $a, b, r > 0$ . Let  $Q_{2n}(z; a, b, r)$  and  $P_{2n+1}(z; a, b, r)$  denote the determinantal polynomials of  $A_{2n}(a, b, r)$  and  $B_{2n+1}(a, b, r)$ , respectively, i.e.,

$$Q_{2n}(z; a, b, r) = \det(zI_{2n} - 2\text{Re}(A_{2n}(a, b, r)))$$

and

$$P_{2n+1}(z; a, b, r) = \det(zI_{2n+1} - 2\text{Re}(B_{2n+1}(a, b, r))).$$

**THEOREM 2.1.** *For  $n \geq 1$ , let  $A_{2n}(a, b, r)$  and  $B_{2n+1}(a, b, r)$  be defined as (1) and (2), respectively. Then*

$$Q_{2n}(z; a, b, r) = zP_{2n-1}(z; a, b, r) - a^2r^{4n-4}Q_{2n-2}(z; a, b, r) \tag{3}$$

and

$$P_{2n+1}(z; a, b, r) = zQ_{2n}(z; a, b, r) - b^2r^{4n-2}P_{2n-1}(z; a, b, r), \tag{4}$$

where  $Q_0(z; a, b, r) = 1$  and  $P_1(z; a, b, r) = z$ .

*Proof.* Here

$$Q_{2n}(z; a, b, r) = \det \begin{pmatrix} z & -a & 0 & 0 & 0 & \dots & \dots & 0 \\ -a & z & -br & 0 & 0 & \dots & \dots & 0 \\ 0 & -br & z & -ar^2 & 0 & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \dots & \vdots \\ \vdots & \dots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \ddots & \ddots & z & -br^{2n-3} & 0 \\ 0 & \dots & \dots & \dots & 0 & -br^{2n-3} & z & -ar^{2n-2} \\ 0 & \dots & \dots & \dots & 0 & 0 & -ar^{2n-2} & z \end{pmatrix}.$$

Expanding the above determinant in terms of the elements of the last row, we get the polynomial (3).

In a similar way, we can prove our required expression as given in (4).  $\square$

**PROPOSITION 2.2.** *For  $n \geq 1$ , let  $A_{2n}(a, b, r)$  and  $B_{2n+1}(a, b, r)$  be defined as (1) and (2), respectively. Then  $Q_{2n}(-z; a, b, r) = Q_{2n}(z; a, b, r)$  and  $P_{2n+1}(-z; a, b, r) = -P_{2n+1}(z; a, b, r)$ .*

*Proof.* We will prove this by using mathematical induction. For  $n = 1$ , we have

$$Q_2(z; a, b, r) = z^2 - a^2 \text{ and } P_3(z; a, b, r) = z(z^2 - b^2r^2 - a^2).$$

Clearly, our statement is true for  $n = 1$ . Assume this be true for  $n = m$ . Now consider,  $n = m + 1$ . Using the expressions (3) and (4), we have

$$\begin{aligned} Q_{2m+2}(-z; a, b, r) &= -zP_{2m+1}(-z; a, b, r) - a^2r^{4m}Q_{2m}(-z; a, b, r) \\ &= Q_{2m+2}(z; a, b, r) \end{aligned}$$

and

$$\begin{aligned} P_{2m+3}(-z; a, b, r) &= -zQ_{2m+2}(-z; a, b, r) - b^2r^{4m+2}P_{2m+1}(-z; a, b, r) \\ &= -P_{2m+3}(z; a, b, r). \end{aligned}$$

Therefore, by mathematical induction on  $n \geq 1$ , we get our required result.  $\square$

REMARK 2.3. For  $n \geq 1$ , the matrices  $A_{2n}(1, 1, r)$  or  $B_{2n+1}(1, 1, r)$  reduce to weighted shift matrices with geometric weights  $(1, r, r^2, \dots, r^{2n-2})$  or  $(1, r, r^2, \dots, r^{2n-1})$ , respectively, discussed in [13]. Here, equations (3) and (4) are same with equation (2.3) of [13].

Now, we proceed to the matrices of the types given in Table 1.

