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COMPOSITION OPERATORS AND THE CLOSURE OF DIRICHLET–MORREY SPACES IN THE BLOCH SPACE

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Abstract. In this paper, we characterize the closure of the Dirichlet-Morrey spaces in the Bloch space by higher-order derivatives. Moreover, the boundedness and compactness of the products of composition and differentiation operators from the Bloch space to the closure of the Dirichlet-Morrey spaces in the Bloch space are investigated. A criterion for an interpolating Blaschke product to be in the closures is given.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space of analytic functions on $\mathbb{D}$. For $0 < p < \infty$, H^p denotes the Hardy space, consisting of all functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

As usual H^∞ denotes the space of bounded analytic functions in $\mathbb{D}$.

For $0 < \alpha < \infty$, the Bloch type space $\mathcal{B}^\alpha$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f'(z)| < \infty.$$

It is a Banach space with the above norm $\|\cdot\|_{\mathcal{B}^\alpha}$. When $\alpha = 1$, then the space $\mathcal{B}^\alpha$ is the classical Bloch space $\mathcal{B}$. It is well known that H^∞ is a subset of the Bloch space. Let n be a positive integer. From [27, p. 1149], we see that $\|f\|_{\mathcal{B}^\alpha}$ is equivalent to

$$\|f\|_{\mathcal{B}^\alpha,n} = |f(0)| + |f'(0)| + \cdots + |f^{(n-1)}(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha+n-1} |f^{(n)}(z)|.$$

The little Bloch type space $\mathcal{B}_0^\alpha$ consists of all $f \in H(\mathbb{D})$ with

$$\lim_{|z|\to 1} (1-|z|^2)^\alpha |f'(z)| = 0.$$

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It is easy to see that the little Bloch type space $\mathcal{B}_0^\alpha$ is the subspace of $\mathcal{B}^\alpha$. We refer to [27] for more results on Bloch type spaces.

Let I be an arc of $\partial\mathbb{D}$ and $|I|$ be the normalized Lebesgue arc length of I. The Carlenson box based on I, denoted by $S(I)$, is defined by

$$S(I) = \{z = re^{i\theta} \in \mathbb{D} : 1 - |I| \leqslant r < 1, e^{i\theta} \in I\}.$$

Let μ be a nonnegative measure on $\mathbb{D}$. For $0 < \alpha < \infty$, μ is said to be an α-Carleson measure if

$$\sup_{I \subset \partial\mathbb{D}} \frac{\mu(S(I))}{|I|^\alpha} < \infty.$$

If μ is an α-Carleson measure and $\lim_{|I|\to 0} \frac{\mu(S(I))}{|I|^\alpha} = 0$, then μ is called a vanishing α-Carleson measure.

For $0 < p < \infty$, $-2 < q < \infty$, $0 \leqslant s < \infty$ and $-1 < q+s < \infty$, recall that the general family of function spaces $F(p,q,s)$ consists of those $f \in H(\mathbb{D})$ such that

$$\|f\|_{F(p,q,s)}^p = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q (1-|\sigma_a(z)|^2)^s dA(z) < \infty,$$

where dA is the normalized Lebesgue area measure on $\mathbb{D}$.

For $0 \leqslant p < \infty$, the weighted Dirichlet space $\mathcal{D}_p$ consists of functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{\mathcal{D}_p} = |f(0)| + \left(\int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^p dA(z)\right)^{1/2} < \infty.$$

For $0 \leqslant p, \lambda \leqslant 1$, the Dirichlet-Morrey space $\mathcal{D}_p^\lambda$ consists of those functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{D}_p^\lambda} = |f(0)| + \sup_{a\in\mathbb{D}} (1-|a|^2)^{\frac{p(1-\lambda)}{2}} \|f\circ\sigma_a - f(a)\|_{\mathcal{D}_p} < \infty,$$

where $\sigma_a = \frac{a-z}{1-\bar{a}z}$, is the Möbius transformation of $\mathbb{D}$. It is known that $\mathcal{D}_p^\lambda$ is a Banach space with the above norm $\|\cdot\|_{\mathcal{D}_p^\lambda}$. When $p = 1$, $\mathcal{D}_p^\lambda$ is the Morrey space $\mathcal{L}^{2,\lambda}$, which was studied in [13, 14]. Moreover,

$$BMOA \subsetneq \mathcal{L}^{2,\lambda} \subsetneq H^2, \quad 0 < \lambda < 1.$$

When $\lambda = 0$ or $\lambda = 1$, $\mathcal{D}_p^\lambda$ reduces to $\mathcal{D}_p$ and $\mathcal{Q}_p$, respectively. Moreover,

$$\mathcal{Q}_p \subsetneq \mathcal{D}_p^\lambda \subsetneq \mathcal{D}_p, \quad 0 < \lambda < 1.$$

By [10, Proposition 2.1], the norm of functions $f \in \mathcal{D}_p^\lambda(\mathbb{D})$ can be defined as follow

$$\|f\|_{\mathcal{D}_p^\lambda} = |f(0)| + \sup_{I\subset\partial\mathbb{D}} \sqrt{\frac{1}{|I|^{p\lambda}} \int_{S(I)} |f'(z)|^2 (1-|z|^2)^p dA(z)}. \tag{1}$$

Let n be a nonnegative integer. Denote by D^n the n-order differential operator. Namely $D^n f = f^{(n)}$ for $f \in H(\mathbb{D})$. Let φ be an analytic self-map of $\mathbb{D}$. The products of composition operators and n-order differential operators $C_\varphi D^n$ are defined by

$$(C_\varphi D^n f)(z) = f^{(n)}(\varphi(z)), \quad f \in H(\mathbb{D}).$$

If $n = 0$, we get the composition operator C_φ. The products of composition operators and differential operators have been studied on some analytic function spaces (see [15, 16, 33, 34]).