### 3. Weighted shift matrices of Type-I with palindromic weights

Consider the following weighted shift matrices

$$R_{4n}^{(1)}(a, b, c, r) = T(a, br, ar^2, \dots, br^{2n-3}, ar^{2n-2}, c, ar^{2n-2}, br^{2n-3}, \dots, ar^2, br, a) \in M_{4n}(\mathbb{C}) \tag{5}$$

and

$$L_{4n+2}^{(1)}(a, b, c, r) = T(a, br, ar^2, \dots, ar^{2n-2}, br^{2n-1}, c, br^{2n-1}, ar^{2n-2}, \dots, ar^2, br, a) \in M_{4n+2}(\mathbb{C}), \tag{6}$$

where  $a, b, c$  and  $r$  are all positive.

THEOREM 3.1. For  $n \geq 1$ , let  $R_{4n}^{(1)}(a, b, c, r)$  be the weighted shift matrix defined in (5). Then

$$F_{4n}^{(1)}(z; a, b, c, r) = \det \left( zI_{4n} - 2\text{Re}(R_{4n}^{(1)}(a, b, c, r)) \right) = U_n(z; a, b, c, r)V_n(z; a, b, c, r),$$

where  $U_n(z; a, b, c, r) = Q_{2n}(z; a, b, r) - cP_{2n-1}(z; a, b, r)$  and  $V_n(z; a, b, c, r) = Q_{2n}(z; a, b, r) + cP_{2n-1}(z; a, b, r)$ .

*Proof.* The determinantal polynomial of  $R_{4n}^{(1)}(a, b, c, r)$  is given by,

$$\det \left( \begin{array}{ccc|ccc|ccc} & & & & 0 & & & & \\ & & & & \vdots & & & & \\ & & & & -ar^{2n-2} & & & & \\ \hline & & zI_{2n-1} - 2\text{Re}(B_{2n-1}(a, b, r)) & & & & & & \mathbf{0}_{(2n-1) \times 2n} \\ \hline 0 & \dots & 0 & -ar^{2n-2} & z & -c & 0 & \dots & 0 \\ \hline & & & & -c & & & & \\ & & & & 0 & & & & \\ & & & & \vdots & & & & \\ & & \mathbf{0}_{2n \times (2n-1)} & & 0 & & & & C_{2n}(z; a, b, r) \end{array} \right),$$

where

$$C_{2n}(z; a, b, r) = \begin{pmatrix} z & -ar^{2n-2} & 0 & \dots & \dots & 0 \\ -ar^{2n-2} & z & -br^{2n-3} & \ddots & \dots & 0 \\ 0 & -br^{2n-3} & z & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \ddots & -br & z & -a \\ 0 & \dots & \dots & 0 & -a & z \end{pmatrix}. \tag{7}$$

Expanding the above determinant in terms of the elements of  $(2n)$ -th row

$$(0, \dots, 0, -ar^{2n-2}, z, -c, 0, \dots, 0),$$

we get

$$\begin{aligned} F_{4n}^{(1)}(z; a, b, c, r) &= z \det \begin{pmatrix} zI_{2n-1} - 2\text{Re}(B_{2n-1}(a, b, r)) & \mathbf{0}_{(2n-1) \times 2n} \\ \mathbf{0}_{2n \times (2n-1)} & C_{2n}(z; a, b, r) \end{pmatrix} \\ &+ ar^{2n-2} \det \begin{pmatrix} D_{2n-1}(z; a, b, r) & \mathbf{0}_{(2n-1) \times 2n} \\ * & C_{2n}(z; a, b, r) \end{pmatrix} \\ &+ c \det \begin{pmatrix} zI_{2n-1} - 2\text{Re}(B_{2n-1}(a, b, r)) & * \\ \mathbf{0}_{2n \times (2n-1)} & E_{2n}(z; a, b, c, r) \end{pmatrix}, \end{aligned} \tag{8}$$

where

$$D_{2n-1}(z; a, b, r) = \begin{pmatrix} zI_{2n-2} - 2\text{Re}(A_{2n-2}(a, b, r)) & \mathbf{0}_{(2n-2) \times 1} \\ * & -ar^{2n-2} \end{pmatrix}$$

and

$$E_{2n}(z; a, b, c, r) = \begin{pmatrix} -c - ar^{2n-2} & 0 & 0 & 0 & \dots & 0 \\ 0 & z & -br^{2n-3} & 0 & 0 & \dots & 0 \\ 0 & -br^{2n-3} & z & -ar^{2n-4} & 0 & \dots & 0 \\ 0 & 0 & -ar^{2n-4} & \ddots & \ddots & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & -br & z & -a \\ 0 & 0 & \dots & \dots & 0 & -a & z \end{pmatrix}. \tag{9}$$

From (8) we have,

$$F_{4n}^{(1)}(z; a, b, c, r) = zP_{2n-1}(z; a, b, r)Q_{2n}(z; a, b, r) - a^2r^{4n-4}Q_{2n-2}(z; a, b, r)Q_{2n}(z; a, b, r) - c^2P_{2n-1}(z; a, b, r)^2.$$

Using (3) in the above equation, we get our required result.  $\square$

The following remark is an easy consequence of Proposition 2.2.

REMARK 3.2. For  $n \geq 1$ , we have  $U_n(-z; a, b, c, r) = V_n(z; a, b, c, r)$ .

THEOREM 3.3. For  $n \geq 1$ , let  $R_{4n}^{(1)}(a, b, c, r)$  be the weighted shift matrix defined in (5). Then the numerical radius

$$\begin{aligned} w(R_{4n}^{(1)}(a, b, c, r)) &= \frac{1}{2} \max \{ |z| : z \text{ is a root of } U_n(z; a, b, c, r) = 0 \} \\ &= \frac{1}{2} \max \{ |z| : z \text{ is a root of } V_n(z; a, b, c, r) = 0 \}. \end{aligned}$$

*Proof.* It is known that  $2w(R_{4n}^{(1)}(a, b, c, r))$  is the maximum positive root of  $F_{4n}^{(1)}(z; a, b, c, r) = 0$ . So,  $2w(R_{4n}^{(1)}(a, b, c, r))$  is either the maximum positive root of  $U_n(z; a, b, c, r) = 0$  or  $V_n(z; a, b, c, r) = 0$ . Let

$$\alpha_1^{(a,b,c,r)}, \alpha_2^{(a,b,c,r)}, \dots, \alpha_{4n-1}^{(a,b,c,r)}, 2w(R_{4n}^{(1)}(a, b, c, r))$$

are the roots of  $V_n(z; a, b, c, r) = 0$  satisfying

$$\alpha_1^{(a,b,c,r)} \leq \alpha_2^{(a,b,c,r)} \leq \dots \leq \alpha_{4n-1}^{(a,b,c,r)} \leq 2w(R_{4n}^{(1)}(a, b, c, r)).$$

Then from Remark 3.2, the roots of  $U_n(z; a, b, c, r) = 0$  are

$$-\alpha_1^{(a,b,c,r)}, -\alpha_2^{(a,b,c,r)}, \dots, -\alpha_{4n-1}^{(a,b,c,r)}, -2w(R_{4n}^{(1)}(a, b, c, r))$$

where

$$-2w(R_{4n}^{(1)}(a, b, c, r)) \leq -\alpha_{4n-1}^{(a,b,c,r)} \leq \dots \leq -\alpha_1^{(a,b,c,r)} < 2w(R_{4n}^{(1)}(a, b, c, r)).$$

Now, if  $\alpha_1^{(a,b,c,r)} \geq 0$ , then our result holds trivially. If  $\alpha_1^{(a,b,c,r)} < 0$  and  $|\alpha_1^{(a,b,c,r)}| > 2w(R_{4n}^{(1)}(a, b, c, r))$ , then  $-\alpha_1^{(a,b,c,r)} > 2w(R_{4n}^{(1)}(a, b, c, r)) > 0$ .

Since  $-\alpha_1^{(a,b,c,r)}$  is also a root of  $F_{4n}^{(1)}(z; a, b, c, r) = 0$ , therefore it contradicts the fact that  $2w(R_{4n}^{(1)}(a, b, c, r))$  is the maximum positive root of  $F_{4n}^{(1)}(z; a, b, c, r) = 0$ . So we have  $|\alpha_1^{(a,b,c,r)}| \leq 2w(R_{4n}^{(1)}(a, b, c, r))$ . Since  $|\alpha_j^{(a,b,c,r)}| \leq 2w(R_{4n}^{(1)}(a, b, c, r))$  for  $j = 1, 2, \dots, 4n - 1$ , therefore

$$\begin{aligned} w(R_{4n}^{(1)}(a, b, c, r)) &= \frac{1}{2} \max \{ |z| : z \text{ is a root of } V_n(z; a, b, c, r) = 0 \} \\ &= \frac{1}{2} \max \{ |z| : z \text{ is a root of } U_n(z; a, b, c, r) = 0 \}. \end{aligned}$$