Let X and Y be two Banach spaces of analytic functions. For simplicity, the closure of $X \cap Y$ in the norm of Y is denoted by $\mathcal{C}_Y(X \cap Y)$. In [1], Anderson, Clunie and Pommerenke posed the follwing problem: what is the closure of H^∞ in the Bloch norm? This problem remains open to this date. Recently, Zhao [31] studied the closures of some Möbius invariant spaces in the Bloch space. Aulaskari and Zhao [2] characterized composition operators from the Bloch space to the closures of some Möbius invariant spaces in the Bloch space. Bao and Göğüş [3] investigated the closure of Dirichlet type spaces $\mathcal{D}_s(-1 < s \leqslant 1)$ in the Bloch space. Hu and Zhu [12] studied the closure of Morrey space in the Bloch space. See [4, 9, 17, 18, 20, 21, 29, 30] for some related results.

It is well known that for $0 < p < 1$,

$$\mathcal{D} \subsetneq Q_p \subsetneq BMOA \subseteq \mathcal{B}.$$

Hence,

$$\mathcal{C}_\mathcal{B}(Q_p) \subseteq \mathcal{C}_\mathcal{B}(\mathcal{D}_p^\lambda \cap \mathcal{B}) \subseteq \mathcal{C}_\mathcal{B}(\mathcal{D}_p \cap \mathcal{B}).$$

From [3], we see that a Bloch function f is in $\mathcal{C}_\mathcal{B}(\mathcal{D}_s \cap \mathcal{B})$, $-1 < s \leqslant 1$ if and only if for any $\varepsilon > 0$,

$$\int_{\Omega_\varepsilon(f)} (1-|z|^2)^{s-2} dA(z) < \infty,$$

where

$$\Omega_\varepsilon(f) = \{z \in \mathbb{D} : (1-|z|^2)|f'(z)| \geqslant \varepsilon\}.$$

It is also know that $F(2,0,s) = Q_s$. From [2] and [31], we see that a Bloch function f is in $\mathcal{C}_\mathcal{B}(Q_s)$, $0 < s \leqslant 1$ if and only if for any $\varepsilon > 0$,

$$\sup_{a \in \mathbb{D}} \int_{\Omega_\varepsilon(f)} (1-|\sigma_a(z)|^2)^s (1-|z|^2)^{-2} dA(z) < \infty,$$

where

$$\Omega_\varepsilon(f) = \{z \in \mathbb{D} : (1-|z|^2)|f'(z)| \geqslant \varepsilon\}.$$

According to the above, it is natural to ask what are the necessary and sufficient conditions for a Bloch function f is in $\mathcal{C}_\mathcal{B}(\mathcal{D}_p^\lambda \cap \mathcal{B})$?

The purpose of this paper is to characterize $\mathcal{C}_\mathcal{B}(\mathcal{D}_p^\lambda \cap \mathcal{B})$. In [29], Zhang characterized the boundedness and compactness of the operator $C_\varphi D^n$ from $\mathcal{B}^\alpha(\mathcal{B}_0^\alpha)$ to

$\mathcal{C}_{\mathcal{B}^\beta}(A^p_\omega \cap \mathcal{B}^\beta)$. Sun et al. [25] studied the boundedness and compactness of the operator $C_\varphi D^n$ from $\mathcal{B}(\mathcal{B}_0)$ to $\mathcal{C}_{\mathcal{B}}(F(p,p-2,s))$. In this work, we will characterize the the boundedness and compactness of the operator $C_\varphi D^n$ from $\mathcal{B}(\mathcal{B}_0)$ to $\mathcal{C}_{\mathcal{B}}(\mathcal{D}^\lambda_p \cap \mathcal{B})$.

The rest of this paper is organized as follows: In Section 2, we characterize the closure of the Dirichlet-Morrey spaces in the Bloch space by higher-order derivatives. In Section 3, we give the characterization of boundedness and compactness of the products of composition and n-th differentiation operators. In Section 4, a criterion for an interpolating Blaschke product to be in $\mathcal{C}_{\mathcal{B}}(\mathcal{D}^\lambda_p \cap \mathcal{B})$ is obtained.

Throughout this paper, we say that $A \lesssim B$ if there exists a constant C such that $A \leqslant CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

2. Characterization of $\mathcal{C}_{\mathcal{B}}(\mathcal{D}^\lambda_p \cap \mathcal{B})$

To state and prove our results in this paper, we need an estimate, which can be found in [28].

LEMMA 2.1. *Let $s > 0$ and $t > -1$. Then there exists a positive costant C such that*

$$\int_{\mathbb{D}} \frac{(1-|\omega|^2)^t}{|1-\overline{z}\omega|^{2+t+s}} dA(\omega) \leqslant \frac{C}{(1-|z|^2)^s}.$$

The following lemma, quoted from Lemma 1 in [32], is an extension of Lemma 2.1. See also [19].

LEMMA 2.2. *Let $s > -1$, $r,t > 0$ and $r+t-s > 2$. If $t < s+t < r$, then there exists a positive costant C such that*

$$\int_{\mathbb{D}} \frac{(1-|\eta|^2)^s}{|1-\overline{\eta}z|^r |1-\overline{\eta}\xi|^t} dA(\eta) \leqslant \frac{C}{(1-|z|^2)^{r-s-2}|1-\overline{\xi}z|^t}.$$

The follow lemma is Lemma 3.1.1 in [26].