Again, if  $2w(R_{4n}^{(1)}(a, b, c, r))$  is the maximum positive root of  $U_n(z; a, b, c, r) = 0$ , then proceeding with similar arguments, we can prove the remaining part of the proof.  $\square$

EXAMPLE 3.4. Let  $R_4^{(1)}(a, b, c, r) = \begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix} \in M_4(\mathbb{C})$  be the weighted shift

matrix defined in (5). Then

$$F_4^{(1)}(z; a, b, c, r) = (a^2 - cz - z^2)(a^2 + cz - z^2).$$

From Theorem 3.3, the numerical radius

$$w(R_4^{(1)}(a, b, c, r)) = \frac{1}{4} (c + \sqrt{4a^2 + c^2}).$$

EXAMPLE 3.5. For  $n = 2$ , take  $a = 1, b = 2, c = 3$  and  $r = 5$ . Then  $R_8^{(1)}(1, 2, 3, 5)$  will be of the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 25 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then Theorem 3.1 gives

$$F_8^{(1)}(z; 1, 2, 3, 5) = (625 + 303z - 726z^2 - 3z^3 + z^4)(625 - 303z - 726z^2 + 3z^3 + z^4).$$

Using Theorem 3.3, we have  $w(R_8^{(1)}(1, 2, 3, 5)) = 14.1361$  (approx.).

COROLLARY 3.6. For  $n \geq 1$ , let  $R_{4n}^{(1)}(a, b, c, r)$  be the weighted shift matrix defined in (5) with  $c = br^{2n-1}$ . Then

$$F_{4n}^{(1)}(z; a, b, c, r) = Q_{2n}(z; a, b, r)^2 - b^2 r^{4n-2} P_{2n-1}(z; a, b, r)^2.$$

*Proof.* Putting  $c = br^{2n-1}$  in Theorem 3.1, we get our required result.  $\square$

THEOREM 3.7. Let  $L_{4n+2}^{(1)}(a, b, c, r)$  be the weighted shift matrix defined in (6). Then

$$G_{4n+2}^{(1)}(z; a, b, c, r) = \det(zI_{4n+2} - 2\text{Re}(L_{4n+2}^{(1)}(a, b, c, r))) = X_n(z; a, b, c, r)Y_n(z; a, b, c, r),$$

where  $X_n(z; a, b, c, r) = P_{2n+1}(z; a, b, r) - cQ_{2n}(z; a, b, r)$  and  $Y_n(z; a, b, c, r) = P_{2n+1}(z; a, b, r) + cQ_{2n}(z; a, b, r)$ .



*Proof.* The determinantal polynomial of  $L_{4n+2}^{(1)}(a, b, c, r)$  is

$$\det \left( \begin{array}{ccc|ccc} & & & 0 & & \\ & & & \vdots & & \\ & & & -c & & \mathbf{0}_{(2n+1) \times 2n} \\ \hline 0 & \dots & 0 & -c & z & -br^{2n-1} & 0 & \dots & 0 \\ \hline & & & -br^{2n-1} & & & & & \\ & & & 0 & & & & & \\ & & & \vdots & & & & & \\ & & & 0 & & & & & \\ & & \mathbf{0}_{2n \times (2n+1)} & & & & & & C_{2n}(z; a, b, r) \end{array} \right)$$

where  $C_{2n}(z; a, b, r)$  is defined in (7).

Expanding the above determinant in terms of the elements of  $(2n + 2)$ -th row

$$(0, \dots, 0, -c, z, -br^{2n-1}, 0, \dots, 0),$$

we have

$$\begin{aligned} G_{4n+2}^{(1)}(z; a, b, c, r) &= z \det \begin{pmatrix} zI_{2n+1} - 2\text{Re}(B_{2n+1}(a, b, r)) & \mathbf{0}_{(2n+1) \times 2n} \\ \mathbf{0}_{2n \times (2n+1)} & C_{2n}(z; a, b, r) \end{pmatrix} \\ &+ c \det \begin{pmatrix} zI_{2n} - 2\text{Re}(A_{2n}(a, b, r)) & \mathbf{0}_{2n \times (2n+1)} \\ * & D_{2n+1}^{(1)}(z; a, b, c, r) \end{pmatrix} \\ &+ br^{2n-1} \begin{pmatrix} zI_{2n+1} - 2\text{Re}(B_{2n+1}(a, b, r)) & * \\ \mathbf{0}_{2n \times (2n+1)} & E_{2n}^{(1)}(z; a, b, r) \end{pmatrix}, \end{aligned} \tag{10}$$

where

$$D_{2n+1}^{(1)}(z; a, b, c, r) = \left( \begin{array}{c|ccc} -c & 0 & \dots & 0 \\ \hline -br^{2n-1} & & & \\ 0 & & & \\ \vdots & & & \\ 0 & & C_{2n}(z; a, b, r) & \end{array} \right)$$

and

$$E_{2n}^{(1)}(z; a, b, r) = \begin{pmatrix} -br^{2n-1} & -ar^{2n-2} & 0 & 0 & 0 & \dots & 0 \\ 0 & z & -br^{2n-3} & 0 & 0 & \dots & 0 \\ 0 & -br^{2n-3} & z & -ar^{2n-4} & 0 & \dots & 0 \\ 0 & 0 & -ar^{2n-4} & z & \ddots & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & -br & z & -a \\ 0 & 0 & \dots & \dots & 0 & -a & z \end{pmatrix}.$$

From (10) we have,

$$G_{4n+2}^{(1)}(z; a, b, c, r) = zP_{2n+1}(z; a, b, r)Q_{2n}(z; a, b, r) - c^2Q_{2n}(z; a, b, r)^2 - b^2r^{4n-2}P_{2n+1}(z; a, b, r)P_{2n-1}(z; a, b, r).$$

Using the expression (4) in the above equation, our result follows.  $\square$

The following remark is a corollary of Proposition 2.2.

REMARK 3.8. For  $n \geq 1$ , we have  $X_n(-z; a, b, c, r) = -Y_n(z; a, b, c, r)$ .

THEOREM 3.9. For  $n \geq 1$ , let  $L_{4n+2}^{(1)}(a, b, c, r)$  be the weighted shift matrix defined in (6). Then

$$w(L_{4n+2}^{(1)}(a, b, c, r)) = \frac{1}{2} \max \{ |z| : z \text{ is a root of } X_n(z; a, b, c, r) = 0 \} \\ = \frac{1}{2} \max \{ |z| : z \text{ is a root of } Y_n(z; a, b, c, r) = 0 \}.$$

*Proof.* Using the above Remark 3.8, the proof is similar to the proof of Theorem 3.3.  $\square$

EXAMPLE 3.10. For  $n = 1$ , take  $a = 1, b = 2, c = 3$  and  $r = 5$ . Then  $L_6^{(1)}(1, 2, 3, 5)$  will be of the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From Theorem 3.7, the determinantal polynomial of  $L_6^{(1)}(1, 2, 3, 5)$  is

$$G_6^{(1)}(z; 1, 2, 3, 5) = (3 - 101z - 3z^2 + z^3)(-3 - 101z + 3z^2 + z^3).$$

Then Theorem 3.9 gives  $w(L_6^{(1)}(1, 2, 3, 5)) = 5.82426$  (approx.).

COROLLARY 3.11. For  $n \geq 1$ , let  $L_{4n+2}^{(1)}(a, b, c, r)$  be the weighted shift matrix defined in (6) with  $c = ar^{2n}$ . Then

$$G_{4n+2}^{(1)}(z; a, b, c, r) = P_{2n+1}(z; a, b, r)^2 - a^2r^{4n}Q_{2n}(z; a, b, r)^2.$$

*Proof.* Putting  $c = ar^{2n}$  in Theorem 3.7, we get our required result.  $\square$

**4. Weighted shift matrices of Type-II with palindromic weights**

Now, consider the following weighted shift matrices,

$$R_{4n+1}^{(2)}(a, b, c, r) = T(a, br, ar^2, \dots, br^{2n-3}, ar^{2n-2}, c, c, ar^{2n-2}, br^{2n-3}, \dots, ar^2, br, a) \in M_{4n+1}(\mathbb{C}) \quad (11)$$

and

$$L_{4n+3}^{(2)}(a, b, c, r) = T(a, br, ar^2, \dots, ar^{2n-2}, br^{2n-1}, c, c, br^{2n-1}, ar^{2n-2}, \dots, ar^2, br, a) \in M_{4n+3}(\mathbb{C}) \quad (12)$$

with  $a, b, c, r > 0$  and  $n \geq 1$ .