LEMMA 2.3. *Let $\alpha, t \in (0,\infty)$ and a nonnegative measure μ on $\mathbb{D}$. The μ is a α-Carleson measure if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^t}{|1-\overline{a}z|^{\alpha+t}} d\mu(z) < \infty.$$

LEMMA 2.4. *Let $0 < p, \lambda < 1$. Then $f \in \mathcal{D}^\lambda_p$ if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^{p(1-\lambda)} (1-|\sigma_a(z)|^2)^{p\lambda} dA(z) < \infty.$$

Moreover,

$$\|f\|_{\mathcal{D}^\lambda_p} \approx |f(0)| + \left(\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^{p(1-\lambda)} (1-|\sigma_a(z)|^2)^{p\lambda} dA(z) \right)^{1/2}.$$

Proof. Denote

$$d\mu(z) = |f'(z)|^2(1-|z|^2)^p dA(z).$$

Then, from (1) we have that $f \in \mathcal{D}_p^\lambda$ if and only if $d\mu$ is a $p\lambda$-Carleson measure. Hence, by Lemma 2.3, we get that $f \in \mathcal{D}_p^\lambda$ if and only if

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^t}{|1-\overline{a}z|^{p\lambda+t}} d\mu(z) \\
=&\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^t}{|1-\overline{a}z|^{p\lambda+t}} |f'(z)|^2(1-|z|^2)^p dA(z) \\
<&\infty.
\end{aligned}$$

Let $t = p\lambda$, we get the desired that. □

The following Theorem A (see [22] and [23]) is the characterizations of $F(p,q,s)$ spaces using higher order derivatives:

THEOREM A. *Let f be analytic on $\mathbb{D}$. Let $0<p<\infty$, $-2<q<\infty$, $0<s<\infty$. Let $n \in \mathbb{N}$ and $q+s>-1$, or $n=0$ and $q+s-p>-1$. Then the following statements are equivalent:*

(i) $f \in F(p,q,s)$;

(ii) $\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f^{(n)}(z)|^p(1-|z|^2)^{np-p+q}(1-|\sigma_a(z)|^2)^s dA(z) < \infty$;

(iii) $\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f^{(n)}(z)|^p(1-|z|^2)^{np-p+q} g^s(z,a) dA(z) < \infty$;

(iv) $d\mu(z) = |f^{(n)}(z)|^p(1-|z|^2)^{np-p+q+s} dA(z)$ *is a bounded s-Carleson measure.*

REMARK 2.5. Note that $F(2,p(1-\lambda),p\lambda) = \mathcal{D}_p^\lambda$ for $0<p,\lambda<1$. From the theorem above, it is easy to see that the following statements are equivalent:

(i) $f \in \mathcal{D}_p^\lambda$;

(ii) $\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f^{(n)}(z)|^2(1-|z|^2)^{p(1-\lambda)+2n-2}(1-|\sigma_a(z)|^2)^{p\lambda} dA(z) < \infty$;

(iii) $d\mu(z) = |f^{(n)}(z)|^2(1-|z|^2)^{p+2n-2} dA(z)$ is a bounded $p\lambda$-Carleson measure.

THEOREM 2.6. *Let $0<p,\lambda<1$ and let n be a positive integer. Suppose $f \in \mathcal{B}$. Then $f \in \mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B})$ if and only if for any $\varepsilon>0$,*

$$\sup_{a\in\mathbb{D}} \int_{\Omega_{n,\varepsilon}(f)} \left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\right)^{p\lambda} \frac{dA(z)}{(1-|z|^2)^{2-p}} < \infty. \tag{2}$$

where

$$\Omega_{n,\varepsilon}(f) = \{z \in \mathbb{D} : (1-|z|^2)^n |f^{(n)}(z)| \geqslant \varepsilon\}.$$

Proof. Take $f \in \mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B})$ and $\varepsilon > 0$. Then there exists a $g \in \mathcal{D}_p^\lambda \cap \mathcal{B}$ such that $\|f-g\|_{\mathcal{B},n} \leqslant \frac{\varepsilon}{2}$. Since

$$
\begin{aligned}
(1-|z|^2)^n|f^{(n)}(z)| &\leqslant \sup_{\omega\in\mathbb{D}}(1-|\omega|^2)^n|f^{(n)}(\omega)-g^{(n)}(\omega)| + (1-|z|^2)^n|g^{(n)}(z)| \\
&\leqslant \frac{\varepsilon}{2} + (1-|z|^2)^n|g^{(n)}(z)|, \quad z \in \mathbb{D}.
\end{aligned}
$$

We have $\Omega_{n,\varepsilon}(f) \subseteq \Omega_{n,\frac{\varepsilon}{2}}(g)$. Consequently,

$$
\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{\Omega_{n,\varepsilon}(f)} \left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\right)^{p\lambda} \frac{dA(z)}{(1-|z|^2)^{2-p}} \\
\leqslant &\sup_{a\in\mathbb{D}} \int_{\Omega_{n,\frac{\varepsilon}{2}}(g)} \frac{(1-|\sigma_a(z)|^2)^{p\lambda}(1-|z|^2)^{2n-p\lambda}|g^{(n)}(z)|^2}{(1-|z|^2)^{2n}|g^{(n)}(z)|^2} \frac{dA(z)}{(1-|z|^2)^{2-p}} \\
\leqslant &\frac{4}{\varepsilon^2} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g^{(n)}(z)|^2 (1-|z|^2)^{p(1-\lambda)+2n-2} (1-|\sigma_a(z)|^2)^{p\lambda} dA(z).
\end{aligned}
$$

Since $g \in \mathcal{D}_p^\lambda$, by Remark 2.5, we have

$$
\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g^{(n)}(z)|^2 (1-|z|^2)^{p(1-\lambda)+2n-2} (1-|\sigma_a(z)|^2)^{p\lambda} dA(z) < \infty.
$$

Hence,

$$
\sup_{a\in\mathbb{D}} \int_{\Omega_{n,\varepsilon}(f)} \left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\right)^{p\lambda} \frac{dA(z)}{(1-|z|^2)^{2-p}} < \infty.
$$

Conversely, suppose that (2) holds. Fix $\varepsilon > 0$ and let f satisfy (2). Since $\mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B})$ is a linear space containing all polynomials, the function

$$
f(z) = \sum_{k=0}^{\infty} \frac{f^k(0)}{k!} z^k \in \mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B})
$$

if and only if

$$
\sum_{k=n}^{\infty} \frac{f^k(0)}{k!} z^k \in \mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B}).
$$