**THEOREM 4.1.** For  $n \geq 1$ , let  $R_{4n+1}^{(2)}(a, b, c, r)$  be the weighted shift matrix defined in (11). Then

$$\begin{aligned} F_{4n+1}^{(2)}(z; a, b, c, r) &= \det(zI_{4n+1} - 2\text{Re}(R_{4n+1}^{(2)}(a, b, c, r))) \\ &= Q_{2n}(z; a, b, r)(zQ_{2n}(z; a, b, r) - 2c^2P_{2n-1}(z; a, b, r)) \end{aligned}$$

and  $2w(R_{4n+1}^{(2)}(a, b, c, r))$  is the maximum positive root of

$$zQ_{2n}(z; a, b, r) - 2c^2P_{2n-1}(z; a, b, r) = 0.$$

*Proof.* The determinantal polynomial of  $R_{4n+1}^{(2)}(a, b, c, r)$  is

$$\det \left( \begin{array}{cccc|c|cccc} & & & & 0 & & & & \\ & & & & \vdots & & & & \\ & & & & -c & & & & \mathbf{0}_{2n \times 2n} \\ \hline 0 & \dots & 0 & -c & z & -c & 0 & \dots & 0 \\ \hline & & & & -c & & & & \\ & & & & 0 & & & & \\ & & & & \vdots & & & & \\ & & & & 0 & & & & C_{2n}(z; a, b, r) \end{array} \right)$$

where  $C_{2n}(z; a, b, r)$  is given by (7).

Expanding the above determinant in terms of the  $(2n + 1)$ -th row

$$(0, \dots, 0, -c, z, -c, 0, \dots, 0),$$

we have

$$\begin{aligned}
 F_{4n+1}^{(2)}(z; a, b, c, r) &= z \det \begin{pmatrix} zI_{2n} - 2\text{Re}(A_{2n}(a, b, r)) & \mathbf{0}_{2n \times 2n} \\ \mathbf{0}_{2n \times 2n} & C_{2n}(z; a, b, r) \end{pmatrix} \\
 &\quad + c \det \begin{pmatrix} D_{2n}^{(2)}(z; a, b, c, r) & \mathbf{0}_{2n \times 2n} \\ * & C_{2n}(z; a, b, r) \end{pmatrix} \\
 &\quad + c \det \begin{pmatrix} zI_{2n} - 2\text{Re}(A_{2n}(a, b, r)) & * \\ \mathbf{0}_{2n \times 2n} & E_{2n}(z; a, b, c, r) \end{pmatrix}, \tag{13}
 \end{aligned}$$

where

$$D_{2n}^{(2)}(z; a, b, c, r) = \left( \begin{array}{ccc|c} & & & 0 \\ & zI_{2n-1} - 2\text{Re}(B_{2n-1}(a, b, r)) & & \vdots \\ & & & 0 \\ \hline 0 & \dots & 0 & -ar^{2n-2} \\ & & & -c \end{array} \right)$$

and  $E_{2n}(z; a, b, c, r)$  is defined in (9). From (13) we have,

$$\begin{aligned}
 F_{4n+1}^{(2)}(z; a, b, c, r) &= zQ_{2n}(z; a, b, r)^2 - 2c^2P_{2n-1}(z; a, b, r)Q_{2n}(z; a, b, r) \\
 &= Q_{2n}(z; a, b, r) (zQ_{2n}(z; a, b, r) - 2c^2P_{2n-1}(z; a, b, r)).
 \end{aligned}$$

Again  $A_{2n}(a, b, r)$  is the compression of  $R_{4n+1}^{(2)}(a, b, c, r)$  and therefore,  $w(A_{2n}(a, b, r)) < w(R_{4n+1}^{(2)}(a, b, c, r))$ . Thus the largest positive root of  $Q_{2n}(z; a, b, r) = 0$  is less than the largest positive root of  $zQ_{2n}(z; a, b, r) - 2c^2P_{2n-1}(z; a, b, r) = 0$ . Hence the result follows.  $\square$

EXAMPLE 4.2. Let  $R_5^{(2)}(a, b, c, r) = \begin{pmatrix} 0 & a & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in M_5(\mathbb{C})$  be the weighted shift

matrix defined in (11). Then the determinantal polynomial of  $R_5^{(2)}(a, b, c, r)$  is

$$F_5^{(2)}(z; a, b, c, r) = (z^2 - a^2)(z^3 - a^2z - 2c^2z).$$

From Theorem 4.1, the maximum positive root of  $F_5^{(2)}(z; a, b, c, r) = 0$  is the maximum positive root of  $z^3 - a^2z - 2c^2z = 0$ . Therefore,  $w(R_5^{(2)}(a, b, c, r)) = \frac{\sqrt{a^2+2c^2}}{2}$ .