Without loss of gennerality, we may assume that

$$
f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0.
$$

For any $z \in \mathbb{D}$, by Proposition 4.27 in [28],

$$
f(z) = \frac{1}{(\beta+2)\cdots(\beta+n)} \int_{\mathbb{D}} \frac{f^{(n)}(\omega)(1-|\omega|^2)^{n+\beta}}{(1-z\overline{\omega})^{2+\beta}\overline{\omega}^n} dA(\omega),
$$

where $\beta > -1$. Following [31], we set $f(z) = f_1(z) + f_2(z)$, where

$$f_1(z) = \frac{1}{(\beta+2)\cdots(\beta+n)} \int_{\Omega_{n,\varepsilon}(f)} \frac{f^{(n)}(\omega)(1-|\omega|^2)^{n+\beta}}{(1-z\overline{\omega})^{2+\beta}\overline{\omega}^n} dA(\omega)$$

and

$$f_2(z) = \frac{1}{(\beta+2)\cdots(\beta+n)} \int_{\mathbb{D}\setminus\Omega_{n,\varepsilon}(f)} \frac{f^{(n)}(\omega)(1-|\omega|^2)^{n+\beta}}{(1-z\overline{\omega})^{2+\beta}\overline{\omega}^n} dA(\omega).$$

After a calculation, we get

$$f_1^{(n)}(z) = (\beta+n+1) \int_{\Omega_{n,\varepsilon}(f)} \frac{f^{(n)}(\omega)(1-|\omega|^2)^{n+\beta}}{(1-z\overline{\omega})^{n+2+\beta}} dA(\omega)$$

and

$$f_2^{(n)}(z) = (\beta+n+1) \int_{\mathbb{D}\setminus\Omega_{n,\varepsilon}(f)} \frac{f^{(n)}(\omega)(1-|\omega|^2)^{n+\beta}}{(1-z\overline{\omega})^{n+2+\beta}} dA(\omega).$$

Let

$$h(z) = f_1(z) - \sum_{k=0}^{n-1} \frac{f_1^k(0)}{k!} z^k.$$

Then $h(0) = h'(0) = \cdots = h^{(n-1)}(0) = 0$ and $(f-h)^{(n)}(z) = f_2^{(n)}(z)$.

Combining the above facts with Lemma 2.1, we obtain

$$\begin{aligned}
\|f-h\|_{\mathcal{B},n} &= \sup_{z\in\mathbb{D}} (1-|z|^2)^n |f_2^{(n)}(z)| \\
&\lesssim \sup_{z\in\mathbb{D}} (1-|z|^2)^n \int_{\mathbb{D}\setminus\Omega_{n,\varepsilon}(f)} \frac{|f^{(n)}(\omega)|(1-|\omega|^2)^{n+\beta}}{(1-z\overline{\omega})^{n+2+\beta}} dA(\omega) \\
&\lesssim \varepsilon \sup_{z\in\mathbb{D}} (1-|z|^2)^n \int_{\mathbb{D}} \frac{(1-|\omega|^2)^{\beta}}{(1-z\overline{\omega})^{n+2+\beta}} dA(\omega) \lesssim \varepsilon.
\end{aligned}$$

Hence $h \in \mathcal{B}$. Using Fubini theorem and Lemma 2.2, we obtain

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |h^{(n)}(z)|^2 (1-|z|^2)^{p(1-\lambda)+2n-2} (1-|\sigma_a(z)|^2)^{p\lambda} dA(z) \\
=& \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_1^{(n)}(z)|^2 (1-|z|^2)^{p(1-\lambda)+2n-2} (1-|\sigma_a(z)|^2)^{p\lambda} dA(z) \\
\leqslant& \|f_1\|_{\mathcal{B},n} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_1^{(n)}(z)| \frac{1}{(1-|z|^2)^n} (1-|z|^2)^{p(1-\lambda)+2n-2} (1-|\sigma_a(z)|^2)^{p\lambda} dA(z) \\
\lesssim& \|f_1\|_{\mathcal{B},n} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (1-|\sigma_a(z)|^2)^{p\lambda} (1-|z|^2)^{p(1-\lambda)+n-2} \\
&\times \left(\int_{\Omega_{n,\varepsilon}(f)} \frac{|f^{(n)}(\omega)|(1-|\omega|^2)^{n+\beta}}{(1-z\overline{\omega})^{n+2+\beta}} dA(\omega) \right) dA(z)
\end{aligned}$$

$$\lesssim \sup_{a\in\mathbb{D}} \int_{\Omega_{n,\varepsilon}(f)} |f^{(n)}(\omega)|(1-|\omega|^2)^{n+\beta}$$
$$\times \left(\int_{\mathbb{D}} \frac{(1-|z|^2)^{p(1-\lambda)+n-2}(1-|\sigma_a(z)|^2)^{p\lambda}}{|1-z\overline{\omega}|^{n+2+\beta}} dA(z) \right) dA(\omega)$$
$$\lesssim \sup_{a\in\mathbb{D}} \int_{\Omega_{n,\varepsilon}(f)} (1-|\omega|^2)^{\beta}(1-|a|^2)^{p\lambda} \left(\int_{\mathbb{D}} \frac{(1-|z|^2)^{p+n-2}}{|1-\overline{z}\omega|^{n+2+\beta}|1-\overline{z}a|^{2p\lambda}} dA(z) \right) dA(\omega)$$
$$\lesssim \sup_{a\in\mathbb{D}} \int_{\Omega_{n,\varepsilon}(f)} \left(\frac{1-|\sigma_a(\omega)|^2}{1-|\omega|^2} \right)^{p\lambda} \frac{dA(\omega)}{(1-|\omega|^2)^{2-p}} < \infty,$$

that is, $h \in \mathcal{D}_p^\lambda$. Thus for any $\varepsilon > 0$, there exists a function $h \in \mathcal{D}_p^\lambda \cap \mathcal{B}$ such that $\|f-g\|_{\mathcal{B},n} \lesssim \varepsilon$, i.e., $f \in \mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B})$. The proof is complete. $\square$

3. The boundedness and compactness of the operator $C_\varphi D^n$

In this section, we characterize the boundedness and compactness of the operator $C_\varphi D^n : \mathcal{B}(\mathcal{B}_0) \to \mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B})$ and on $\mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B})$.

THEOREM 3.1. *Let φ be an analytic self-map of $\mathbb{D}$ and let n be a nonnegative integer. Suppose that $0 < p, \lambda < 1$. Then $C_\varphi D^n : \mathcal{B} \to \mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B})$ is bounded if and only if for any $\varepsilon > 0$,*

$$\sup_{a\in\mathbb{D}} \int_{\Gamma_\varepsilon(\varphi)} \left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2} \right)^{p\lambda} \frac{dA(z)}{(1-|z|^2)^{2-p}} < \infty, \tag{3}$$

where $\Gamma_\varepsilon(\varphi) = \{z \in \mathbb{D} : \frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{n+1}} |\varphi'(z)| \geqslant \varepsilon\}$.

Proof. Assume that $C_\varphi D^n : \mathcal{B} \to \mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B})$ is bounded. From [24], we see that exists two functions $f_1, f_2 \in \mathcal{B}$ such that

$$|f_1^{(n+1)}(z)| + |f_2^{(n+1)}(z)| \geqslant \frac{C}{(1-|z|^2)^{n+1}}.$$

By the boundedness of $C_\varphi D^n$, we get $f_1^{(n)} \circ \varphi, f_2^{(n)} \circ \varphi \in \mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B})$. Hence, Theorem 2.6 implies that, for any $\varepsilon > 0$,

$$\sup_{a\in\mathbb{D}} \int_{\Omega_{\frac{\varepsilon}{2}}(f_1^{(n)}\circ\varphi)} \left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2} \right)^{p\lambda} \frac{dA(z)}{(1-|z|^2)^{2-p}} < \infty$$

and

$$\sup_{a\in\mathbb{D}} \int_{\Omega_{\frac{\varepsilon}{2}}(f_2^{(n)}\circ\varphi)} \left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2} \right)^{p\lambda} \frac{dA(z)}{(1-|z|^2)^{2-p}} < \infty.$$

When $z \in \Gamma_\varepsilon(\varphi)$, we get

$$\begin{aligned}
&(|(f_1^{(n)} \circ \varphi)'(z)| + |(f_2^{(n)} \circ \varphi)'(z)|)(1-|z|^2)\\
=&(|f_1^{(n+1)}(\varphi(z))| + |f_2^{(n+1)}(\varphi(z))|)|\varphi'(z)|(1-|z|^2)\\
\geqslant&\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{n+1}}|\varphi'(z)| \geqslant \varepsilon,
\end{aligned}$$

which implies that either

$$|(f_1^{(n)} \circ \varphi)'(z)|(1-|z|^2) \geqslant \frac{\varepsilon}{2}$$

or

$$|(f_2^{(n)} \circ \varphi)'(z)|(1-|z|^2) \geqslant \frac{\varepsilon}{2}.$$

Hence

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{\Gamma_\varepsilon(\varphi)} \left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\right)^{p\lambda} \frac{dA(z)}{(1-|z|^2)^{2-p}}\\
\leqslant&\sup_{a\in\mathbb{D}} \int_{\Omega_{\frac{\varepsilon}{2}}(f_1^{(n)}\circ\varphi)\cup\Omega_{\frac{\varepsilon}{2}}(f_2^{(n)}\circ\varphi)} \left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\right)^{p\lambda} \frac{dA(z)}{(1-|z|^2)^{2-p}}\\
\leqslant&\sup_{a\in\mathbb{D}} \int_{\Omega_{\frac{\varepsilon}{2}}(f_1^{(n)}\circ\varphi)} \left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\right)^{p\lambda} \frac{dA(z)}{(1-|z|^2)^{2-p}}\\
&+\sup_{a\in\mathbb{D}} \int_{\Omega_{\frac{\varepsilon}{2}}(f_2^{(n)}\circ\varphi)} \left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\right)^{p\lambda} \frac{dA(z)}{(1-|z|^2)^{2-p}}\\
<&\infty.
\end{aligned}$$

Conversely, suppose that (3) holds. Let $f \in \mathcal{B}$. Then

$$\begin{aligned}
|(C_\varphi D^n f)'(z)|(1-|z|^2) =&|f^{(n+1)}(\varphi(z))|(1-|\varphi(z)|^2)^{n+1}\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{n+1}}|\varphi'(z)|\\
\leqslant&\|f\|_{\mathcal{B},n+1}\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{n+1}}|\varphi'(z)|.
\end{aligned}$$

Therefore, for any $\delta > 0$, if $|(C_\varphi D^n f)'(z)|(1-|z|^2) > \delta$, we have that

$$\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{n+1}}|\varphi'(z)| \geqslant \frac{\delta}{\|f\|_{\mathcal{B},n+1}} = \varepsilon.$$

Hence,

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{\Omega_\delta(C_\varphi D^n f)} \left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\right)^{p\lambda} \frac{dA(z)}{(1-|z|^2)^{2-p}}\\
\leqslant&\sup_{a\in\mathbb{D}} \int_{\Gamma_\varepsilon(\varphi)} \left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\right)^{p\lambda} \frac{dA(z)}{(1-|z|^2)^{2-p}} < \infty.
\end{aligned}$$

From Theorem 2.6, we have

$$C_\varphi D^n f \in \mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B}),$$

i.e., $C_\varphi D^n : \mathcal{B} \to \mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B})$ is bounded. The proof is complete. $\square$

THEOREM 3.2. *Let φ be an analytic self-map of $\mathbb{D}$ and let n be a nonnegative integer. Suppose that $0 < p, \lambda < 1$. Then $C_\varphi D^n : \mathcal{B}_0 \to \mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B})$ is bounded if and only if $\varphi \in \mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B})$ and*

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{n+1}} |\varphi'(z)| < \infty. \tag{4}$$