COROLLARY 4.3. For  $n \geq 1$ , let  $R_{4n+1}^{(2)}(a, b, c, r)$  be the weighted shift matrix defined in (11) with  $c = br^{2n-1}$ . Then

$$F_{4n+1}^{(2)}(z; a, b, c, r) = Q_{2n}(z; a, b, r) (zQ_{2n}(z; a, b, r) - 2b^2r^{4n-2}P_{2n-1}(z; a, b, r))$$

and  $2w(R_{4n+1}^{(2)}(a, b, c, r))$  is the maximum positive root of

$$zQ_{2n}(z; a, b, r) - 2b^2r^{4n-2}P_{2n-1}(z; a, b, r) = 0.$$

*Proof.* Putting  $c = br^{2n-1}$  in Theorem 4.1, we get our required result.  $\square$

**THEOREM 4.4.** For  $n \geq 1$ , let  $L_{4n+3}^{(2)}(a, b, c, r)$  be the weighted shift matrix defined in (12). Then

$$\begin{aligned} G_{4n+3}^{(2)}(z; a, b, c, r) &= \det \left( zI_{4n+3} - 2\text{Re}(L_{4n+3}^{(2)}(a, b, c, r)) \right) \\ &= P_{2n+1}(z; a, b, r) (zP_{2n+1}(z; a, b, r) - 2c^2 Q_{2n}(z; a, b, r)) \end{aligned}$$

and  $2w(L_{4n+3}^{(2)}(a, b, c, r))$  is the maximum positive root of

$$zP_{2n+1}(z; a, b, r) - 2c^2 Q_{2n}(z; a, b, r) = 0.$$

*Proof.* The determinantal polynomial of  $L_{4n+3}^{(2)}(a, b, c, r)$  is

$$\det \left( \begin{array}{ccc|c|ccc} & & & 0 & & & \\ & & & \vdots & & & \\ & zI_{2n+1} - 2\text{Re}(B_{2n+1}(a, b, r)) & & -c & & \mathbf{0}_{(2n+1) \times (2n+1)} & \\ \hline 0 & \cdots & 0 & -c & z & -c & 0 & \cdots & 0 \\ \hline & & \mathbf{0}_{(2n+1) \times (2n+1)} & & -c & & & & \\ & & & & 0 & & & & \\ & & & & \vdots & & C_{2n+1}^{(2)}(z; a, b, r) & & \\ & & & & 0 & & & & \end{array} \right)$$

where

$$C_{2n+1}^{(2)}(z; a, b, r) = \left( \begin{array}{c|ccc} z & -br^{2n-1} & 0 & \cdots & 0 \\ \hline -br^{2n-1} & & & & \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & C_{2n}(z; a, b, r) \end{array} \right)$$

and  $C_{2n}(z; a, b, r)$  is given by (7).

Expanding the above determinant in terms of the elements of the  $(2n + 2)$ -th row

$$(0, \dots, 0, -c, z, -c, 0, \dots, 0),$$

we have

$$\begin{aligned} G_{4n+3}^{(2)}(z; a, b, c, r) &= z \det \left( \begin{array}{cc} zI_{2n+1} - 2\text{Re}(B_{2n+1}(a, b, r)) & \mathbf{0}_{(2n+1) \times (2n+1)} \\ \mathbf{0}_{(2n+1) \times (2n+1)} & C_{2n+1}^{(2)}(z; a, b, r) \end{array} \right) \\ &+ c \det \left( \begin{array}{cc} D'_{2n+1}(z; a, b, c, r) & \mathbf{0}_{(2n+1) \times (2n+1)} \\ * & C_{2n+1}^{(2)}(z; a, b, r) \end{array} \right) \\ &+ c \det \left( \begin{array}{cc} zI_{2n+1} - 2\text{Re}(B_{2n+1}(a, b, r)) & * \\ \mathbf{0}_{(2n+1) \times (2n+1)} & E'_{2n+1}(z; a, b, c, r) \end{array} \right) \quad (14) \end{aligned}$$