Proof. Suppose $\varphi \in \mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B})$ and

$$K := \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{n+1}} |\varphi'(z)| < \infty.$$

Let $f \in \mathcal{B}$. For any $\varepsilon > 0$, there is a constant $r (0 < r < 1)$ such that

$$|f^{(n)}(z)|(1-|z|^2)^n < \frac{\varepsilon}{K},$$

whenever $|z| > r$. Let $z \in \Omega_\varepsilon(C_\varphi D^n)$. Then, by the assumed condition, we have

$$\begin{aligned}
\varepsilon &\leqslant |f^{(n+1)}(\varphi(z))||\varphi'(z)|(1-|z|^2) \\
&\leqslant |f^{(n+1)}(\varphi(z))|(1-|\varphi(z)|^2)^{n+1} \frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{n+1}} |\varphi'(z)| \\
&\leqslant K|f^{(n+1)}(\varphi(z))|(1-|\varphi(z)|^2)^{n+1},
\end{aligned}$$

which implies that $|\varphi(z)| < r$. Thus,

$$\begin{aligned}
\varepsilon &\leqslant |f^{(n+1)}(\varphi(z))||\varphi'(z)|(1-|z|^2) \\
&\leqslant \|f\|_{\mathcal{B},n+1} \frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{n+1}} |\varphi'(z)| \\
&\leqslant \frac{\|f\|_{\mathcal{B},n+1}}{(1-r^2)^{n+1}} (1-|z|^2)|\varphi'(z)|.
\end{aligned}$$

Let $\delta = \frac{\varepsilon(1-r^2)^{n+1}}{\|f\|_{\mathcal{B},n+1}}$. Then $(1-|z|^2)|\varphi'(z)| \geqslant \delta$. Hence, $\Omega_\varepsilon(C_\varphi D^n f) \subseteq \Omega_\delta(\varphi)$. Since $\varphi \in \mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B})$, by Theorem 2.6 we get

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{\Omega_\varepsilon(C_\varphi D^n f)} \left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\right)^{p\lambda} \frac{dA(z)}{(1-|z|^2)^{2-p}} \\
\leqslant &\sup_{a\in\mathbb{D}} \int_{\Omega_\delta(\varphi)} \left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\right)^{p\lambda} \frac{dA(z)}{(1-|z|^2)^{2-p}} \\
< &\infty.
\end{aligned}$$

By Theorem 2.6, we see that $C_\varphi D^n f \in \mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B})$. Hence, $C_\varphi D^n : \mathcal{B}_0 \to \mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B})$ is bounded.

Conversely, suppose that $C_\varphi D^n : \mathcal{B}_0 \to \mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B})$ is bounded. Noting that $f_n(z) = z^{n+1}/(n+1)! \in \mathcal{B}$, we have

$$\varphi = C_\varphi D^n(f_n) \in \mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B}).$$

Since $\mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B})$ is a subspace of $\mathcal{B}$ and $C_\varphi D^n : \mathcal{B}_0 \to \mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B})$ is bounded, then $C_\varphi D^n : \mathcal{B}_0 \to \mathcal{B}$ is bounded. It is easy to see (4) holds according to [34, Theorem 2.1] with $\alpha = \beta = 1$. The proof is complete. □

THEOREM 3.3. *Let φ be an analytic self-map of $\mathbb{D}$ and let n be a nonnegative integer. Suppose that $0 < p, \lambda < 1$. Then the following statements are equivalent.*

(i) $C_\varphi D^n : \mathcal{B} \to \mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B})$ *is compact;*

(ii) $C_\varphi D^n : \mathcal{B}_0 \to \mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B})$ *is compact;*

(iii) $\varphi \in \mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B})$ *and*

$$\lim_{|\varphi(z)| \to 1} \frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{n+1}} |\varphi'(z)| = 0. \tag{5}$$

Proof. $(i) \Rightarrow (ii)$ It is clear.

$(ii) \Rightarrow (iii)$ Assume that $C_\varphi D^n : \mathcal{B}_0 \to \mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B})$ is compact. Then $C_\varphi D^n : \mathcal{B}_0 \to \mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B})$ is bounded. By Theorem 3.2, we see that $\varphi \in \mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B})$. Since $\mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B}) \subseteq \mathcal{B}$, we get that $C_\varphi D^n : \mathcal{B}_0 \to \mathcal{B}$ is compact. This implies that (5) holds by [34, Theorem 2.2].

$(iii) \Rightarrow (i)$ By the assumption, we see that there exists r $(0 < r < 1)$, such that

$$\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{n+1}} |\varphi'(z)| < \frac{\varepsilon}{2}, \quad \text{when } |\varphi(z)| > r.$$

Let $z \in \Gamma_\varepsilon(\varphi)$. Then $|\varphi(z)| \leqslant r$. Therefore,

$$\varepsilon \leqslant \frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{n+1}} |\varphi'(z)| \leqslant \frac{(1-|z|^2)}{(1-r^2)^{n+1}} |\varphi'(z)|.$$

Thus $\varepsilon(1-r^2)^{n+1} \leqslant (1-|z|^2)|\varphi'(z)|$. Let $\delta = \varepsilon(1-r^2)^{n+1}$. Then $z \in \Omega_\delta(\varphi)$. Hence

$$\begin{aligned}
&\sup_{a \in \mathbb{D}} \int_{\Gamma_\varepsilon(\varphi)} \left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2} \right)^{p\lambda} \frac{dA(z)}{(1-|z|^2)^{2-p}} \\
\leqslant &\sup_{a \in \mathbb{D}} \int_{\Omega_\delta(\varphi)} \left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2} \right)^{p\lambda} \frac{dA(z)}{(1-|z|^2)^{2-p}}.
\end{aligned}$$

Since $\varphi \in \mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda \cap \mathcal{B})$, by Theorem 2.6 we have

$$\sup_{a \in \mathbb{D}} \int_{\Omega_\delta(\varphi)} \left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2} \right)^{p\lambda} \frac{dA(z)}{(1-|z|^2)^{2-p}} < \infty.$$