where

$$D'_{2n+1}(z; a, b, c, r) = \left( \begin{array}{ccc|c} zI_{2n} - 2\text{Re}(A_{2n}(a, b, r)) & & & 0 \\ & & & \vdots \\ & & & 0 \\ \hline 0 & \dots & 0 & -br^{2n-1} \\ & & & -c \end{array} \right)$$

and

$$E'_{2n+1}(z; a, b, c, r) = \left( \begin{array}{c|cccc} -c & -br^{2n-1} & 0 & \dots & 0 \\ \hline 0 & & & & \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \end{array} \right) C_{2n}(z; a, b, r).$$

From (14) we have,

$$\begin{aligned} G_{4n+3}^{(2)}(z; a, b, c, r) &= zP_{2n+1}(z; a, b, r)^2 - c^2P_{2n+1}(z; a, b, r)Q_{2n}(z; a, b, r) \\ &\quad - c^2P_{2n+1}(z; a, b, r)Q_{2n}(z; a, b, r) \\ &= P_{2n+1}(z; a, b, r) (zP_{2n+1}(z; a, b, r) - 2c^2Q_{2n}(z; a, b, r)). \end{aligned}$$

Since  $B_{2n+1}(a, b, r)$  is the compression of  $L_{4n+3}^{(2)}(a, b, c, r)$ , therefore  $w(B_{2n+1}(a, b, r)) < w(L_{4n+3}^{(2)}(a, b, c, r))$ . So, the largest positive root of  $P_{2n+1}(z; a, b, r) = 0$  is less than the largest positive root of  $zP_{2n+1}(z; a, b, r) - 2c^2Q_{2n}(z; a, b, r) = 0$ . Hence the result follows.  $\square$

EXAMPLE 4.5. Let  $L_7^{(2)}(a, b, c, r) = \begin{pmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & br & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & br & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in M_7(\mathbb{C})$  be a weighted

shift matrix defined in (12). Then the determinantal polynomial of  $L_7^{(2)}(a, b, c, r)$  is

$$G_7^{(2)}(z; a, b, c, r) = (z^3 - (a^2 + b^2r^2)z) (2a^2c^2 - (a^2 + 2c^2 + b^2r^2)z^2 + z^4).$$

The maximum positive root of  $G_7^{(2)}(z; a, b, c, r) = 0$  is the maximum positive root of  $2a^2c^2 - (a^2 + 2c^2 + b^2r^2)z^2 + z^4 = 0$ . Now, the maximum positive root of  $2a^2c^2 - (a^2 + 2c^2 + b^2r^2)z^2 + z^4 = 0$  is

$$\frac{\sqrt{a^2 + 2c^2 + b^2r^2} + \sqrt{-8a^2c^2 + (-a^2 - 2c^2 - b^2r^2)^2}}{\sqrt{2}}.$$

Hence  $w(L_7^{(2)}(a, b, c, r)) = \frac{\sqrt{a^2 + 2c^2 + b^2r^2} + \sqrt{-8a^2c^2 + (a^2 + 2c^2 + b^2r^2)^2}}{2\sqrt{2}}.$

COROLLARY 4.6. For  $n \geq 1$ , let  $L_{4n+3}^{(2)}(a, b, c, r)$  be the weighted shift matrix defined in (12) with  $c = ar^{2n}$ . Then

$$G_{4n+3}^{(2)}(z; a, b, c, r) = P_{2n+1}(z; a, b, r)(zP_{2n+1}(z; a, b, r) - 2a^2r^{4n}Q_{2n}(z; a, b, r))$$

and  $2w(L_{4n+3}^{(2)}(a, b, c, r))$  is the maximum positive root of

$$zP_{2n+1}(z; a, b, r) - 2a^2r^{4n}Q_{2n}(z; a, b, r) = 0.$$

*Proof.* Putting  $c = ar^{2n}$  in Theorem 4.4, we get our required result.  $\square$

**Particular cases**

1. Putting  $r = 1$  in Theorem 3.1, 3.7, 4.1 and 4.4, our results reduce to Theorem 3.4, 3.1, 3.6 and 3.7, respectively of [4].
2. Putting  $a = b = 1$  in Corollary 3.6, 3.11, 4.3, and 4.6, our results reduce to the results given by Undrakh [12].

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