Therefore,

$$\sup_{a\in\mathbb{D}}\int_{\Gamma_\varepsilon(\varphi)}\left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\right)^{p\lambda}\frac{dA(z)}{(1-|z|^2)^{2-p}}<\infty.$$

By Theorem 3.1, $C_\varphi D^n:\mathcal{B}\to\mathcal{C}_\mathcal{B}(\mathcal{D}_p^\lambda\cap\mathcal{B})$ is bounded. It is easy to know that $C_\varphi D^n:\mathcal{B}\to\mathcal{B}$ is compact by [34, Theorem 2.2]. Therefore $C_\varphi D^n:\mathcal{B}\to\mathcal{C}_\mathcal{B}(\mathcal{D}_p^\lambda\cap\mathcal{B})$ is compact. The proof is complete. □

From Theorem 3.3, we immediately get the following corollary.

COROLLARY 3.4. *Let φ be an analytic self-map of $\mathbb{D}$ and let n be a nonnegative integer. Suppose that $0<p,\lambda<1$ and $\varphi\in\mathcal{C}_\mathcal{B}(\mathcal{D}_p^\lambda\cap\mathcal{B})$. Then the following statements are equivalent.*

(i) $C_\varphi D^n:\mathcal{B}\to\mathcal{C}_\mathcal{B}(\mathcal{D}_p^\lambda\cap\mathcal{B})$ *is compact;*
(ii) $C_\varphi D^n:\mathcal{B}_0\to\mathcal{C}_\mathcal{B}(\mathcal{D}_p^\lambda\cap\mathcal{B})$ *is compact;*
(iii) $C_\varphi D^n:\mathcal{B}\to\mathcal{B}$ *is compact;*
(iv) $C_\varphi D^n:\mathcal{B}_0\to\mathcal{B}$ *is compact.*

THEOREM 3.5. *Let φ be an analytic self-map of $\mathbb{D}$ and let n be a nonnegative integer. Suppose that $0<p,\lambda<1$. Then $C_\varphi D^n:\mathcal{C}_\mathcal{B}(\mathcal{D}_p^\lambda\cap\mathcal{B})\to\mathcal{C}_\mathcal{B}(\mathcal{D}_p^\lambda\cap\mathcal{B})$ is compact if and only if $\varphi\in\mathcal{C}_\mathcal{B}(\mathcal{D}_p^\lambda\cap\mathcal{B})$ and*

$$\lim_{|\varphi(z)|\to1}\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{n+1}}|\varphi'(z)|=0. \tag{6}$$

Proof. Suppose that $\varphi\in\mathcal{C}_\mathcal{B}(\mathcal{D}_p^\lambda\cap\mathcal{B})$ and (6) holds. By Thoerem 3.3, $C_\varphi D^n:\mathcal{B}\to\mathcal{C}_\mathcal{B}(\mathcal{D}_p^\lambda\cap\mathcal{B})$ is compact. Since $\mathcal{C}_\mathcal{B}(\mathcal{D}_p^\lambda\cap\mathcal{B})\subseteq\mathcal{B}$, we get that $C_\varphi:\mathcal{C}_\mathcal{B}(\mathcal{D}_p^\lambda\cap\mathcal{B})\to\mathcal{C}_\mathcal{B}(\mathcal{D}_p^\lambda\cap\mathcal{B})$ is compact, as desired.

Conversely, assume that $C_\varphi D^n:\mathcal{C}_\mathcal{B}(\mathcal{D}_p^\lambda\cap\mathcal{B})\to\mathcal{C}_\mathcal{B}(\mathcal{D}_p^\lambda\cap\mathcal{B})$ is compact. It is clear that $\varphi\in\mathcal{C}_\mathcal{B}(\mathcal{D}_p^\lambda\cap\mathcal{B})$ since $f_n(z)=z^{n+1}/(n+1)!\in\mathcal{C}_\mathcal{B}(\mathcal{D}_p^\lambda\cap\mathcal{B})$. Since $\mathcal{B}_0$ is closure of all polynomials in $\mathcal{B}$ and the space $\mathcal{D}_p^\lambda$ contains all polynomials, we get that $C_\varphi:\mathcal{B}_0\to\mathcal{C}_\mathcal{B}(\mathcal{D}_p^\lambda\cap\mathcal{B})$ is compact. By [34, Theorem 2.2], we see that (6) holds. The proof is complete. □

4. Interpolating Blaschke products in $\mathcal{C}_\mathcal{B}(\mathcal{D}_p^\lambda\cap\mathcal{B})$

If a sequence $\{a_k\}_{k=1}^\infty$ in $\mathbb{D}$ satisfies

$$\sum_{k=1}^\infty(1-|a_k|)<\infty,$$

then we say that $\{a_k\}_{k=1}^\infty$ in $\mathbb{D}$ is a Blaschke sequence. The corresponding Blaschke product B is given by

$$B(z)=\prod_{k=1}^\infty\frac{|a_k|}{a_k}\frac{a_k-z}{1-\overline{a_k}z},$$

where $|a_k|/a_k = -1$ if $a_k = 0$. It is clear that $B \in H^\infty$. Blaschke products play an important role in the study of zero sets and inner functions in analytic function spaces. They also can be used to construct examples in various function spaces. See [5] for example.

A sequence $\{a_k\}_{k=1}^\infty$ in $\mathbb{D}$ is called an interpolating sequence if for any bounded sequence $\{\omega_k\}_{k=1}^\infty$ in $\mathbb{C}$, there is a function f in H^∞ satisfying $f(a_k) = \omega_k$ for every k. A significant result of Carleson [6] showed that $\{a_k\}_{k=1}^\infty$ is an interpolating sequence if and only if $\{a_k\}_{k=1}^\infty$ is uniformly separated, that is, there exists a $\delta > 0$, such that

$$\inf_m \prod_{n \neq m} \rho(a_n, a_m) \geqslant \delta. \tag{7}$$

Here ρ denotes the pseudo-hyperbolic distance:

$$\rho(z,\omega) = \left| \frac{z-\omega}{1-\overline{\omega}z} \right|, \; z, \omega \in \mathbb{D}.$$

Also, for $a \in \mathbb{D}$ and $0 < r < 1$, $\Delta(a,r)$ will denote the pseudo-hyperbolic disc of center a and radius r:

$$\Delta(a,r) = \{z \in \mathbb{D} : \rho(z,a) < r\}.$$

If the zero set of a Blaschke product B is uniformly separated, then we say that B is an interpolating Blaschke product. Equivalently, B is an interpolating Blaschke product if and only if its zero set $\{z_n\}$ satisfying

$$\inf_n (1-|z_n|)|B'(z_n)| > 0.$$

See [7, 8] for interpolating Blaschke products, and [3, 4, 9] for the characterization of interpolating Blaschke products in closure in the Bloch norm of some analytic function spaces.

Now, we characterize interpolating Blaschke products in $\mathcal{C}_\mathcal{B}(\mathcal{D}_p^\lambda \cap \mathcal{B})$ as follows.

THEOREM 4.1. *Let* $0 < p, \lambda < 1$ *and* B *be an interpolating Blaschke product with zero set* $\{z_n\}_{n=1}^\infty$. *Then* $B \in \mathcal{C}_\mathcal{B}(\mathcal{D}_p^\lambda \cap \mathcal{B})$ *if and only if*

$$\sum_{n=1}^\infty (1-|z_n|)^{p(1-\lambda)} < \infty. \tag{8}$$

Proof. Suppose (8) holds. Note that

$$|B'(z)| \leqslant \sum_{n=1}^\infty \frac{1-|z_n|^2}{|1-\overline{z_n}z|^2}, \; z \in \mathbb{D}.$$

Hence, for any $\varepsilon > 0$, we know that

$$z \in \Omega_\varepsilon(B) = \{z \in \mathbb{D} : (1-|z|^2)|B'(z)| \geqslant \varepsilon\}.$$

Then, using Lemma 2.2, we obtain

$$\begin{aligned}
&\int_{\Omega_\varepsilon(B)}\left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\right)^{p\lambda}\frac{dA(z)}{(1-|z|^2)^{2-p}}\\
\leqslant&\frac{1}{\varepsilon}\sum_{n=1}^{\infty}(1-|z_n|^2)\int_{\Omega_\varepsilon(B)}\frac{(1-|\sigma_a(z)|^2)^{p\lambda}(1-|z|^2)^{p-p\lambda-1}}{|1-\overline{z_n}z|^2}dA(z)\\
\leqslant&\frac{1}{\varepsilon}\sum_{n=1}^{\infty}(1-|z_n|^2)(1-|a|^2)^{p\lambda}\int_{\mathbb{D}}\frac{(1-|z|^2)^{p-1}}{|1-\overline{z}z_n|^2|1-\overline{z}a|^{2p\lambda}}dA(z)\\
\leqslant&\frac{1}{\varepsilon}\sum_{n=1}^{\infty}\frac{C(1-|a|^2)^{p\lambda}}{(1-|z_n|^2)^{-p}|1-\overline{a}z_n|^{2p\lambda}}\\
\lesssim&\sum_{n=1}^{\infty}(1-|z_n|)^{p(1-\lambda)}(1-|\sigma_a(z_n)|^2)^{p\lambda}\\
\lesssim&\sum_{n=1}^{\infty}(1-|z_n|)^{p(1-\lambda)}<\infty.
\end{aligned}$$

Applying Theorem 2.6, we see that $B\in\mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda\cap\mathcal{B})$.

Conversely, suppose $B\in\mathcal{C}_{\mathcal{B}}(\mathcal{D}_p^\lambda\cap\mathcal{B})$. Since B is an interpolating Blaschke product with zero set $\{z_n\}_{n=1}^\infty$, there exists $\delta>0$ such that condition (7) holds. Girela et al. [11] proved that

$$(1-|z|)|B'(z)|\geqslant\frac{\delta(1-\delta)}{8},$$

where $z\in\Delta(z_n,\delta/4)$ for every positive integer n. Hence

$$\bigcup_{n=1}^{\infty}\Delta(z_n,\delta/4)\subseteq\left\{z\in\mathbb{D}:(1-|z|)|B'(z)|\geqslant\frac{\delta(1-\delta)}{8}\right\}.$$

By [28], we know that $|\Delta(z_n,\delta/4)|\approx(1-|z_n|)^2$ and $1-|z|\approx1-|z_n|\approx|1-\overline{z_n}z|$ for all $z\in\Delta(z_n,\delta/4)$. Clearly, $\{\Delta(z_n,\delta/4)\}_{n=1}^\infty$ are pairwise disjoint. These together with Theorem 2.6, we have

$$\begin{aligned}
\infty>&\int_{\{z\in\mathbb{D}:(1-|z|)|B'(z)|\geqslant\frac{\delta(1-\delta)}{8}\}}\left(\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\right)^{p\lambda}\frac{dA(z)}{(1-|z|^2)^{2-p}}\\
\geqslant&\int_{\bigcup_{n=1}^\infty\Delta(z_n,\delta/4)}\left(\frac{1-|\sigma_{z_n}(z)|^2}{1-|z|^2}\right)^{p\lambda}\frac{dA(z)}{(1-|z|^2)^{2-p}}\\
\geqslant&\sum_{n=1}^{\infty}\int_{\Delta(z_n,\delta/4)}\left(\frac{1-|\sigma_{z_n}(z)|^2}{1-|z|^2}\right)^{p\lambda}\frac{dA(z)}{(1-|z|^2)^{2-p}}\\
\approx&\sum_{n=1}^{\infty}(1-|z_n|)^{p(1-\lambda)}.
\end{aligned}$$

The proof is complete. □

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